Symmetric Functions

Putnam Practice

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Although there is no general formula that takes us from coefficients $c_0, c_1, ..., c_n$ of the polynomial equation

$$c_0 x^n + c_1 x^{n-1} + ... + c_n = 0$$

to its roots $x_1, ...x_n$, there are formulas that take us from $c_0, c_1, ...c_n$ to a large and important class of symmetric functions. A *symmetric function* of $x_1, ..., x_n$ is one whose value is unchanged if $x_1, x_2, ..., x_n$ are permuted arbitrarily. For example, the following are symmetric functions of three variables:

$$Q(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

$$R(x_1, x_2, x_3) = \frac{x_1 + x_2}{x_3} + \frac{x_2 + x_3}{x_1} + \frac{x_3 + x_1}{x_2}$$

Certain symmetric functions serve as building blocks for all the rest. Let

$$\sigma_k = \sum x_{i_1} x_{i_2} ... x_{i_k}$$

where the sum is taken over all C_k^n choices of the indices $i_1, ...i_k$ from $\{1, 2, ...n\}$. Then σ_k is called the *k-th elementary symmetric function* of $x_1, ..., x_n$.

Theorem 1 *Every symmetric polynomial function of $x_1, x_2, ..., x_n$ is a polynomial function of $\sigma_1, \sigma_2, ..., \sigma_n$. The same conclusion holds if polynomial is replaced by rational function.*

For $n = 3$,

$$\sigma_1 = x_1 + x_2 + x_3$$

$$\sigma_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$$

$$\sigma_3 = x_1 x_2 x_3$$

It is easy to check that $Q(x_1, x_2, x_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$ and $R(x_1, x_2, x_3) = \frac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}$.

Theorem 2 *Let $x_1, x_2, ..., x_n$ be the roots of the polynomial equation*

$$x^n + c_1 x^{n-1} + ... + c_n = 0$$

and let σ_k be the k-th elementary function of the x_i. Then $\sigma_k = (-1)^k c_k$, $k = 1, 2, ..., n$.

Proof:

$$x^n + c_1 x^{n-1} + ... + c_n = (x - x_1)(x - x_2)...(x - x_n)$$

Example 1: Find all solutions of the system of equations

$$x + y + z = 0$$

$$x^2 + y^2 + z^2 = 6ab$$

$$x^3 + y^3 + z^3 = 3(a^3 + b^3)$$

Theorem 3 *Let $S_p = x_1^p + x_2^p + ... + x_n^p$, where $x_1, ...x_n$ are roots of the polynomial $x^n + c_1 x^{n-1} + ... + c_n = 0$. Then*

$$S_1 + c_1 = 0$$

$$S_2 + c_1 S_1 + 2c_2 = 0$$

$$S_n + c_1 S_{n-1} + ... + n c_n = 0$$

$$S_p + c_1 S_{p-1} + ... + c_n S_{p-n} = 0, p > n$$

Example 2: If $x + y + z = 1$, $x^2 + y^2 + z^2 = 2$ and $x^3 + y^3 + z^3 = 3$, find $x^4 + y^4 + z^4$.

Problems:

1. If x, y, z satisfy $x + y + z = 3$, $x^2 + y^2 + z^2 = 5$ and $x^3 + y^3 + z^3 = 12$ determine $x^4 + y^4 + z^4$.
2. If $x^2 + y^2 = 9$ and $x^3 + y^3 = 27$ determine all possible values of $x^4 + y^4$.
3. Let a, b, c be real numbers such that $a + b + c = 0$. show that

$$\frac{a^5 + b^5 + c^5}{5} = (\frac{a^2 + b^2 + c^2}{2})(\frac{a^3 + b^3 + c^3}{3}).$$