1. Group Actions on Trees

Definition 1.1. A group $G$ acts on a graph $X$, denoted by $G \times X \to X$, if $G$ acts on the vertices and edges of $X$:

1. $G \times \text{vert}(X) \to \text{vert}(X)$
2. $G \times \text{edge}(X) \to \text{edge}(X)$

and the action commutes with the usual incidence functions $o, t : \text{edge}(X) \to \text{vert}(X)$:

\[
\begin{align*}
o(gy) &= g(o(y)) \\
t(gy) &= g(t(y))
\end{align*}
\]

where $g \in G$, $x \in \text{vert}(X)$, and $y \in \text{edge}(X)$.

Definition 1.2. Let $G$ be a group and $X$ a graph upon which $G$ acts.

a. An inversion is a pair consisting of some $g \in G$ and an edge $y$ of $X$ such that $gy = \overline{y}$ (where $\overline{y}$ is the reverse edge of $y$).

b. If no such pair exists we say that $G$ acts without inversion on $X$.

In other words, the action does not map any edge to its reverse edge (and thus preserves the orientation of $X$).

If $G$ acts on $X$ without inversion, then we can define the quotient graph $G \backslash X$ (read: $X$ mod $G$) in an obvious way:

- The vertex set of $G \backslash X$ is the quotient of $\text{vert}(X)$ under the action of $G$: $\text{vert}(G \backslash X) = \{ Gx : x \in \text{vert}(X) \}$
- The edge set of $G \backslash X$ is the quotient of $\text{edge}(X)$ under the action of $G$: $\text{edge}(G \backslash X) = \{ Gy : y \in \text{edge}(X) \}$

Definition 1.3. A tree is a connected, nonempty graph without circuits.

Definition 1.4. Let $G$ be a group acting on a tree $X$ without inversion. A fundamental domain of $G \backslash X$ is a subgraph $T$ of $X$ such that $T \to G \backslash X$ is an isomorphism.

Proposition 1.5. Let $G$ be a group acting upon a tree $X$ without inversion. Then every subtree $T'$ of $G \backslash X$ lifts to a subtree $T$ of $X$.

Proposition 1.6. Let $G$ be a group acting without inversion on a tree $X$. A fundamental domain of $G \backslash X$ exists if and only if $G \backslash X$ is a tree.

Proof. ($\Rightarrow$) Let $T$ be a fundamental domain of $G \backslash X$. Since $X$ is connected and non-empty, then $G \backslash X \cong T$ is connected and non-empty. So $T$ is a tree as a non-empty, connected subgraph of a tree. Thus $G \backslash X$ is a tree.

($\Leftarrow$) Suppose $G \backslash X$ is a tree. By Proposition 1.5, $G \backslash X$ is isomorphic to a subtree of $X$, call it $T$. This is the desired fundamental domain. □
2. Free Products with Amalgamated Subgroups

Definition 2.1. Suppose the $G = \langle S_G | R_G \rangle$ is a presentation of $G$ where $S_G$ is a set of generators and $R_G$ is a set of relations. Similarly, suppose that $H = \langle S_H | R_H \rangle$ is a presentation of $H$. Then the **free product of $G$ and $H$**, denoted $G * H$, is given by:

$$G * H = \langle S_G \cup S_H | R_G \cup R_H \rangle$$

Definition 2.2. Suppose that $G$ and $H$ are as defined above and contain an isomorphic copy of the group $F$. Let $i_G : F \hookrightarrow G$ and $i_H : F \hookrightarrow H$ be the respective inclusions. Let $R_F = \{ i_G(f)i_H(f)^{-1} : f \in F \}$. Then the **free product of $G$ and $H$ with amalgamated subgroup $F$** (also called the **amalgam of $G$ and $H$ over $F$**), denoted $G *_F H$, is given by:

$$G *_F H = \langle S_G \cup S_H | R_G \cup R_H \cup R_F \rangle$$

Example 2.3. Take

$$G = \mathbb{Z}/4\mathbb{Z} = \langle a | a^4 \rangle$$
$$H = \mathbb{Z}/6\mathbb{Z} = \langle b | b^6 \rangle$$
$$F = \mathbb{Z}/2\mathbb{Z} = \langle c | c^2 \rangle$$

along with the inclusions:

- $i_G : \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}$, where $i_G(\mathbb{Z}/2\mathbb{Z}) = \{0, 2\} \subset \mathbb{Z}/4\mathbb{Z}$
- $i_H : \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z}$, where $i_H(\mathbb{Z}/2\mathbb{Z}) = \{0, 3\} \subset \mathbb{Z}/6\mathbb{Z}$

So we have that $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} = \langle a, b | a^4, b^6, i_G(c)i_H(c)^{-1} \rangle$. 
3. Trees and Amalgams

As it turns out, every group that acts without inversion on a graph with a segment as fundamental domain is an amalgam of two groups, and the graph is a tree:

**Theorem 3.1.** Let $G$ be a group acting without inversion on a graph $X$, and let $T$ be a segment of $X$ that has edge $y$ (reverse edge $\overline{y}$) with $o(y) = P$, $t(y) = Q$. Suppose that $T$ is a fundamental domain of $G \setminus X$. Let $G_P$, $G_Q$, and $G_y = G_\overline{y}$ be the respective stabilizers of $y$, $P$, and $Q$ under the action of $G$. Then $X$ is a tree if and only if the homomorphism $G_P *_{G_y} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is an isomorphism. 

*Note:* This amalgam makes sense because $G_P \cap G_Q = G_y$.

**Proof.** We shall need the following two lemmas:

**Lemma 3.2.** $X$ is connected if and only if $G$ is generated by $G_P \cup G_Q$.

**Proof.** Let $X'$ be the component of $X$ containing $T$, and let $G'$ be the set of elements that stabilize $X'$, i.e. $G' = \{ g \in G : gX' = X' \}$. Let $G''$ be the subgroup of $G$ generated by $G_P \cup G_Q$.

We want to show that $G = G''$. Let $h \in G_P \cup G_Q$. Then $T$ and $hT$ share a common vertex, which gives that $hT \subset X' \Rightarrow hX' = X' \Rightarrow h \in G'$. Thus $G'' \subset G'$.

On the other hand, notice that $G''T$ and $(G - G'')T$ are disjoint subgraphs of $X$ whose union is $X$. So either $X' \subset G''T$ or $X' \subset (G - G'')T$. Since $T = 1_{G''}T \in G''T$ then $X' \subset G''T$. Thus $G' \subset G'' \Rightarrow G = G' = G''$.

Now $X = X'$ (i.e. $X$ is connected) if and only if $G = G' = G''$. \hfill \Box

**Lemma 3.3.** $X$ contains no circuit if and only if $G_P *_{G_y} G_Q \to G$ is injective.

**Proof.** We know that $X$ contains a circuit if and only if there is a path $c = (w_0, \ldots, w_n)$, $n \geq 1$, in $X$ without backtracking such that $o(c) = t(c)$. Write $w_i = h_iy_i$, where $h_i \in G$ and $y_i = y$ or $\overline{y}$. Passing to the quotient $G \setminus X \cong T$ we get that $\overline{y}_i = y_{i-1}$. Let $P_i = o(y_i) = t(y_{i-1})$. Notice that:

$$h_iP_i = h_i o(y_i) = o(h_iy_i) = t(h_{i-1}y_{i-1}) = h_{i-1}t(y_{i-1}) = h_{i-1}P_i$$

So $g_i \in G_P$, where $h_i = h_{i-1}g_i$. Also notice that $h_iy_i \neq h_{i-1}y_{i-1}$ so that $g_i \notin G_y$. Notice that $o(c) = t(c)$ is equivalent to writing $t(y_n) = P_0$, which is also equivalent to:

$$h_0P_0 = h_nP_0 = h_{n-1}g_nP_0 = h_{n-2}g_{n-1}g_nP_0 = \ldots = h_0g_1g_2 \cdots g_nP_0$$
i.e. \( g_1 g_2 \cdots g_n \in G_{P_0} \).

Thus, \( X \) contains a circuit if and only if we can find a sequence of vertices of \( T \) with \( \{ P_{i-1}, P_i \} = \{ P, Q \} \) for all \( i \) and a sequence of elements \( g_i \in G_{P_i} - G_y \) (\( 0 \leq i \leq n \)), such that \( g_0 g_1 \cdots g_n = 1 \). So \( G_P *_{G_y} G_Q \to G \) is not injective. \( \square \)

These two lemmas together form the statement: \( X \) is a tree if and only if \( G_P *_{G_y} G_Q \to G \) is an isomorphism. \( \square \)

The converse is also true: every amalgam of two groups acts on a tree with a segment as fundamental domain:

**Theorem 3.4.** Let \( G = G_1 *_{A} G_2 \) be the amalgam of \( G_1 \) and \( G_2 \) over \( A \).
Then there is a tree \( X \) (unique up to isomorphism) on which \( G \) acts, with the segment \( T \) (as defined in the previous theorem) as fundamental domain, where \( G_P = G_1, G_Q = G_2, \) and \( G_y = A \) are the respective stabilizers of the vertices and edges.

**Proof.** Up to isomorphism, \( X \) is the following graph:

\[
\text{vert}(X) = (G/G_1) \coprod (G/G_2) \\
\text{edge}(X) = (G/A) \coprod (G/A)
\]

with the inclusions \( A \to G_1 \) and \( A \to G_2 \) inducing the maps \( o : G/A \to G/G_1 \) and \( t : G/A \to G/G_2 \). Put \( P = 1 \cdot G_1, Q = 1 \cdot G_2, \) and \( y = 1 \cdot G_A \).

\( T \) is then a fundamental domain for the natural action of \( G \) on \( X \). The preceding theorem gives that \( X \) is a tree. \( \square \)

These theorems establish an equivalence between amalgams of two groups and groups acting on trees with a segment as a fundamental domain.

**Example 3.5.** To show that \( SL(2, \mathbb{Z}) \) is an amalgam of two groups, we need to

1. find a tree with a segment as a fundamental domain upon which \( SL(2, \mathbb{Z}) \) acts without inversion
2. compute the stabilizers of the vertices and the edge of the fundamental domain

Fortunately, \( SL(2, \mathbb{Z}) \) acts in a well-known way on the upper half plane via Möbius transformations. Let \( y \) be the circular arc consisting of the points \( z = e^{i\theta} \) for \( \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \). Let \( P = o(y) = e^{\pi i/3} \) and \( Q = t(y) = e^{\pi i/2} = i \).

Define the graph \( X \) to be the union of the transforms of \( y \) by the action of \( G \). Thus \( PQ \) is our desired fundamental domain.

It can be easily shown the action of \( G \) is without inversion and that \( X \) is, in fact, a tree. Theorem 3.1 implies that \( G \) is an amalgam of the stabilizers of \( P \) and \( Q \) over the stabilizer of \( y \):
$G_P$: Computing the stabilizer of $P$ as such $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot e^{\pi i/3} = e^{\pi i/3}$ yields a cyclic subgroup of $G = SL(2, \mathbb{Z})$ of order 6 generated by $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Thus $G_P = \mathbb{Z}/6\mathbb{Z}$.

$G_Q$: Computing the stabilizer of $Q$ as such $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot i = i$ yields a cyclic subgroup of $G = SL(2, \mathbb{Z})$ of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus $G_P = \mathbb{Z}/4\mathbb{Z}$.

$G_y$: Since $G_y = G_P \cap G_Q$, we can see that $G_y = \mathbb{Z}/2\mathbb{Z}$.

We are now able to express $SL(2, \mathbb{Z})$ as an amalgam of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}/2\mathbb{Z}$:

$$SL(2, \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \ast_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$