Honors Research Thesis: Isomorphisms Between the Tensor Products of Crystal Graphs

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Abstract

This paper provides an introduction to the concept of crystal graphs, with particular emphasis on the tensor products between the crystal graphs of modules over the special linear algebra. Covered are the fundamental topics needed to introduce the reader to crystal graphs, including an introduction to Lie algebras and quantum algebras over the special linear algebra.

1 Lie Algebras

Definition 1. *A **Lie algebra** over a field F is an F-vector space L, together with a bilinear map $[-,-]$ from $L \times L \to L$ such that*

1. $[x, x] = 0 \; \forall x \in L$
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \; \forall x, y, z \in L$

Now, given this general definition of Lie algebras, we examine two particular examples that will form the core of our future study. We call them the general linear algebra and the special linear algebra, respectively.

Example 1. *The **general linear algebra** is the vector space of all $n \times n$ matrices over F, written $\mathfrak{gl}_n$, with $[-,-]$ defined by*

$$[x, y] \equiv xy - yx$$

Example 2. *The **special linear algebra** is the subspace of $\mathfrak{gl}_n$ of all elements that have trace 0. It's denoted $\mathfrak{sl}_n$. For $n = 2$ and $F = \mathbb{C}$, it has the following basis elements:*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And for $n > 2$, it has basis elements $e_i, f_i, h_i, 0 \leq i < n$, where each e_i is the matrix with all 0 entries except for the entries about the i-th term along the diagonal, which have values given by e above (similarly for f_i, h_i). For example, the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ has bases generated by

$$e_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$
$$f_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$
$$h_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3}$$

We call the set of all iI.

To examine the relationship and structure of these two Lie algebras, we first need to define several concepts familiar to familiar algebraic structures, as they pertain to Lie algebras

Definition 2. *Let L be a Lie algebra, and let K be a vector subspace of L. We say K is a* ***Lie subalgebra*** *of$L$ if $\forall x, y \in K$,*

$$[x, y] \in K$$

Definition 3. *Let L_1, L_2 be Lie algebras, with a linear map $\varphi : L_1 \to L_2$. We say φ is a* ***homomorphism*** *of Lie algebras if for all $x, y \in L_1$,*

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

Definition 4. *An* ***ideal*** *of Lie algebra L is a subspace $I \subseteq L$ such that $\forall x \in L, y \in I$,*

$$[x, y] \in I$$

Now, we note that for all ideals I, J of Lie algebra L,

$$I + J = \{x + y | x \in I, y \in J\}$$

and

$$[I, J] = \text{span}\{[x, y] | x \in I, y \in J\}$$

are ideals as well. Next since our definition of ideals of Lie algebras is similar to those of ideals on other algebraic structures, it makes sense to examine quotient algebras as well:

Definition 5. *If I is an ideal of Lie algebra L, then the **quotient algebra** is the set*

$$L/I = \{z + I | z \in L\}$$

With the Lie bracket

$$[w + I, z + I] = [w, z] + I$$

As in other algebraic structures with ideals and quotient groups, the three Isomorphism Theorems apply, the first of which tells us

Lemma 1. *Let $\varphi : L_1 \to L_2$ be a Lie algebra homomorphism. Then $ker(\varphi)$ is an ideal of L_1, $im(\varphi)$ is a subalgebra of L_2, and*

$$L_1/ker(\varphi) \cong im(\varphi)$$

An immediate result of this theorem tells us that

$$\mathfrak{gl}_n/\mathfrak{sl}_n \cong F$$

under the map $\varphi(x) = \text{trace}(x)$. Thus we also have that $\mathfrak{sl}_n$ is an ideal of $\mathfrak{gl}_n$.

Now we move on to introduce the related concepts of representations and modules of Lie algebras, on which we will build the rest of our study.

Definition 6. *Let L be a Lie algebra over F. A **representation** of L is a Lie algebra homomorphism*

$$\varphi : L \to \mathfrak{gl}(V)$$

where V is a finite-dimensional vector space over F.

Definition 7. *Let L be a Lie algebra over F. A **Lie module** for L is a finite-dimensional F-vector space V together with a map $L \times V \to V$ such that*

1. $(\lambda x + \mu y).v = \lambda(x.v) + \mu(y.v)$
2. $x.(\lambda v + \mu w) = \lambda(x.v) + \mu(x.w)$

3. $[x, y].v = x(y.v) - y(x.v)$

These two concepts are related since given a representation $\varphi : L \to \mathfrak{gl}(V)$, we may make V into an L-module by defining

$$x.v \equiv \varphi(x)(v)$$

From here we wish to examine the structure of modules directly, particularly their behavior under the action of our Lie algebra. To do so, we need to define the concept of submodules:

Definition 8. *Suppose V is a module for Lie algebra L. A **submodule** of V is a subspace W of V which is invariant under the action of L. That is to say, for all $x \in L, w \in W$*

$$x.w \in W$$

*Similarly, W induces a **subrepresentation**.*

Definition 9. *A Lie-module V is said to be **irreducible** or **simple** if it is nontrivial and it has no submodules other than $\{0\}$ and V.*

Next, we describe all modules of $\mathfrak{sl}_2(\mathbb{C})$, building upon two lemmas:

Lemma 2. *Let $\mathbb{C}[X, Y]$ be the vector space of polynomials in X, Y with complex coefficients. Then, let*

$$V_d = \{p \in \mathbb{C}[X, Y] | p \text{ has degree } d\}$$

Then V_d is spanned by

$$X^d, X^{d-1}Y, ..., Y^d$$

So V_d has dimension $d + 1$. Then V_d is a Lie module of $\mathfrak{sl}_2(\mathbb{C})$, given by the map

$$\begin{aligned}
\varphi(e) &\equiv X\frac{\partial}{\partial Y} \\
\varphi(f) &\equiv Y\frac{\partial}{\partial X} \\
\varphi(h) &\equiv X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}
\end{aligned}$$

Further, V_d is irreducible as a $\mathfrak{sl}_2(\mathbb{C})$-module, and for any finite-dimensional, irreducible $\mathfrak{sl}_2(\mathbb{C})$-module V, V is isomorphic to one of the V_d.

We build upon this with our much stronger theorem, known as Weyl's theorem:

Lemma 3. *Let L be a complex Lie algebra such that*

$$L = \bigoplus L_n$$

*Where each L_n is simple (we call L **semisimple** in this case). Then, every finite-dimensional representation of L is completely reducible (can be written as the direct sum of irreducible submodules).*

This gives us our important classification of the modules over $\mathfrak{sl}_2(\mathbb{C})$: Every finite-dimensional module over $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the direct sum of some V_d.

2 Quantum Algebras

Next we want to introduce the concept of quantum algebras, structures that further build upon Lie algebras and the subject of our future study. First, a few combinatorial results:

Definition 10. *We define the operator $[-]$ for a variable v over $\mathbb{Q}$ as*

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}}$$

*Then, we may define the **Gaussian binomial coefficients** by*

$$\binom{a}{n}_q = \frac{[a][a-1]...[a-n+1]}{[1][2]...[n]}$$

For the sake of simplicity, we write

$$\binom{a}{n}_q = \frac{[a]!}{[n]![a-n]!}$$

These combinatorial definitions allow us to examine in depth the quantum algebra of $\mathfrak{sl}_2$:

Definition 11. *Let k be a field, with $q \in k$ where $q \neq 0$ and $q^2 \neq 1$. Then we define the **quantum algebra** of $\mathfrak{sl}_2$, denoted $U_q(\mathfrak{sl}_2)$ as the associateive algebra over k with generators E, F, K, K^{-1} such that*

1. $KK^{-1} = K^{-1}K$
2. $KEK^{-1} = q^2 E$
3. $KFK^{-1} = q^{-2} F$
4. $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

The significance of this quantum algebra lies in the fact that it has well-defined bases elements and no zero divisors.

Lemma 4. *$U_q(\mathfrak{sl}_2)$ has PBW-type bases and no zero divisors. PBW-type bases are monomials of the form*

$$F^s K^n E^r$$

Where $r, s, n \in \mathbb{Z}$, $r, s \geq 0$, that form a basis for $U_q(\mathfrak{sl}_2)$ as a vector space.

Now we want to examine the behavior of $U_q(\mathfrak{sl}_2)$-Modules, which can be broken into the direct sum of what we call weight spaces, which are as follows:

Definition 12. *If M is a $U_q(\mathfrak{sl}_2)$-Module, set for all $\lambda \in k, \lambda \neq 0$*

$$M_\lambda = \{m \in M | Km = q^\lambda m\}$$

That is to say, M_λ is the eigenspace of K acting on M for the eigenvalue of λ. We call M_λ a ***weight space*** *of M. All λ where $M_\lambda \neq 0$ are the* ***weights*** *of M, and the set of all λ which define unique M_λ is the* ***weight lattice*** *P of M.*

And this definition gives us our crucial characterization of these modules:

Lemma 5. *Suppose q is not a root of unity and that char(k) $\neq 2$. Let M be a finite-dimensional $U_q(\mathfrak{sl}_2)$-Module. Then M is the direct sum of its weight spaces, and all weights of M have the form $\pm q^a$, with $a \in \mathbb{Z}$. Further, M is semi-simple as a $U_q(\mathfrak{sl}_2)$-Module.*

3 Crystal Graphs

Now, having introduced Lie algebras and quantum algebras over them, we may finally introduce the core subject of our study: crystal graphs. To do so, we are going to consider $U_q(\mathfrak{sl}_2)$-Modules M^q that satisfy the following criteria:

1. $M^q = \bigoplus_{\lambda \in k} M^q_\lambda$, where each M^q_λ represents the weight space of M, with

   $$dim_{F(q)} M^q_\lambda < \infty$$

   $\forall \lambda \in P$.

2. There exists finitely many $\lambda_1, ..., \lambda_s \in P$ such that

   $$wt(M^q) \subseteq D(\lambda_1) \cup ... \cup D(\lambda_s)$$

   where $D(\lambda) = \{\mu \in P | \mu \leq \lambda\}$

3. The operators e_i and f_i are nilpotent on M^q for all $i \in I$

We now define the modified root operators $\tilde{e}_i$ and $\tilde{f}_i$, known as the Kashiwara Operators. First, a lemma:

Lemma 6. *Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a U-Module. Then for all $i \in I$, each weight vector $u \in M_\lambda$ may be written as*

$$u = u_0 + f_i u_1 + ... + f_i^{(N)} u_N$$

Where $N \in \mathbb{Z}_{\geq 0}$ and $u_k \in M_{\lambda + k\alpha_i} \cap ker(e_i)$.

This lemma tells us that the following definitions are well-defined:

Definition 13. *The* ***Kashiwara Operators*** *$\tilde{e_i}$ and $\tilde{f_i}$ on M are defined by:*

$$\tilde{e_i}.u = \sum_{k=1}^{N} f_i^{(k-1)} u_k$$
$$\tilde{f_i}.u = \sum_{k=0}^{N} f_i^{(k+1)} u_k$$

Now, we note that this implies that for all $i \in I$ and $\lambda \in P$

$$\tilde{e_i} M_\lambda = e_i.M_\lambda \subseteq M_{\lambda+\alpha_i}$$
$$\tilde{f_i} M_\lambda = f_i.M_\lambda \subseteq M_{\lambda-\alpha_i}$$

And further $\tilde{e_i}, \tilde{f_i}$ commute with U-Module homomorphisms.

Definition 14. *We now define A_0 to be*

$$\begin{aligned} A_0 &= \{f(q) \in F(q) | f \text{ is regular at } q = 0\} \\ &= \{g/h | g, h \in F[q], h(0) \neq 0\} \end{aligned}$$

This allows us to define the concept of a crystal lattice of M, as follows:

Definition 15. *Let M be a U-Module. A free A_0-submodule $\mathscr{L}$ of M is called a* ***crystal lattice*** *if*

1. *$\mathscr{L}$ generates M as a vector space over $F(q)$*
2. *$\mathscr{L} = \bigoplus_{\lambda \in P} \mathscr{L}_\lambda$, where $\mathscr{L} = \mathscr{L} \cap M_\lambda$, $\forall \lambda \in P$*
3. *$\forall i \in I, \tilde{e_i}.\mathscr{L} \subseteq \mathscr{L}$, $\tilde{f_i}.\mathscr{L} \subseteq \mathscr{L}$.*

Further, we call

$$\mathscr{L}/q\mathscr{L} \cong F \bigotimes_{A_0} \mathscr{L}$$

the crystal limit.

Note that since $\tilde{e}_i, \tilde{f}_i$ preserve $\mathscr{L}$, they act as operators on $\mathscr{L}/q\mathscr{L}$.

Finally, we may introduce the subject of our study: the crystal basis of a U-Module and the related concept of the crystal graph.

Definition 16. *A **crystal basis** of a U-Module M is a pair $(\mathscr{L}, \mathscr{B})$ such that*

1. *$\mathscr{L}$ is a crystal lattice of M*
2. *$\mathscr{B}$ is a f-basis of $\mathscr{L}/q\mathscr{L}$*
3. *$\mathscr{B} = \bigsqcup_\lambda \mathscr{B}_\lambda$ where $\mathscr{B}_\lambda = \mathscr{B} \cap (\mathscr{L}_\lambda/q\mathscr{L}_\lambda)$*
4. *$\forall i \in I$,*

$$\tilde{e}_i.\mathscr{B} \subseteq \mathscr{B} \cup \{0\}$$
$$\tilde{f}_i.\mathscr{B} \subseteq \mathscr{B} \cup \{0\}$$

5. *$\forall b, b' \in \mathscr{B}$ and $i \in I$, $\tilde{f}_i.b = b'$ if and only if $b = \tilde{e}_i.b'$*

From which, we build the crystal graph:

Definition 17. *Let $(\mathscr{L}, \mathscr{B})$ be a crystal basis. We define the **crystal graph** of M by representing $\mathscr{B}$ as a set of vertices, connected by I-colored arrows on $\mathscr{B}$ according to the rule that $b \to b'$ if and only if $b = \tilde{f}_i b'$.*

Now, let's draw the crystal graph over a $U_q(\mathfrak{sl}_2)$-Module:

Example 3. *For $m \in \mathbb{Z}_{\geq 0}$, let $V(m)$ be the $(m+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-Module with basis $\{u, f.u, ...f^{(m)}.u\}$, where*

$$E.u = 0$$
$$K.u = q^m u$$
$$f^{(k)}.u = \frac{1}{[k]!} f^k u$$

Now define

$$\mathscr{L}(m) = \bigoplus_{k=0}^{m} A_0 f^{(k)}.u$$
$$\mathscr{B}(m) = \{\overline{u}, \overline{fu}, ..., \overline{f^{(m)}u}\}$$

Where $\overline{f^{(k)}u}$ denotes the image of $f^{(k)}$ under the crystal limit. By the definition of Kashiwara operators, we have

$$\tilde{e}.f^{(k)}.u = f^{(k-1)}.u$$
$$\tilde{f}.f^{(k)}.u = f^{(k+1)}.u$$

So, $(\mathscr{L}(m), \mathscr{B}(m))$ *is a crystal basis of* $V(m)$*, with the crystal graph:*

$$\overline{u} \to \overline{f.u} \to ... \to \overline{f^{(m)}.u}$$

Now, we finally introduce the tensor product of crystal graphs:

Definition 18. *Let* M_j *be a* $U_q(\mathfrak{sl}_2)$*-Module with* $(\mathscr{L}_j, \mathscr{B}_j)$ *a crystal basis of* M_j $(j = 1, 2)$*. Set* $\mathscr{L} = \mathscr{L}_1 \bigotimes_{A_0} \mathscr{L}_2$ *and* $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$*. Then* $(\mathscr{L}, \mathscr{B})$ *is a crystal basis of* $M_1 \bigotimes_{F(q)} M_2$*, where*

$$\tilde{e_i}(b_1 \otimes b_2) = \begin{cases} \tilde{e_i} b_1 \otimes b_2 & \textit{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ \tilde{b_1} \otimes \tilde{e_i} b_2 & \textit{else} \end{cases}$$

$$\tilde{f_i}(b_1 \otimes b_2) = \begin{cases} \tilde{f_i} b_1 \otimes b_2 & \textit{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ \tilde{b_1} \otimes \tilde{f_i} b_2 & \textit{else} \end{cases}$$

References

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