Formulas in the C_2 Lie algebra Weight System

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Abstract

It is known that one can use the weight system of a Lie algebra or superalgebra as an invariant on chords by relating the chords to elements of the enveloping algebra of the Lie algebra or superalgebra. Recursive formulas for these invariants have already been found for the Lie algebra sl_2 (of type A_1) [3] and the Lie superalgebra $gl(1|1)$ [5]. These formulas were proved by finding relations between specific elements of the enveloping algebra or superalgebra. In this paper we show some relations for weight systems for the Lie algebra of type C_2, as well as results on linear independence of chord diagrams.

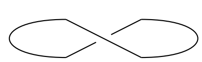
1 Introduction

In this paper, we investigate a kind of knot invariants called Vassiliev invariants. First, we must define a knot and an invariant. For the following section, we rely heavily on [2].

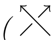
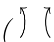
Definition 1 *A* ***knot*** *is an embedding of the circle* S^1 *into the Euclidean space* $\mathbb{R}^3$.

Example 1 *The knot*  *is called the* ***trefoil knot****,*
and  *is called the* ***unknot****.*

Definition 2 *We say two knots* K_1 *and* K_2 *are* ***equivalent*** *if we can continuously deform* K_1 *through* $\mathbb{R}^3$ *to obtain* K_2 *such that the deformations do not allow the knot to pass through itself.*

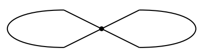
Example 2  *is equivalent to the unknot because in* $\mathbb{R}^3$*, we can move the top strand of the knot to below the knot in order to obtain a circle.*

Definition 3 *A* ***knot invariant*** *is a map from the set of knots to a set of values that is equal on equivalent knots.*

Example 3 *[1] The* ***Alexander-Conway polynomial*** *C is an invariant of oriented knots taking values in the ring $\mathbb{Z}[t]$ defined by the following two properties:*
(1) C ( *) $= 1$*
(2) C ( *) $-C($*  *) $= tC$ (*  *)*
where in (2), the diagrams are identical everywhere except for the crossing shown above.

From here, we can generalize knots to singular knots.

Definition 4 *A* ***singular knot*** *is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an embedding. We only consider singular knots with the simplest singularities, a finite number of points of intersection between two strands.*

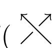
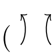
Example 4  *has a single point of self intersection, making it a singular knot.*

As with knots, we can define invariants on singular knots as well. Any knot invariant V can be extended to a singular knot invariant by means of the **Vassiliev skein relation**:

V () $= V$ () $-V($ ).

Definition 5 *A singular knot invariant $V : \mathcal{K} \to \mathbb{C}$ (where $\mathcal{K}$ is the space of knots) is said to be a* ***Vassiliev invariant*** *(or a* ***finite type invariant****) of order n if it vanishes on all knots with more than n singularities.*

Consider what happens when we apply this relation to the Alexander-Conway polynomial invariant above:

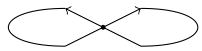
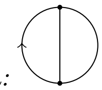
C () $= C$ () $-C($ ) $= tC$ ().

From here, we can see that the smallest power of t is bounded below by the number of singularites in a knot. Now we can easily define a Vassiliev invariant V_k from this by setting $V_k(K)$ to be the coefficient of t^k from the resulting polynomial given by the Alexander-Conway invariant. These are exactly the invariants given by the Vassiliev invariant arising from $gl(1|1)$ [5].

By work done in [7], the value of a Vassiliev invariant does not depend at all on the nonsingular crossings of the knot, so we can redraw a singular knot in a much simpler way: as a chord diagram.

Definition 6 *A* ***chord diagram*** *of order n (or degree n) is an oriented circle with $2n$ distinct points paired with lines.*

We obtain a chord diagram from an oriented singular knot (say, with n singularities) by marking on the parameterizing circle of the chord diagram the n pairs of points whose images are the n singularities of the knot.

Example 5  *becomes the chord diagram:* 

Hopefully, the results achieved below will lead to the calculation of a chord invariant similar to those found in [3] and [5].

2 Background

To understand the connection between invariants and Lie algebras, we must first define a Lie algebra.

Definition 7 *[4] Let F be a field. A **Lie algebra** over F is an F-vector space L with a bilinear map $L \times L \to L$ denoted $[x, y]$ where $x, y \in L$. This bilinear map is called the **Lie bracket** of L and it satisfies the following properties:*
$[x, x] = 0 \ \forall x \in L$
$[x, y] = -[y, x]$
$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \ \forall x, y, z \in L$

*The bracket is often called the **commutator** of x and y.*

Example 6 *The Lie algebra of type A_1 is $sl_2 := \{A \in M_2(\mathbb{C}) \,|Tr(A) = 0\}$ with the bracket $[A, B] = AB - BA$ where $M_2(\mathbb{C})$ is the set of 2×2 matrices with complex entries and AB denotes matrix multiplication.*

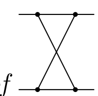
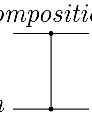
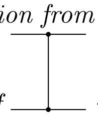
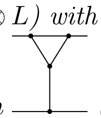
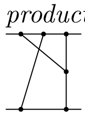
The definition of the Lie algebra of type C_n is $sp_{2n}(\mathbb{C})$, the set of $2n \times 2n$ symplectic matrices, or, more explicitly, $sp_{2n} = \left\{A \in M_{2n}(\mathbb{C}) \,\middle|\, \Omega A + A^T\Omega = 0\right\}$ where A^T denotes the transpose of the matrix A, and $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. [4]

The formulas we have in Section 3 are for chord diagrams with respect to the weight system of the Lie algebra of type C_2. The universal weight system for a Lie algebra is a function from chord diagrams with n vertices to $U(L)$, the universal enveloping algebra of L. We define $L^{\otimes n}$ to be the tensor product of the Lie algebra L with itself n times. To define this function, we define the process to create a linear map $f : \mathbb{C} \to L^{\otimes n}$. From there, by vector space properties, we only need to understand the value of $f(1)$, which will give us a vector in $L^{\otimes n}$. To obtain the universal weight system, we then embed $L^{\otimes n}$ in $\bigoplus_{n \geq 0} L^{\otimes n}$, which is an algebra via concatenation of tensors, and then mod out by the ideal generated by $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in L\}$ which gives us the universal enveloping algebra of L. To create the function f, we need to understand the connection of these with Feynman diagrams.

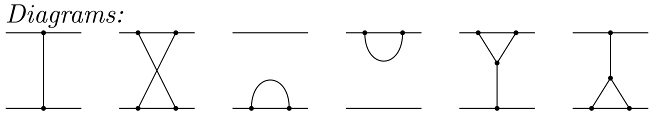
Definition 8 *A **Feynman diagram** $F_{n,m}$ is a graph with n upper vertices, m lower vertices, k inner vertices. $0 \leq m, n, k < \infty$ The upper and lower vertices have one edge attached to them, and the inner vertices have three edges.*

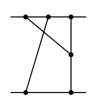
We can consider a Feynman Diagram $F_{n,m}$ as a function from $L^{\otimes n}$ to $L^{\otimes m}$ where L is our Lie algebra. In our case this is C_2, but this holds for any Lie algebra. As a note, zero vertices, $L^{\otimes 0}$, corresponds to the field of complex numbers, so a diagram $F_{0,m}$ is a function from $\mathbb{C}$ to $L^{\otimes m}$. Likewise, a diagram $F_{n,0}$ is a function from $L^{\otimes n}$ to $\mathbb{C}$.

Now, suppose we have two diagrams $F_{a,b}$ and $F_{c,d}$. If we want to create the graph of $F_{a+c,b+d}$, we place the graphs side by side: $F_{a,b}$ first, then $F_{c,d}$. To create the corresponding function, we take the function $F_{a+c.b+d}$ to be $F_{a,b} \otimes F_{c,d}$ where $\otimes$ denotes the tensor product. If we have the function and graph for $F_{a,b}$ and $F_{b,c}$, we can create the graph for $F_{a,c}$ by putting $F_{a,b}$ on top of $F_{b,c}$ matching up the b vertices in order. The function we get is $F_{a,c} = F_{b,c} \circ F_{a,b}$ where $\circ$ is the composition of the functions.

Example 7 *If we have the composition of the tensor product of*  *(which is a function from $L \otimes L$) with*  *(a function from L to L), and the tensor product of*  *with*  *(a function from $L \otimes L$ to L), we get the diagram*  *which is a function from $L^{\otimes 3}$ to $L^{\otimes 2}$.*

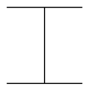
Theorem 1 *[6] Any Feynman Diagram can be written as a combination (tensor product or composition) of a finite number of the six elementary Feynman Diagrams:*

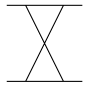


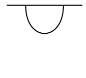
Example 8 *If we consider the diagram*  *from above, we see that it can be broken down into the four diagrams we used to create it.*

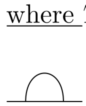
Because it is simple to tell the difference between an inner vertex and a crossing, from this point on, we no longer include the dots to indicate vertices. Now we have a way to combine graphs and a finite number of elementary diagrams from which we can build any diagram we like. As said earlier, Feynman diagrams can be thought of as functions, and we also know how to combine these functions in the same way as building a diagram. Thus, if we define the functions for the six basic diagrams, we can create explicit formulas for any diagram.

Because the Lie algebras we consider are algebras of matrices, we define the elementary diagrams with their functions as follows:

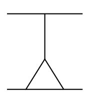
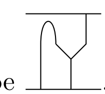
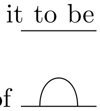
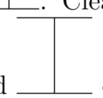
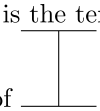
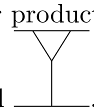
 is the identity map, $L \to L$.

 is a function $L \otimes L \to L \otimes L$ which takes the element $v \otimes w \mapsto w \otimes v$.

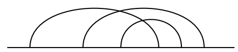
 is a function from $L \otimes L$ to our field, $\mathbb{C}$ which is called the Killing form, κ. To understand this function, first we must define the adjoint function, $ad : L \to M_d(\mathbb{C})$, where $M_d(\mathbb{C})$ denotes the $d \times d$ square matrices over the complex field (d is the dimension of L). For $x \in L$, $ad(x)$ is a function from L to L given by $ad(x)(y) = [x, y]$. This allows us to define $\kappa(x, y) := Tr(ad(x)ad(y))$, where Tr denotes the trace of the matrix.

 is a function $\kappa^* : \mathbb{C}$ to $L \otimes L$. Because $\mathbb{C}$ and $L \otimes L$ are vector spaces and $\mathbb{C}$ is one dimensional, we only need to know what this function does to the complex number, 1. In order to do this, L needs to be semisimple so that the killing form is nondegenerate. To do this, we define a basis for L, which we call $\{e_i\}_{i=1}^d$ (where d is the dimension of L). For each e_i, we then define e_i^* to be the element in L such that $\kappa(e_i^*, e_j) = \delta_{ij}$, where κ is as above, and δ_{ij} is the Kronecker delta function. Because the killing form is nondegenerate, this means that e_i^* is defined. From here, we define $\kappa^*(1) = \sum_{i=1}^d e_i \otimes e_i^*$.

 is the bracket operation from the Lie algebra, $A \otimes B \mapsto [A, B] := AB - BA$.

Looking at  topologically, we can deform it to be . Clearly, from here we can see that this is the tensor product of  and  composed with the tensor product of  and . This is how we define this function. This also demonstrates an important fact about these diagrams: isotopic diagrams yield equal functions. [6]

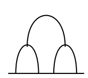
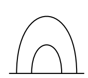
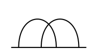
We can create a injection between chord diagrams and Feynman Diagrams with zero upper vertices and $2n$ lower vertices by choosing a non vertex point of the circle containing the vertices, and the points on either side of this non vertex point become the first and last of the $2n$ lower vertices of our diagram. We create edges in the diagram such that two points are connected in the Feynman diagram if and only if they are connected in the chord diagram. Because we have no vertices on the top, we do not draw the top line for convenience sake. Note that this is not well defined until we consider this in $U(L)$, the universal enveloping algebra. We work in $L^{\otimes n}$ instead of $U(L)$ because the relations we find in the tensor algebra hold in general, ie if we find a diagram relation, we can use this relation for subdiagrams as well without disturbing the rest of the diagram.

Example 9 *Intuitively we can find a point that is not a vertex and "cut" the diagram there, and "roll it flat."*
 *becomes*  *when we choose the point at the top of the circle.*

Thus, we have a Feynman Diagram as a function from our field to the Lie algebra. As noted in the definition of the functions above, we only need to know what this does on the complex number, 1. We apply our function to 1 to give us an element in $L^{\otimes n}$. This gives us an element of $L^{\otimes n}$ associated with the chord diagram. If we find sets of linearly dependent vectors, then these relations between the vectors of $L^{\otimes n}$ give us relations between the weight system on the chords. These relations can be used to prove recursive formulas for the weight system of a chord diagram giving us the universal Vassiliev invariant on the space of chords, as done in [3] for sl_2 and in [5] for $gl(1|1)$. The universal Vassiliev invariant is a function from the space of chord diagrams to the particular universal enveloping Lie algebra.

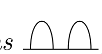
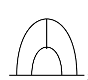
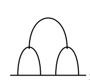
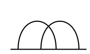
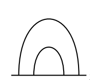
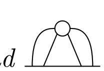
3 Results

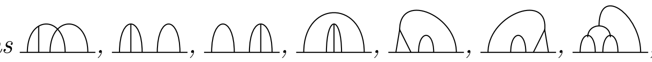
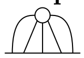
Using the MAGMA Computer Algebra System, we defined the six elementary diagrams in the general setting. Using those, we programmed several Lie algebras which would specify the space the diagrams would act on. We then programmed in corresponding Feynman diagrams built up from the elementary diagrams for the chords. Using MAGMA, we first verified the relations in [3, Theorem 6].

Theorem 2 *For L of type A_1,*  $= 2$  -2 

As we see in [3], this is all that is needed to inductively prove the Vassiliev invariant formula of [3, Theorem 1]. This is an example that our technique works.

Next we showed linear independence of certain diagrams.

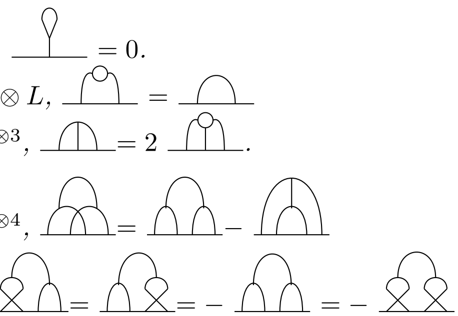
Proposition 1 *Diagrams* , , , , , *and*  *are linearly independent in $L^{\otimes 4}$ where L is the type C_2 Lie algebra.*

Proposition 2 *Diagrams* 
 *are linearly independent in $L^{\otimes 5}$ in type C_2.*

If a particular set was not linearly dependent, we found (mainly by trial and error) which diagrams were linearly dependent on others. Once we had a set of maximal linearly independent vectors, we created a subspace (using the subspace command in MAGMA) with this set of elements as a basis. Then we found the coordinates of the linearly dependent vector with respect to this basis, giving us relations between the chord diagrams.

Proposition 3 *We have the following relations where L is the type C_2 Lie algebra:*

In L, In $L \otimes L$, In $L^{\otimes 3}$, In $L^{\otimes 4}$, and



4 Future Work

The relations we have do not seem sufficient to find a recursive formula for the Vassiliev invariant for the Lie algebra of type C_2. Future work includes finding more relations of weight systems, particularly by inputting more Feynman diagrams into MAGMA to find more linear dependences among the diagrams. Once we believe we have the sufficient relations, then we hope to find the recursive formula for the Vassiliev invariant.

5 Acknowledgments

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6

References

[1] J.W. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* **30** (1928) 275–306.

[2] S. Chmutov, S. Duzhin, J. Mostovoy, Introduction to Vassiliev Knot Invariants, First Draft, July 20, 2010.

[3] S.V. Chmutov, A.N. Varchenko, Remarks on the Vassiliev Knot Invariants Coming from sl_2, *Topology* **36** (1997), no 1, 153–178.

[4] K. Erdmann, M.J. Wildon, Introduction to Lie Algebras, Springer Undergraduate Mathematics Series, Springer-Verlag London, (2006).

[5] J.M. Figueroa-O'Farrill, T. Kimura, A. Vaintrob, The universal Vassiliev invariant for the Lie superalgebra $gl(1|1)$, preprint, *Comm. Math. Phys.* **185** (1997), no. 1, 93127.

[6] Vaintrob, Arkady, Vassiliev Knot Invariants and Lie S-Algebras, *Mathematical Research Letters*, **1** (1994), 579-95.

[7] V.A. Vassiliev, Cohomology of knot spaces, Theory of Singularities and Its Applications, *adv. Soviet Math.* **1**, (1990) 23-69.