Sharp Sobolev Trace Inequalities on Riemannian Manifolds with Boundaries

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Abstract

In this paper, we establish some sharp Sobolev trace inequalities on n-dimensional, compact Riemannian manifolds with smooth boundaries. More specifically, let

$$q = 2(n-1)/(n-2)\,, \qquad \frac{1}{S} = \inf\left\{\int_{\mathbb{R}^n_+} |\nabla u|^2 : \nabla u \in L^2(\mathbb{R}^n_+)\,,\ \int_{\partial\mathbb{R}^n_+} |u|^q = 1\right\}.$$

We establish for any Riemannian manifold with a smooth boundary, denoted as (M,g), that there exists some constant $A = A(M,g) > 0$, $(\int_{\partial M} |u|^q\, ds_g)^{2/q} \le S\int_M |\nabla_g u|^2\, dv_g + A\int_{\partial M} u^2\, ds_g$, for all $u \in H^1(M)$. The inequality is sharp in the sense that the inequality is false when S is replaced by any smaller number. © 1997 John Wiley & Sons, Inc.

0 Introduction

It is well-known that sharp Sobolev-type inequalities are important in the study of partial differential equations, especially those that arise in geometry and physics. There has been much work on such inequalities and their applications (see, for example, Trudinger [35], Moser [30], Aubin [3], Talenti [34], Lieb [27, 28], Brezis-Nirenberg [9], Cherrier [13], Brezis-Lieb [8], Carleson-Chang [11], Escobar [14, 16], Beckner [6], Adimurthi and Yadava [1], Hebey and Vaugon [21, 20], Hebey [19], and the references therein).

For $n \ge 3$, it was shown by Aubin [3] and Talenti [34] that, for $p = 2n/(n-2)$,

$$\frac{1}{S_1} = \inf\left\{\frac{\int_{\mathbb{R}^n} |\nabla u|^2}{(\int_{\mathbb{R}^n} |u|^p)^{2/p}}\ \middle|\ u \in L^p(\mathbb{R}^n)\setminus\{0\},\ \nabla u \in L^2(\mathbb{R}^n)\right\}$$

is achieved and the extremal functions are found. In particular,

$$\text{(A)} \qquad \frac{1}{S_1} = \pi n(n-2)\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2/n}.$$

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It was shown by P. L. Lions [29] that, for $q = 2(n-1)/(n-2)$,

$$\frac{1}{S} = \inf\left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2}{(\int_{\partial\mathbb{R}^n_+} |u|^q)^{2/q}} \,\middle|\, \nabla u \in L^2(\mathbb{R}^n_+),\ u \in L^q(\partial\mathbb{R}^n_+) \setminus \{0\} \right\} \tag{0.1}$$

is achieved. The extremal functions were found independently by Escobar [16] and Beckner [6]. In particular,

$$\frac{1}{S} = \frac{n-2}{2}\sigma_n^{1/(n-1)}\,, \tag{B}$$

where σ_n denotes the volume of the unit sphere in $\mathbb{R}^n$.

In this paper we study some Sobolev-type trace inequalities on Riemannian manifolds with boundaries. Throughout this paper we denote $p = 2n/(n-2)$, $q = 2(n-1)/(n-2)$, and S_1 and S as in (A) and (B), respectively.

THEOREM 0.1 (Main Theorem) *For $n \geq 3$, let (M, g) be some smooth n-dimensional, compact, Riemannian manifold with a smooth boundary. Then there exists some constant $A = A(M, g) > 0$ such that, for all $u \in H^1(M)$,*

$$\left(\int_{\partial M} |u|^q \, ds_g\right)^{2/q} \leq S \int_M |\nabla_g u|^2 \, dv_g + A \int_{\partial M} u^2 \, ds_g\,, \tag{0.2}$$

where dv_g denotes the volume form of (M, g) and ds_g denotes the induced volume form on ∂M.

REMARK 0.2 The constant S in front of $\int_M |\nabla_g u|^2 \, dv_g$ is sharp. It cannot be replaced by any smaller number.

REMARK 0.3 In general, $\int_{\partial M} u^2 \, ds_g$ cannot be replaced by $\int_{\partial M} u^r \, ds_g$ for $r < 2$. For instance, this is the case for any bounded domain in $\mathbb{R}^n$ with the flat metric.

REMARK 0.4 The above theorem in the special case $n \geq 5$ and (M, g) a bounded, smooth domain in $\mathbb{R}^n$ with the Euclidean metric was obtained by Adimurthi and Yadava in [1]. Our method in proving Theorem 0.1 is different from theirs.

REMARK 0.5 Clearly we only need to consider the case when M is connected. Throughout the paper, we assume this.

The present work is stimulated by some recent work of Hebey and Vaugon [21], where they proved a conjecture of Aubin [4]: For $n \geq 3$ and (M, g) any smooth n-dimensional, compact manifold without boundary, there exists some constant $C > 0$ such that, for all $u \in H^1(M)$,

$$\left(\int_M |u|^p \, dv_g\right)^{2/p} \le S_1 \int_M |\nabla_g u|^2 \, dv_g + C \int_M u^2 \, dv_g \,. \tag{0.3}$$

One of the main ingredients in their proof of (0.3) is, through the Moser iteration technique, to obtain an appropriate upper bound for blowup minimum-type solutions to certain critical exponent equations with the zero Dirichlet boundary condition. Such asymptotic analysis was obtained by Han in [18], using the Moser iteration technique, for blowup minimum-type solutions to certain critical exponent equations with the zero Dirichlet boundary conditions in general domains in $\mathbb{R}^n$, which extend results of Atkinson and Peletier [2] and Brezis and Peletier [10] on balls in $\mathbb{R}^n$. Such extension was conjectured by Brezis and Peletier and was proven by Rey [31] using a different method.

During the past few years, energy-independent asymptotic analysis for blowup solutions to certain critical exponent equations has been obtained. See Schoen [32], Zhang [36], Chang, Gursky, and Yang [12], Li [24, 25, 23], Schoen and Zhang [33], and the references therein.

As in [21], one of the main ingredients in our proof of Theorem 0.1 is some asymptotic analysis for blowup minimum-type solutions. However, we need to overcome new difficulties since what we encounter here are certain nonlinear Neumann boundary conditions rather than zero Dirichlet boundary conditions as in [18] and [21]. Moreover, Theorem 0.1 for $n = 3$ is subtler: In addition to the upper bound of solutions obtained by the Moser iteration technique, we also need to obtain an appropriate lower bound.

Another main ingredient is local balance checking via the Pohozaev identity. Using similar methods, we have established some other Sobolev-type inequalities. In particular, we have extended theorem 1 in [1] from dimension $n \ge 5$ to $n \ge 3$. This will be addressed in a forthcoming paper.

1 Preliminary Estimates

We first present two weaker inequalities from which one can deduce that minimum-type solutions can blow up at only one point. Although this step is well-known, we include a proof here for the reader's convenience.

PROPOSITION 1.1 *For all $\varepsilon > 0$, there exists some constant $B(\varepsilon)$ depending only on ε, M, and g such that*

$$\left(\int_{\partial M} |u|^q \, ds_g\right)^{2/q} \\ \leq (S+\varepsilon) \int_M |\nabla_g u|^2 \, dv_g + B(\varepsilon) \int_M u^2 \, dv_g \,, \quad \forall u \in H^1(M) \,.$$

PROOF: By partition of unity, it follows easily from (0.1). We omit the details. ∎

PROPOSITION 1.2 *For all $\varepsilon > 0$, there exists some constant $A(\varepsilon)$ depending only on ε, M, and g such that for all $u \in H^1(M)$,*

$$\left(\int_{\partial M} |u|^q \, ds_g\right)^{2/q} \leq (S+\varepsilon) \int_M |\nabla_g u|^2 \, dv_g + A(\varepsilon) \int_{\partial M} u^2 \, ds_g \,. \tag{1.1}$$

PROOF: We prove this proposition using an argument by contradiction. Suppose the contrary of (1.1), namely, that there exists some constant $\delta > 0$ such that for all $\alpha > 1$,

$$\xi_\alpha := \inf_{H^1(M)\setminus\{0\}} \frac{\int_M |\nabla_g u|^2 \, dv_g + \alpha \int_{\partial M} u^2 \, ds_g}{(\int_{\partial M} |u|^q \, ds_g)^{2/q}} \leq \frac{1}{S} - \delta \,. \tag{1.2}$$

Claim. There exists some nonnegative function $u_\alpha \in H^1(M)$ satisfying

$$\xi_\alpha = \int_M |\nabla_g u_\alpha|^2 \, dv_g + \alpha \int_{\partial M} u_\alpha^2 \, ds_g \,, \quad \int_{\partial M} u_\alpha^q \, ds_g = 1 \,. \tag{1.3}$$

PROOF OF CLAIM: We sketch this well-known proof for the reader's convenience. Let $\{u^{(m)}\}$ be a minimizing sequence with

$$\|u^{(m)}\|_{q,\partial M} = \left(\int_{\partial M} |u^{(m)}|^q \, ds_g\right)^{1/q} = 1$$

and $u^{(m)} \geq 0$. Clearly, $\|u^{(m)}\|_{H^1(M)} \leq C$. After passing to a subsequence, $u^{(m)}$ converges weakly to some $u \in H^1(M)$, $u \geq 0$. It is not difficult to see that

$$\int_{\partial M} \left(|u^{(m)}|^q - |u^{(m)} - u|^q\right) \, ds_g = \int_{\partial M} u^q \, ds_g + o(1) \,,$$

and consequently

$$\int_{\partial M} |u^{(m)} - u|^q \, ds_g \leq 1 + o(1) \,, \quad \int_{\partial M} u^q \, ds_g \leq 1 \,,$$

where $o(1)$ denotes some quantity tending to zero as m tends to ∞.

Therefore, by the Sobolev embedding theorems and Proposition 1.1, we have, for $\varepsilon_0 > 0$,

$$\begin{aligned}
\xi_\alpha &= \int_M |\nabla u^{(m)}|^2 + \alpha \int_{\partial M} |u^{(m)}|^2 + o(1) \\
&= \int_M |\nabla (u^{(m)} - u)|^2 + \int_M |\nabla u|^2 + \alpha \|u\|^2_{2,\partial M} + o(1) \\
&= \int_M |\nabla (u^{(m)} - u)|^2 + \frac{B(\varepsilon_0)}{S+\varepsilon_0} \int_M |u^{(m)} - u|^2 + \int_M |\nabla u|^2 \\
&\quad + \alpha \|u\|^2_{2,\partial M} + o(1) \\
&\geq \frac{1}{S+\varepsilon_0} \left(\int_{\partial M} |u^{(m)} - u|^q \right)^{2/q} + \xi_\alpha \left(\int_{\partial M} u^q \right)^{2/q} + o(1) \\
&\geq \frac{1}{S+\varepsilon_0} \int_{\partial M} |u^{(m)} - u|^q + \xi_\alpha \int_{\partial M} u^q + o(1) \\
&= \left(\frac{1}{S+\varepsilon_0} - \xi_\alpha \right) \int_{\partial M} |u^{(m)} - u|^q + \xi_\alpha + o(1)\,.
\end{aligned}$$

Choose $\varepsilon_0 > 0$ small so that $\frac{1}{S+\varepsilon_0} - \xi_\alpha \geq \delta/2$; we have from the above that $\int_{\partial M} |u^{(m)} - u|^q = o(1)$. It follows easily that u is a minimum of (1.2). ∎

Now let u_α be some nonnegative function in $H^1(M)$ satisfying (1.3). It is easy to see from (1.3) that $\|u_\alpha\|_{H^1(M)}$ is bounded by some constant independent of α. It follows that, after passing to some subsequence, u_α weakly converges to some $\overline{u} \in H^1(M)$. This leads to

$$\int_M |u_\alpha - \overline{u}|^2 \, dv_g + \int_{\partial M} |u_\alpha - \overline{u}|^2 \, ds_g = o(1)\,, \tag{1.4}$$

and therefore, in view of (1.3),

$$\overline{u} = 0 \quad \text{on } \partial M\,. \tag{1.5}$$

Here and in the following, $o(1)$ denotes some quantity tending to zero as α tends to ∞.

Therefore, by Proposition 1.1, (1.4), and (1.5), we have, for $\varepsilon_0 > 0$,

$$\begin{aligned}
\xi_\alpha &= \int_M |\nabla u_\alpha|^2 + \alpha \int_{\partial M} |u_\alpha|^2 \\
&= \int_M |\nabla (u_\alpha - \overline{u})|^2 + \int_M |\nabla \overline{u}|^2 + \alpha \|u_\alpha\|^2_{2,\partial M} + o(1)
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_M |\nabla(u_\alpha - \overline{u})|^2 + o(1) \\
&\geq \frac{1}{S+\varepsilon_0}\left(\int_{\partial M} |u_\alpha - \overline{u}|^q\right)^{2/q} + o(1) \\
&= \frac{1}{S+\varepsilon_0} + o(1)\,.
\end{aligned}
$$

Sending α to ∞, we obtain from the above and (1.2) that

$$
\frac{1}{S} - \delta \geq \frac{1}{S+\varepsilon_0}\,.
$$

Sending ε_0 to zero, we reach a contradiction. ∎

2 Asymptotic Analysis

From now on, we begin to prove Theorem 0.1 through an argument by contradiction. Suppose the contrary of Theorem 0.1 is true; then we have, for all $\alpha \geq 1$,

$$
\xi_\alpha < \frac{1}{S}\,, \tag{2.1}
$$

where ξ_α is defined in (1.2). As in Section 1, there exists some nonnegative function $u_\alpha \in H^1(M)$ satisfying (1.3). It follows that u_α satisfies

$$
\begin{cases} -\Delta_g u_\alpha = 0 & \text{in } M \\ \frac{\partial_g u_\alpha}{\partial \nu} = \xi_\alpha u_\alpha^{q-1} - \alpha u_\alpha & \text{on } \partial M\,. \end{cases} \tag{2.2}
$$

In this section we establish, by using the Moser iteration technique, an appropriate upper bound for u_α.

For all $\varepsilon > 0$, it follows from (1.3), (2.1), and Proposition 1.2 that there exists some $A(\varepsilon)$ such that

$$
\begin{aligned}
1 + \frac{\varepsilon}{S} &> (S+\varepsilon)\xi_\alpha \\
&= (S+\varepsilon)\|\nabla_g u_\alpha\|_{2,M}^2 + \alpha(S+\varepsilon)\|u_\alpha\|_{2,\partial M}^2 \\
&\geq \left(\int_{\partial M} |u_\alpha|^q\right)^{2/q} + [\alpha(S+\varepsilon) - A(\varepsilon)]\,\|u_\alpha\|_{2,\partial M}^2 \\
&= 1 + [\alpha(S+\varepsilon) - A(\varepsilon)]\,\|u_\alpha\|_{2,\partial M}^2\,.
\end{aligned}
$$

Sending α to ∞, we have

$$(S+\varepsilon)\liminf_{\alpha\to\infty}\xi_\alpha \ge 1 \quad \text{and} \quad 1+\frac{\varepsilon}{S} \ge 1+(S+\varepsilon)\limsup_{\alpha\to\infty}\alpha\|u_\alpha\|^2_{2,\partial M}\,.$$

Sending ε to 0, we have, by using (2.1), that

$$\lim_{\alpha\to\infty}\xi_\alpha = \frac{1}{S} \tag{2.3}$$

and

$$\lim_{\alpha\to\infty}\alpha\|u_\alpha\|^2_{2,\partial M} = 0\,. \tag{2.4}$$

PROPOSITION 2.1 *There exists $\overline{x}_\alpha \in \partial M$ such that for all $\delta > 0$,*

$$\lim_{\alpha\to\infty}\int_{B_\delta(\overline{x}_\alpha)\cap\partial M} u_\alpha^q = 1\,.$$

Before proving the previous proposition, we present a well-known lemma (see, e.g., [5] for results of this type).

LEMMA 2.2 *Suppose $\{y_\alpha\} \in \partial M$ for a sequence of $\alpha\to\infty$ satisfies, for some $0<\beta<1$, $\delta>0$,*

$$\int_{B_\delta(y_\alpha)\cap\partial M} u_\alpha^q \le \beta\,. \tag{2.5}$$

Then

$$\lim_{\alpha\to\infty}\int_{B_{\delta/2}(y_\alpha)\cap\partial M} u_\alpha^q = 0\,. \tag{2.6}$$

PROOF: Let $\eta=\eta_\alpha \in C^\infty(\overline{M})$ be some cutoff function satisfying

$$\eta = \begin{cases} 1 & \text{in } B_{\delta/2}(y_\alpha)\cap\overline{M} \\ 0 & \text{in } \overline{M}\setminus B_\delta(y_\alpha) \end{cases}$$

and

$$|\nabla_g\eta| + |\nabla_g^2\eta| \le C(\delta, M, g)\,.$$

For $1 < r \le q-1$, multiplying the first equation in (2.2) by $\eta^2 u_\alpha^r$ and integrating by parts, we obtain, by using the boundary condition of u_α in (2.2), that

$$\int_M \nabla_g u_\alpha \cdot \nabla_g\left(\eta^2 u_\alpha^r\right) dv_g = \xi_\alpha \int_{\partial M} \eta^2 u_\alpha^{q-1+r}\, ds_g - \alpha\int_{\partial M}\eta^2 u_\alpha^{r+1}\, ds_g\,.$$

Direct calculation yields

$$\begin{aligned}
&\int_M \nabla_g u_\alpha \cdot \nabla_g\big(\eta^2 u_\alpha^r\big) dv_g \\
(2.7) &= \frac{4r}{(r+1)^2}\int_M \big|\nabla_g\big(u_\alpha^{(r+1)/2}\eta\big)\big|^2\,dv_g + \frac{r-1}{(r+1)^2}\int_M u_\alpha^{r+1}\Delta_g(\eta^2)\,dv_g \\
&\quad - \frac{4r}{(r+1)^2}\int_M u_\alpha^{r+1}|\nabla_g\eta|^2\,dv_g - \frac{r-1}{(r+1)^2}\int_{\partial M} u_\alpha^{r+1}\frac{\partial_g(\eta^2)}{\partial\nu}\,ds_g\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_M \big|\nabla_g\big(u_\alpha^{(r+1)/2}\eta\big)\big|^2 \\
&\quad = -\frac{r-1}{4r}\int_M u_\alpha^{r+1}\Delta_g(\eta^2) + \int_M u_\alpha^{r+1}|\nabla_g\eta|^2 + \frac{r-1}{4r}\int_{\partial M} u_\alpha^{r+1}\frac{\partial_g(\eta^2)}{\partial\nu} \\
&\qquad + \frac{\xi_\alpha(r+1)^2}{4r}\int_{\partial M} u_\alpha^{q-1+r}\eta^2 - \frac{\alpha(r+1)^2}{4r}\int_{\partial M} u_\alpha^{r+1}\eta^2 \\
&\quad \le \frac{\xi_\alpha(r+1)^2}{4r}\int_{\partial M} u_\alpha^{q-1+r}\eta^2 + C(\delta,r)\left\{\int_M u_\alpha^{r+1} + \int_{\partial M} u_\alpha^{r+1}\right\}.
\end{aligned}$$

Using (1.3), the fact that $r+1 \le q < p$, and the Sobolev embedding theorems, we know that

$$\begin{aligned}
&\int_M u_\alpha^{r+1} + \int_{\partial M} u_\alpha^{r+1} \\
(2.8) \qquad &\quad \le C(r,M)\left\{\int_M |\nabla_g u_\alpha|^2 + \int_{\partial M} u_\alpha^2\right\}^{(r+1)/2} \\
&\quad \le C(r,M)\,.
\end{aligned}$$

Consequently,

$$(2.9)\ \int_M \big|\nabla_g\big(u_\alpha^{(r+1)/2}\eta\big)\big|^2 \le \frac{\xi_\alpha(r+1)^2}{4r}\int_{\partial M} u_\alpha^{q-1+r}\eta^2 + C(\delta,r,M)\,.$$

Applying Hölder's inequality and then Proposition 1.1 to $u = u_\alpha^{(r+1)/2}\eta$ gives, for all $\varepsilon > 0$,

$$\begin{aligned}
(2.10) \qquad &\int_{\partial M} u_\alpha^{q-1+r}\eta^2 \\
&\quad \le \left(\int_{B_\delta(y_\alpha)\cap\partial M} u_\alpha^q\right)^{(q-2)/q}\left(\int_{\partial M}\big(u_\alpha^{(r+1)/2}\eta\big)^q\right)^{2/q}
\end{aligned}$$

$$\leq \Big(\int_{B_\delta(y_\alpha)\cap\partial M} u_\alpha^q \Big)^{(q-2)/q} (S+\varepsilon) \Big\| \nabla_g \big(u_\alpha^{(r+1)/2}\eta \big) \Big\|_{2,M}^2 + B(\varepsilon) \Big(\int_{B_\delta(y_\alpha)\cap\partial M} u_\alpha^q \Big)^{(q-2)/q} \Big\| u_\alpha^{(r+1)/2}\eta) \Big\|_{2,M}^2 .$$

Using the fact that $\beta < 1$, we now fix some $r \in (1, q-1]$ and $\varepsilon > 0$ satisfying

$$\frac{S+\varepsilon}{S-\varepsilon}\, \beta^{(q-2)/q}\, \frac{(r+1)^2}{4r} \leq 1 - \varepsilon .$$

Consequently, in view of (2.3), we have for α large

$$\xi_\alpha (S+\varepsilon)\, \beta^{(q-2)/q}\, \frac{(r+1)^2}{4r} \leq 1 - \varepsilon .$$

Combining (2.9), (2.10), (2.8), (2.5), and the above, we obtain

$$\int_M \big| \nabla_g \big(u_\alpha^{(r+1)/2}\eta \big) \big|^2 \leq C(\delta, r, \varepsilon, \beta, M) . \tag{2.11}$$

It follows from the Sobolev embedding theorems, (2.8), and (2.11) that

$$\begin{aligned} \int_{B_{\delta/2}(y_\alpha)\cap\partial M} & u_\alpha^{(r+1)q/2} \\ & \leq \int_{\partial M} \big(u_\alpha^{(r+1)/2}\eta \big)^q \\ & \leq C(M) \Big\{ \big\| \nabla_g \big(u_\alpha^{(r+1)/2}\eta \big) \big\|_{2,M}^q + \big\| u_\alpha^{(r+1)/2}\eta \big\|_{2,M}^q \Big\} \\ & \leq C(\delta, r, \beta, \varepsilon, M) . \end{aligned} \tag{2.12}$$

Since $(r+1)q/2 > q$, we can derive (2.6) from (2.4), (2.12), and Hölder's inequality. Lemma 2.2 is thereby established. ∎

PROOF OF PROPOSITION 2.1: For $x \in \partial M$, we define $\delta_{x,\alpha} > 0$ by

$$\int_{B_{\delta_{x,\alpha}}(x)\cap\partial M} u_\alpha^q = \frac{1}{2} . \tag{2.13}$$

Clearly, $\inf_{x\in\partial M} \delta_{x,\alpha} > 0$. We pick $\overline{x}_\alpha \in \partial M$ satisfying

$$\delta_{\overline{x}_\alpha,\alpha} \leq 2 \inf_{x\in\partial M} \delta_{x,\alpha} . \tag{2.14}$$

We claim that $\{\overline{x}_\alpha\}$ satisfies the property stated in the proposition. Suppose the contrary; then there exists some $\delta > 0$, $0 < \beta < 1$, and a sequence of $\alpha \to \infty$ such that

$$\int_{B_\delta(\overline{x}_\alpha)\cap\partial M} u_\alpha^q \le \beta\,.$$

This, according to Lemma 2.2, implies

$$\lim_{\alpha\to\infty}\int_{B_{\delta/2}(\overline{x}_\alpha)\cap\partial M} u_\alpha^q = 0\,.$$

Therefore, in view of (2.13), we have for large α

$$\delta_{\overline{x}_\alpha,\alpha} \ge \frac{\delta}{2}\,.$$

This, together with (2.14), yields for large α

$$\delta_{x,\alpha} \ge \frac{\delta}{4}\,, \quad \forall x \in \partial M\,. \tag{2.15}$$

Clearly, $\bigcup_{x\in\partial M} B_{\delta/8}(x)$ is an open cover of ∂M. Due to the compactness of ∂M, there exist $x_1,\dots,x_m \in \partial M$ such that $\partial M \subset \bigcup_{i=1}^m B_{\delta/8}(x_i)$. We see from (2.15) and (2.13) that

$$\int_{B_{\delta/4}(x_i)\cap\partial M} u_\alpha^q \le \frac{1}{2}\,, \quad 1 \le i \le m\,.$$

We can then apply Lemma 2.2 with δ replaced by $\delta/4$, $\beta = 1/2$, and $y_\alpha = x_i$ to conclude that

$$\lim_{\alpha\to\infty}\int_{\partial M} u_\alpha^q \le \lim_{\alpha\to\infty}\sum_{i=1}^m\int_{B_{\delta/8}(x_i)\cap\partial M} u_\alpha^q = 0\,.$$

This contradicts (1.3). Proposition 2.1 is thus established. ∎

Let $x_\alpha \in \overline{M}$ be some maximum point of u_α, that is,

$$u_\alpha(x_\alpha) = \max_{\overline{M}} u_\alpha\,.$$

It follows from the maximum principle that $x_\alpha \in \partial M$ unless u_α is identically equal to a constant. It is easy to see from (1.2) and (1.3) that u_α is not identically equal to a constant for α large. Therefore, $x_\alpha \in \partial M$ for large

α. Set $\mu_\alpha = u_\alpha(x_\alpha)^{-2/(n-2)}$. Since $\frac{\partial_g u_\alpha}{\partial \nu}(x_\alpha) \geq 0$, we see from (2.2) that $\alpha u_\alpha(x_\alpha) \leq \xi_\alpha u_\alpha(x_\alpha)^{q-1}$, that is,

$$\alpha\mu_\alpha \leq \xi_\alpha \leq C\,. \tag{2.16}$$

It follows that

$$\lim_{\alpha\to\infty} \mu_\alpha = 0\,.$$

Let $(y^1, \ldots, y^{n-1}, y^n)$ denote some geodesic normal coordinates given by the exponential map $\exp_{x_\alpha}$ with $\frac{\partial}{\partial y^n}$ being the unit inner normal of M at $y = 0$. In this coordinate system, the metric g is given by $g_{ij}(y)\, dy^i\, dy^j$. For suitably small $\delta_1 > 0$ (independent of α), we define v_α in a neighborhood of $z = 0$ by

$$v_\alpha(z) = u_\alpha(x_\alpha)^{-1} u_\alpha(\exp_{x_\alpha}(\mu_\alpha z))\,, \quad z \in O_\alpha \subset \mathbb{R}^n\,,$$

where

$$O_\alpha = \left\{ z \in \mathbb{R}^n : |z| < \frac{\delta_1}{\mu_\alpha}\,,\ \exp_{x_\alpha}(\mu_\alpha z) \in M \right\}\,. \tag{2.17}$$

We write $\partial O_\alpha = \Gamma^1_\alpha \cup \Gamma^2_\alpha$, where

$$\begin{aligned} \Gamma^1_\alpha &= \{ z \in \partial O_\alpha : \exp_{x_\alpha}(\mu_\alpha z) \in \partial M \}\,, \\ \Gamma^2_\alpha &= \{ z \in \partial O_\alpha : \exp_{x_\alpha}(\mu_\alpha z) \in M \}\,. \end{aligned}$$

It follows from (2.2) that v_α satisfies

$$\begin{cases} -\Delta_{g_\alpha} v_\alpha = 0 & \text{in } O_\alpha \\ \frac{\partial_{g_\alpha} v_\alpha}{\partial \nu} = \xi_\alpha v_\alpha^{q-1} - \alpha\mu_\alpha v_\alpha & \text{on } \Gamma^1_\alpha \\ v_\alpha(0) = 1\,, \quad 0 \leq v_\alpha \leq 1\,, \end{cases} \tag{2.18}$$

where g_α denotes the metric on O_α given by $g_\alpha = g_{ij}(\mu_\alpha z)\, dz^i\, dz^j$. It follows from (2.18), (2.16), and standard elliptic estimates (see, e.g., [17]) that for all $R > 1$,

$$\|v_\alpha\|_{C^3(B_R \cap \overline{O}_\alpha)} \leq C(R)\,, \quad \forall \alpha \geq 1\,. \tag{2.19}$$

Notice that because $v_\alpha(0) = 1$, we know from (2.19) that

$$\begin{cases} \int_{B_1(0)\cap\Gamma^1_\alpha} v_\alpha^q\, ds_{g_\alpha} \geq 1/C > 0 \\ \int_{B_1(0)\cap\Gamma^1_\alpha} v_\alpha^2\, ds_{g_\alpha} \geq 1/C > 0\,. \end{cases} \tag{2.20}$$

It follows from the first inequality in (2.20) and Proposition 2.1 that

$$\lim_{\alpha\to\infty} |x_\alpha - \overline{x}_\alpha| = 0\,. \tag{2.21}$$

By change of variables, we have

$$\alpha \|u_\alpha\|^2_{2,\partial M} \ge \alpha \int_{B_{\mu_\alpha}(x_\alpha)\cap\partial M} u_\alpha^2 = \alpha\mu_\alpha \int_{B_1\cap\Gamma^1_\alpha} v_\alpha^2 \, ds_{g_\alpha}\,.$$

We derive from (2.4), the second inequality in (2.20), and the above that

$$\lim_{\alpha\to\infty} \alpha\mu_\alpha = 0\,. \tag{2.22}$$

It follows from (2.19) that there exists $v \in C^2(\overline{\mathbb{R}}^n_+)$ such that along some subsequence,

$$\lim_{\alpha\to\infty} \|v_\alpha - v\|_{C^2(B_R\cap\overline{O}_\alpha)} = 0\,, \quad \forall R > 0\,, \tag{2.23}$$

where $B_R = \{z \in \mathbb{R}^n : |z| < R\}$. Clearly, in view of (2.18), (2.22), and (2.3), v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+ \\ \frac{\partial v}{\partial \nu} = \frac{1}{S} v^{q-1} & \text{on } \partial\mathbb{R}^n_+ \\ v(0) = 1\,, \quad 0 < v \le 1\,. \end{cases} \tag{2.24}$$

It follows from our earlier work [26] that

$$v(z', z_n) = \left(\frac{(n-2)^2 S^2}{|z'|^2 + (z_n + (n-2)S)^2} \right)^{(n-2)/2}, \tag{2.25}$$

where $z' = (z_1, \ldots, z_{n-1})$.

Due to the uniqueness of the limit function v, we know that (2.23) holds for the full limit $\alpha \to \infty$.

PROPOSITION 2.3 *For $\delta_1 = \delta_1(M, g) > 0$ small enough,*

$$\lim_{\alpha\to\infty} \int_{\Gamma^1_\alpha} |v_\alpha - v|^q = 0\,.$$

PROOF: Multiplying (2.24) by v and integrating by parts, we have

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 = \frac{1}{S} \int_{\partial\mathbb{R}^n_+} v^q\,.$$

We also know that v is a minimum of (0.1), namely,

$$S \int_{\mathbb{R}^n_+} |\nabla v|^2 = \left(\int_{\partial \mathbb{R}^n_+} v^q \right)^{2/q} .$$

Thus,

$$\int_{\partial \mathbb{R}^n_+} v^q = 1 . \tag{2.26}$$

It follows from (2.21) and Proposition 2.1 that

$$\lim_{\alpha \to \infty} \int_{\Gamma^1_\alpha} v^q_\alpha = \lim_{\alpha \to \infty} \int_{B_{\delta_1}(x_\alpha) \cap \partial M} u^q_\alpha = 1 . \tag{2.27}$$

It is easy to see from (2.26) and the explicit form of v in (2.25) that

$$\lim_{\alpha \to \infty} \int_{\Gamma^1_\alpha} v^q = 1 .$$

Therefore, for all $\varepsilon > 0$, there exists $R = R(\varepsilon) > 1$ such that for α large,

$$\int_{\Gamma^1_\alpha \cap B_R} v^q > 1 - \varepsilon , \qquad \int_{\Gamma^1_\alpha \setminus B_R} v^q < 2\varepsilon . \tag{2.28}$$

Consequently, using the strong convergence of v_α to v given in (2.23), we have, for α large, that

$$\int_{\Gamma^1_\alpha \cap B_R} |v_\alpha - v|^q < \varepsilon , \qquad \int_{\Gamma^1_\alpha \cap B_R} v^q_\alpha > 1 - 2\varepsilon . \tag{2.29}$$

We derive from (2.27) and the second inequality in (2.29) that for large α,

$$\int_{\Gamma^1_\alpha \setminus B_R} v^q_\alpha \le 3\varepsilon . \tag{2.30}$$

Combining the first inequality in (2.29), (2.30), and the second inequality in (2.28), we have for large α that

$$\begin{aligned} \int_{\Gamma^1_\alpha} |v_\alpha - v|^q &\le \int_{\Gamma^1_\alpha \cap B_R} |v_\alpha - v|^q + 2^q \int_{\Gamma^1_\alpha \setminus B_R} v^q_\alpha + 2^q \int_{\Gamma^1_\alpha \setminus B_R} v^q \\ &\le (1 + 2^{q+3}) \varepsilon . \end{aligned}$$

Proposition 2.3 follows immediately. ∎

Recall that the conformal Laplacian operator L_g and the conformal boundary operator B_g are given by (see, e.g., [15])

$$\begin{cases} L_g\psi = \Delta_g\psi - a(n)R_g\psi \\ B_g\psi = \frac{\partial_g\psi}{\partial\nu} + b(n)H_g\psi\,, \end{cases}$$

where $a(n) = \frac{n-2}{4(n-1)}$, $b(n) = \frac{n-2}{2}$, R_g is the scalar curvature of M, and H_g is the mean curvature of ∂M with respect to the inner normal of ∂M (e.g., the unit ball in $\mathbb{R}^n$ has positive mean curvature).

Let φ be some C^2-positive function on $\overline{M}$, and let $\hat{g} = \varphi^{4/(n-2)}g$. It is well-known (see, e.g., [15]) that for all $\psi \in H^1(M)$,

$$\begin{cases} L_{\hat{g}}(\psi/\varphi) = \varphi^{-(n+2)/(n-2)}L_g(\psi) & \text{in } M \\ B_{\hat{g}}(\psi/\varphi) = \varphi^{-n/(n-2)}B_g(\psi) & \text{on } \partial M\,. \end{cases} \tag{2.31}$$

Rewrite (2.2) as

$$\begin{cases} \Delta_g u_\alpha = 0 & \text{in } M \\ \frac{\partial_g u_\alpha}{\partial\nu} + b(n)H_g u_\alpha & \\ \quad = \xi_\alpha u_\alpha^{q-1} - \alpha u_\alpha + b(n)H_g u_\alpha & \text{on } \partial M\,. \end{cases} \tag{2.32}$$

Setting $w_\alpha = u_\alpha/\varphi$, it follows from (2.31) that

$$\begin{cases} \Delta_g u_\alpha - a(n)R_g u_\alpha & \\ \quad = \varphi^{(n+2)/(n-2)}(\Delta_{\hat{g}}w_\alpha - a(n)R_{\hat{g}}w_\alpha) & \text{in } M \\ \frac{\partial_g u_\alpha}{\partial\nu} + b(n)H_g u_\alpha & \\ \quad = \varphi^{n/(n-2)}(\frac{\partial_{\hat{g}}w_\alpha}{\partial\nu} + b(n)H_{\hat{g}}w_\alpha) & \text{on } \partial M\,. \end{cases} \tag{2.33}$$

We will choose an appropriate $\varphi = \varphi_\alpha$ and then apply the Moser iteration technique to show that w_α is bounded above by some constant independent of α. Without loss of generality, we assume (M, g) is a smooth, bounded open set of a slightly larger Riemannian manifold $(\widetilde{M}, g)$. Let γ be the geodesic in $\widetilde{M}$ with $\gamma(0) = x_\alpha$, $\gamma'(0) = \nu$. Set $P_\alpha = \gamma(t_\alpha\mu_\alpha)$ with $t_\alpha = (n-2)/\xi_\alpha$. Let $(y^1, \ldots, y^{n-1}, y^n)$ be some geodesic normal coordinate system of $T_{P_\alpha}\widetilde{M}$ with $\frac{\partial}{\partial y^n} = -\gamma'(t_\alpha\mu_\alpha)$, $\exp_{P_\alpha} : T_{P_\alpha}\widetilde{M} \to \widetilde{M}$ denoting the exponential map, and $g_{ij}(y) = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle$ denoting the metric of $\widetilde{M}$, with $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$, where Γ_{ij}^k is the Christoffel symbol. We define G_{P_α} by

$$\begin{cases} -\Delta_g G_{P_\alpha} = n(n-2)\omega_n\delta_{P_\alpha} & \text{in } \widetilde{M} \\ G_{P_\alpha} = 0 & \text{on } \partial\widetilde{M}\,, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It follows from Appendix B that

$$G_{P_\alpha} \circ \exp_{P_\alpha}(y) = |y|^{2-n} + E(y)\,,$$

where $E(y)$ satisfies

$$|y|^{n-3}|E(y)| + |y|^{n-2}|\nabla_g E(y)| \le C(\delta_1)\,, \quad \forall |y| \le \delta_1\,. \tag{2.34}$$

Define $\varphi_\alpha : \overline{M} \to \mathbb{R}$ by

$$\varphi_\alpha = t_\alpha^{n-2} \mu_\alpha^{(n-2)/2} G_{P_\alpha}\,.$$

Clearly, φ_α satisfies

$$-\Delta_g \varphi_\alpha = 0 \quad \text{in } M\,.$$

PROPOSITION 2.4 *There exists some constant C depending only on (M, g) such that for all $\alpha \ge 1$,*

$$u_\alpha \le C\varphi_\alpha \quad \textit{on } M\,.$$

PROOF: We only need to prove the proposition for α large. Set $w_\alpha = u_\alpha/\varphi_\alpha$ and $\hat{g} = \varphi_\alpha^{4/(n-2)} g$. Equation (2.33) holds in M for $\varphi = \varphi_\alpha$. Setting $\psi = \varphi = \varphi_\alpha$ in (2.31), we have

$$\begin{cases} -a(n)[R_{\hat{g}} - R_g \varphi_\alpha^{-4/(n-2)}] = 0 & \text{in } M \\ \frac{\partial_g \varphi_\alpha}{\partial \nu} + b(n) H_g \varphi_\alpha = b(n) H_{\hat{g}} \varphi_\alpha^{n/(n-2)} & \text{on } \partial M\,. \end{cases} \tag{2.35}$$

Combining (2.32), (2.33), and (2.35), we have

$$\begin{cases} \Delta_{\hat{g}} w_\alpha = 0 & \text{in } M\,, \\ \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} = \xi_\alpha w_\alpha^{q-1} - \big(\alpha \varphi_\alpha^{-2/(n-2)} + \frac{\partial_g \varphi_\alpha}{\partial \nu} \varphi_\alpha^{-n/(n-2)}\big) w_\alpha & \text{on } \partial M\,. \end{cases} \tag{2.36}$$

We need the following lemma to simplify (2.36):

LEMMA 2.5 *For α large,*

$$\alpha \varphi_\alpha^{-2/(n-2)} + \frac{\partial_g \varphi_\alpha}{\partial \nu} \varphi_\alpha^{-n/(n-2)} \ge 0 \quad \textit{on } \partial M\,.$$

PROOF: Clearly Lemma 2.5 is equivalent to

$$-\frac{\partial_g \varphi_\alpha}{\partial \nu} \le \alpha \varphi_\alpha \quad \text{on } \partial M\,. \tag{2.37}$$

Let $0 < \delta_2 \ll \delta_1$. It is clear from the proof how small we need δ_2 to be. It is independent of α. Notice that G_{P_α} is bounded below by some positive constant independent of α in $M\setminus B_{\delta_2}(x_\alpha)$; also, the absolute values of its first derivatives are bounded above by some constant independent of α in the same region. It is clear that (2.37) holds in $M\setminus B_{\delta_2}(x_\alpha)$ for large α.

In the y-coordinate, ∂M near x_α is given by

$$y^n = t_\alpha\mu_\alpha + f(y')\,, \quad |y'|\le \delta_1\,,$$

with $f(0)=0$, $|\nabla_g^2 f(y')|\le C(\delta_1)$, $\forall |y'|\le\delta_1$. By the choice of coordinates, $\frac{\partial}{\partial y^n}$ is orthogonal to the tangent space of $\widetilde{M}$ at x_α, which is spanned by $\frac{\partial}{\partial y^i}+\frac{\partial f}{\partial y^i}(0)\frac{\partial}{\partial y^n}$, $1\le i\le n-1$. Consequently, for $1\le i\le n-1$,

$$\frac{\partial f}{\partial y^i}(0) = -\frac{g_{in}}{g_{nn}} = O(\mu_\alpha^2)\,.$$

It follows from the mean value theorem that

$$\begin{cases} f(y') = O(|y'|^2+\mu_\alpha^2|y'|) \\ \frac{\partial f}{\partial y^i}(y') = O(|y'|+\mu_\alpha^2)\,, \quad 1\le i\le n-1\,. \end{cases} \tag{2.38}$$

Therefore,

$$\mu_\alpha \le C|y|\,, \quad \forall |y|\le\delta_1\,,\ y^n = t_\alpha\mu_\alpha+f(y')\,. \tag{2.39}$$

It is not difficult to see that at $\exp_{P_\alpha}(y', t_\alpha\mu_\alpha+f(y'))\in\partial M$,

$$\nu = \sum_{i=1}^{n-1}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^n} + O(|y'|^2+\mu_\alpha^2)\,,$$

$$\begin{aligned}\frac{\partial\varphi_\alpha}{\partial y^i} &= -(n-2)t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}y^i|y|^{-n} \\ &\quad + t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}\frac{\partial E}{\partial y^i}(y)\,, \quad 1\le i\le n\,.\end{aligned} \tag{2.40}$$

It is easy to see from the definition of φ_α and (2.34) that

$$\varphi_\alpha(y) \ge C^{-1}\mu_\alpha^{(n-2)/2}|y|^{2-n}\,, \quad |y|\le\delta_2\,. \tag{2.41}$$

It follows that for all $|y|\le\delta_2$, $y^n = t_\alpha\mu_\alpha+f(y')$, we have

$$\begin{aligned}\frac{\partial_g\varphi_\alpha}{\partial\nu} &= \nabla_g\varphi_\alpha\cdot\nu = \sum_{l,i=1}^{n} g^{li}\frac{\partial\varphi_\alpha}{\partial y^i}\frac{\partial}{\partial y^l}\cdot\nu \\ &= -\frac{\partial\varphi_\alpha}{\partial y^n} + O(|y||\nabla\varphi_\alpha|)\,.\end{aligned} \tag{2.42}$$

Also, from (2.34), (2.40), and (2.41), we have that

$$|y|\,|\nabla\varphi_\alpha| \le C\varphi_\alpha(y)\,, \quad \forall |y| \le \delta_2\,,\ y^n = t_\alpha\mu_\alpha + f(y')\,. \tag{2.43}$$

Combining (2.42) and (2.43), we have

$$\frac{\partial\varphi_\alpha}{\partial\nu} \ge -\frac{\partial\varphi_\alpha}{\partial y^n} - C\varphi_\alpha(y)\,, \quad \forall|y| \le \delta_1\,,\ y^n = t_\alpha\mu_\alpha + f(y')\,. \tag{2.44}$$

It follows from (2.34) and (2.41) that

$$\mu_\alpha^{(n-2)/2}|\nabla E(y)| \le C\varphi_\alpha(y)\,.$$

Using (2.38) and the above, we have, for $|y| \le \delta_1,\ y^n = t_\alpha\mu_\alpha + f(y')$, that

$$\begin{aligned}
-\frac{\partial\varphi_\alpha}{\partial y^n} &\ge (n-2)t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}y^n|y|^{-n} - C\varphi_\alpha(y)\\
&= (n-2)t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}(t_\alpha\mu_\alpha)|y|^{-n}\\
&\quad + (n-2)t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}f(y')|y|^{-n} - C\varphi_\alpha(y)\\
&\ge (n-2)t_\alpha^{n-2}\mu_\alpha^{(n-2)/2}(t_\alpha\mu_\alpha)|y|^{-n} - C\varphi_\alpha(y)\\
&\ge -C\varphi_\alpha(y)\,.
\end{aligned}$$

Lemma 2.5 follows from (2.44) and the above since, as pointed out earlier, (2.37) easily holds in $M \setminus B_{\delta_2}(x_\alpha)$ for large α. ∎

It follows from Lemma 2.5 and (2.36) that w_α satisfies

$$\begin{cases} \Delta_{\hat g} w_\alpha = 0 & \text{in } M\\ \frac{\partial_{\hat g} w_\alpha}{\partial\nu} \le \xi_\alpha w_\alpha^{q-1} & \text{on } \partial M\,. \end{cases} \tag{2.45}$$

Let η be some smooth, nonnegative cutoff function. Multiplying (2.45) by $w_\alpha^k\eta^2$ for $k > 1$ and integrating by parts, we obtain

$$\int_M \nabla_{\hat g} w_\alpha \nabla_{\hat g}\big(w_\alpha^k\eta^2\big)dv_{\hat g} \le \xi_\alpha \int_{\partial M} w_\alpha^{q-1+k}\eta^2\, ds_{\hat g}\,.$$

Here and in the following, C denotes some constant independent of α. As in (2.7), we have

$$\begin{aligned}
&\int_M \nabla_{\hat g} w_\alpha \nabla_{\hat g}\big(w_\alpha^k \eta^2\big) dv_{\hat g}\\
&\quad = \frac{4k}{(k+1)^2}\int_M \big|\nabla_{\hat g}\big(w_\alpha^{(k+1)/2}\eta\big)\big|^2 dv_{\hat g} + \frac{k-1}{(k+1)^2}\int_M w_\alpha^{k+1}\Delta_{\hat g}(\eta^2) dv_{\hat g}\\
&\qquad - \frac{4k}{(k+1)^2}\int_M w_\alpha^{k+1}|\nabla_{\hat g}\eta|^2 dv_{\hat g} - \frac{k-1}{(k+1)^2}\int_{\partial M} w_\alpha^{k+1}\frac{\partial_{\hat g}(\eta^2)}{\partial \nu} ds_{\hat g}.
\end{aligned}$$

We deduce from the last two formulae that

$$\begin{aligned}
&\int_M \big|\nabla_{\hat g}\big(w_\alpha^{(k+1)/2}\eta\big)\big|^2 dv_{\hat g}\\
&\quad \le -\frac{k-1}{4k}\int_M w_\alpha^{k+1}\Delta_{\hat g}(\eta^2) dv_{\hat g} + \int_M w_\alpha^{k+1}|\nabla_{\hat g}\eta|^2 dv_{\hat g}\\
&\qquad + \frac{k-1}{4k}\int_{\partial M} w_\alpha^{k+1}\frac{\partial_{\hat g}(\eta^2)}{\partial \nu} ds_{\hat g} + \frac{\xi_\alpha (k+1)^2}{4k}\int_{\partial M} w_\alpha^{q-1+k}\eta^2 ds_{\hat g}\,.
\end{aligned} \tag{2.46}$$

We still need the following lemma to start the Moser iteration process:

LEMMA 2.6 *There exists some $0 < \delta_0 \ll 1$, $s_0 > q$, and $C > 1$ independent of α such that*

$$\int_{\partial M \setminus B_{\mu_\alpha/\delta_0}(x_\alpha)} w_\alpha^{s_0} ds_{\hat g} \le C\,. \tag{2.47}$$

PROOF: For all $\varepsilon > 0$, it follows from Proposition 2.1 and Proposition 2.3 that there exists $0 < \delta_0 = \delta_0(\varepsilon) < 1$ such that

$$\int_{\partial M \setminus B_{\mu_\alpha/\delta_0}(x_\alpha)} w_\alpha^q ds_{\hat g} = \int_{\partial M \setminus B_{\mu_\alpha/\delta_0}(x_\alpha)} u_\alpha^q ds_g < \varepsilon\,.$$

Since $\hat g^{ij} \sim \mu_\alpha^2 \delta^{ij}$ in $B_{2\mu_\alpha/\delta_0}(x_\alpha) \setminus B_{\mu_\alpha/(4\delta_0)}(x_\alpha)$, we can choose η to be some cutoff function satisfying

$$\begin{cases} \eta(x) = 1\,,\ d(x_\alpha, x) \ge \mu_\alpha/\delta_0\,; \quad \eta(x) = 0\,,\ d(x_\alpha, x) \le \mu_\alpha/(2\delta_0)\\ |\nabla_{\hat g}\eta| + |\nabla_{\hat g}^2\eta| \le C\,. \end{cases}$$

We also take some $1 < k \le q-1$. It follows from (2.46) and Theorem A.1 in Appendix A that

$$\begin{aligned}
&\int_M \left|\nabla_{\hat g}\left(w_\alpha^{(k+1)/2}\eta\right)\right|^2 dv_{\hat g} \\
&\quad \le C(k,\delta_0) + \frac{\xi_\alpha (k+1)^2}{4k}\int_{\partial M} w_\alpha^{q-1+k}\eta^2\, ds_{\hat g} \\
&\quad \le C(k,\delta_0) \\
&\qquad + \frac{\xi_\alpha (k+1)^2}{4k}\left(\int_{\partial M}\left(w_\alpha^{(k+1)/2}\eta\right)^q ds_{\hat g}\right)^{2/q}\left(\int_{\partial M} w_\alpha^q ds_{\hat g}\right)^{(q-2)/q} \\
&\quad \le C(k,\delta_0) + C\varepsilon^{(q-2)/q}\int_M \left|\nabla_{\hat g}\left(w_\alpha^{(k+1)/2}\eta\right)\right|^2 dv_{\hat g}\,.
\end{aligned}$$

Taking $\varepsilon > 0$ small, we have

$$\int_M \left|\nabla_{\hat g}\left(w_\alpha^{(k+1)/2}\eta\right)\right|^2 dv_{\hat g} \le C\,.$$

It follows from Theorem A.1 in Appendix A that

$$\int_{\partial M}\left(w_\alpha^{(k+1)/2}\eta\right)^q ds_{\hat g} \le \left(\int_M \left|\nabla_{\hat g}\left(w_\alpha^{(k+1)/2}\eta\right)\right|^2 dv_{\hat g}\right)^{q/2} \le C\,.$$

Lemma 2.6 is established. ∎

REMARK 2.7 Without loss of generality, we can assume that δ_0 in Lemma 2.6 is small enough so that $B_{\mu_\alpha/\delta_0}(x_\alpha) \subset B_{4\mu_\alpha/\delta_0}(P_\alpha)$.

Set, for $\delta = \delta_0/10$,

$$R_i = \mu_\alpha \frac{\left(2 - \frac{1}{2^{i-1}}\right)}{\delta}\,, \quad i = 1, 2, 3, \dots.$$

Clearly

$$B_{\mu_\alpha/\delta_0}(x_\alpha) \subset B_{R_i}(P_\alpha) \quad \forall i\,. \tag{2.48}$$

Recall that for $\mu_\alpha/\delta < |y| \le 2\mu_\alpha/\delta$,

$$\frac{\mu_\alpha^{(2-n)/2}}{C} \le \varphi_\alpha(y) \le C\mu_\alpha^{(2-n)/2}, \quad C^{-1}\mu_\alpha^{-2} g \le \hat{g} \le C\mu_\alpha^{-2} g\,. \tag{2.49}$$

We can choose some smooth cutoff function η_i satisfying

$$\begin{cases} \eta_i(y) = 1\,,\ |y| > R_{i+1}; \quad \eta_i(y) = 0\,,\ |y| < R_i \\ |\nabla_{\hat{g}}\eta_i| \le C2^i\,, \quad |\nabla_{\hat{g}}^2 \eta_i| \le C4^i\,. \end{cases}$$

Taking $\eta = \eta_i$ in (2.46), we have

$$\begin{aligned} &\int_M \left|\nabla_{\hat{g}}\big(w_\alpha^{(k+1)/2}\eta_i\big)\right|^2 dv_{\hat{g}} \\ &\quad \le C4^i \int_{M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\, dv_{\hat{g}} + C2^i \int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\, ds_{\hat{g}} \\ &\qquad + \frac{C(k+1)^2}{k} \int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{q-1+k}\, ds_{\hat{g}}\,. \end{aligned} \tag{2.50}$$

It follows from (2.48), (2.49), and Theorem A.1 in Appendix A that

$$\begin{aligned} &\left[\int_{M\setminus B_{R_i}(P_\alpha)} \big(w_\alpha^{(k+1)/2}\eta_i\big)^p\, dv_{\hat{g}}\right]^{2/p} \\ &\quad \le C \int_{M\setminus B_{R_i}(P_\alpha)} \left|\nabla_{\hat{g}}\big(w_\alpha^{(k+1)/2}\eta_i\big)\right|^2 dv_{\hat{g}}\,, \end{aligned} \tag{2.51}$$

$$\begin{aligned} &\left[\int_{\partial M\setminus B_{R_i}(P_\alpha)} \big(w_\alpha^{(k+1)/2}\eta_i\big)^q\, ds_{\hat{g}}\right]^{2/q} \\ &\quad \le C \int_{M\setminus B_{R_i}(P_\alpha)} \left|\nabla_{\hat{g}}\big(w_\alpha^{(k+1)/2}\eta_i\big)\right|^2 dv_{\hat{g}}\,. \end{aligned} \tag{2.52}$$

Using (2.50), we can derive from (2.51) and (2.52) that

$$\begin{aligned} &\left[\int_{M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)p/2}\, dv_{\hat{g}}\right]^{2/p} \\ &\quad \le C4^i \int_{M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\, dv_{\hat{g}} + C2^i \int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\, ds_{\hat{g}} \\ &\qquad + \frac{C(k+1)^2}{k} \int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{q-1+k}\, ds_{\hat{g}}\,, \end{aligned} \tag{2.53}$$

and

$$\begin{aligned}
&\left[\int_{\partial M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)q/2}\,ds_{\hat g}\right]^{2/q}\\
&\le C4^i\int_{M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\,dv_{\hat g}+C2^i\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\,ds_{\hat g}\\
&\quad+\frac{C(k+1)^2}{k}\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{q-1+k}\,ds_{\hat g}\,.
\end{aligned}\tag{2.54}$$

Set $r_0=s_0/(q-2)$ where s_0 is given in Lemma 2.6. It follows from (2.47) and Hölder's inequality that

$$\begin{aligned}
&\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{q-1+k}\,ds_{\hat g}\\
&\quad=\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{q-2}w_\alpha^{k+1}\,ds_{\hat g}\\
&\quad\le C\left(\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)}\,ds_{\hat g}\right)^{(r_0-1)/r_0}.
\end{aligned}\tag{2.55}$$

It follows from (2.53), (2.54), and (2.55) that

$$\begin{aligned}
&\left[\int_{M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)p/2}\,dv_{\hat g}\right]^{2/p}+\left[\int_{\partial M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)q/2}\,ds_{\hat g}\right]^{2/q}\\
&\le C4^i\int_{M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\,dv_{\hat g}+C2^i\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\,ds_{\hat g}\\
&\quad+\frac{C(k+1)^2}{k}\left[\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)}\,ds_{\hat g}\right]^{(r_0-1)/r_0}.
\end{aligned}\tag{2.56}$$

By setting $\beta=q(r_0-1)/(2r_0)$, it is easy to see from $s_0>q$ that $\beta>1$. Since we can take s_0 close to q from the beginning, we can assume without loss of generality that $\beta\le p/2$. It follows from Hölder's inequality, (2.48), and (2.49) that

$$\begin{aligned}
&\left[\int_{M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)\beta}\,dv_{\hat g}\right]^{1/\beta}\\
&\qquad\qquad\le C\left[\int_{M\setminus B_{R_{i+1}}(P_\alpha)} w_\alpha^{(k+1)p/2}\,dv_{\hat g}\right]^{2/p}
\end{aligned}\tag{2.57}$$

and

$$\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{k+1}\, ds_{\hat g} \le \Big[\int_{\partial M\setminus B_{R_i}(P_\alpha)} w_\alpha^{(k+1)r_0/(r_0-1)}\, ds_{\hat g}\Big]^{(r_0-1)/r_0}. \tag{2.58}$$

Set $q_0 = 2r_0/(r_0-1) < q$, $q_i = \beta q_{i-1} = \beta^{i-1}q$, and $p_i = q_i(r_0-1)/r_0 = 2\beta^i$, where $i \ge 1$. Taking $k = p_i - 1$ $(i \ge 1)$ in (2.56) and using (2.57) and (2.58), we obtain

$$\begin{aligned}\|w_\alpha\|^{p_i}_{p_{i+1},M\setminus B_{R_{i+1}}(P_\alpha)} &+ \|w_\alpha\|^{p_i}_{q_{i+1},\partial M\setminus B_{R_{i+1}}(P_\alpha)}\\ &\le \left(C4^i + \frac{Cp_i^2}{(p_i-1)}\right)\Big(\|w_\alpha\|^{p_i}_{p_i,M\setminus B_{R_i}(P_\alpha)} + \|w_\alpha\|^{p_i}_{q_i,\partial M\setminus B_{R_i}(P_\alpha)}\Big).\end{aligned}$$

Since $\beta > 1$, we have $a^\beta + b^\beta \le (a+b)^\beta$ for all $a, b \ge 0$. It follows that

$$\begin{aligned}\Big(\|w_\alpha\|^{p_{i+1}}_{p_{i+1},M\setminus B_{R_{i+1}}(P_\alpha)} &+ \|w_\alpha\|^{p_{i+1}}_{q_{i+1},\partial M\setminus B_{R_{i+1}}(P_\alpha)}\Big)^{1/p_{i+1}}\\ &\le \Big(\|w_\alpha\|^{p_i}_{p_{i+1},M\setminus B_{R_{i+1}}(P_\alpha)} + \|w_\alpha\|^{p_i}_{q_{i+1},\partial M\setminus B_{R_{i+1}}(P_\alpha)}\Big)^{1/p_i}\\ &\le \left(C4^i + \frac{Cp_i^2}{p_i-1}\right)^{1/p_i}\Big(\|w_\alpha\|^{p_i}_{p_i,M\setminus B_{R_i}(P_\alpha)}\\ &\quad + \|w_\alpha\|^{p_i}_{q_i,\partial M\setminus B_{R_i}(P_\alpha)}\Big)^{1/p_i}.\end{aligned} \tag{2.59}$$

It is easy to see that

$$\left(C4^i + \frac{Cp_i^2}{p_i-1}\right)^{1/p_i} \le [C(4^i + 2\beta^i)]^{1/(2\beta^i)} \le C^{1/(2\beta^i)}(4+\beta)^{i/(2\beta^i)}.$$

Thus

$$\prod_{i=1}^{\infty}\left(C4^i + \frac{Cp_i^2}{p_i-1}\right)^{1/p_i} \le C < \infty.$$

It follows that

$$\begin{aligned}\|w_\alpha\|_{p_{i+1},M\setminus B_{R_{i+1}}(P_\alpha)} &\le C\Big(\|w_\alpha\|^{2\beta}_{p_1,M\setminus B_{R_1}(P_\alpha)} + \|w_\alpha\|^{2\beta}_{q_1,\partial M\setminus B_{R_1}(P_\alpha)}\Big)^{1/(2\beta)}\\ &\le C.\end{aligned}$$

Sending i to ∞, we have

$$\|w_\alpha\|_{L^\infty(M\setminus B_{2\mu_\alpha/\delta}(P_\alpha))} \le C(\delta)\,. \tag{2.60}$$

It is easy to see that inside $B_{2\mu_\alpha/\delta}(P_\alpha)$, $|y| \le C\mu_\alpha$. Therefore, it follows from (2.41) that $\forall y \in B_{2\mu_\alpha/\delta}(P_\alpha) : \varphi_\alpha(y) \ge C^{-1}\mu_\alpha^{-(n-2)/2}$. It follows that for all $y \in B_{2\mu_\alpha/\delta}(P_\alpha)$,

$$w_\alpha = \frac{u_\alpha}{\varphi_\alpha} \le C\mu_\alpha^{(n-2)/2}u_\alpha = Cu_\alpha/u_\alpha(x_\alpha) \le C\,. \tag{2.61}$$

Proposition 2.4 follows from (2.60) and (2.61). ∎

3 Balance Checking via Pohozaev Identity

In this section, we derive a contradiction by using the Pohozaev identity to do a balance checking in a ball centered at x_α of radius $1/\alpha$. The upper bound obtained in Section 2 plays a crucial role. For $n = 3$, it is subtler since we need to obtain an appropriate lower bound of u_α in order to reach a contradiction. This lower bound is obtained in this section by use of the maximum principle.

By choosing an appropriate coordinate system centered at x_α, we can assume without loss of generality that $x_\alpha = 0$, $g_{ij}(0) = \delta_{ij}$, $B_1^+(0) \subset M$, and $\{(x', 0) : |x'| < 1\} \subset \partial M$.

Let $R_\alpha = 1/(\alpha\mu_\alpha)$, $h_\alpha = g_{ij}(\mu_\alpha x)\,dx^i\,dx^j$ in $B^+_{10R_\alpha}(0)$, and

$$\bar v_\alpha(x) = \mu_\alpha^{(n-2)/2}u_\alpha(\mu_\alpha x) \quad \text{for } x \in B^+_{10R_\alpha}(0)\,.$$

It follows from (2.22) and (2.2) that $R_\alpha \to \infty$ as $\alpha \to \infty$, and $\bar v_\alpha$ satisfies

$$\begin{cases} \Delta_{h_\alpha}\bar v_\alpha = 0 & \text{in } B^+_{10R_\alpha}(0) \\ \frac{\partial_{h_\alpha}\bar v_\alpha}{\partial\nu} = \xi_\alpha \bar v_\alpha^{q-1} - \alpha\mu_\alpha\bar v_\alpha & \text{on } \{(x',0) : |x'| < 10R_\alpha\} \\ \bar v_\alpha(0) = 1\,, \quad 0 < \bar v_\alpha \le 1\,. \end{cases} \tag{3.1}$$

Clearly

$$|h_\alpha^{ij}(x) - \delta^{ij}| \le C|\mu_\alpha x|\,,\ |\Gamma^k_{ij}(x)| \le C\mu_\alpha \quad \text{in } B^+_{10R_\alpha}(0)\,, \tag{3.2}$$

where Γ^k_{ij} is the Christoffel symbol of h_α and C is, as always, some constant independent of α.

As explained in Section 2,

$$\lim_{\alpha\to\infty} \|\overline{v}_\alpha - v\|_{C^3(\overline{B^+_R(0)})} = 0 \quad \forall R > 0\,, \tag{3.3}$$

where v is the function defined in $\mathbb{R}^n_+$ given in (2.25). It is not difficult to see from Proposition 2.4 that

$$\bar{v}_\alpha(x) \le \frac{C}{1+|x|^{n-2}} \quad \text{for } x \in \bar{B}^+_{10R_\alpha}(0)\,. \tag{3.4}$$

We need some further estimates on $\bar{v}_\alpha$.

PROPOSITION 3.1 *For all $\alpha \ge 1$, $x \in B^+_{R_\alpha}(0)$, we have*

$$|\nabla \bar{v}_\alpha(x)| \le \frac{C}{1+|x|^{n-1}}\,, \quad |\nabla^2 \bar{v}_\alpha(x)| \le \frac{C}{1+|x|^{n}}\,,$$

where $|\nabla^2 \bar{v}_\alpha| = \sum_{i,j=1}^n |\partial^2 \bar{v}_\alpha / \partial x^i \partial x^j|$ and C is some constant independent of α and x.

PROOF: It follows from (3.3) that

$$|\nabla \bar{v}_\alpha(x)| < C\,,\ |\nabla^2 \bar{v}_\alpha(x)| \le C \quad \text{in } B^+_1(0)\,.$$

So we only need to show Proposition 3.1 for $|x| > 1$. For all $x_0 \in B^+_{R_\alpha}(0) \setminus B^+_1(0)$, set $R = |x_0|$, $\tilde{u}(x) = R^{n-2}\bar{v}_\alpha(Rx)$, and $\tilde{g}_{ij}(x) = (h_\alpha)_{ij}(Rx)$. It follows from (3.1) that

$$\begin{cases} \Delta_{\tilde{g}} \tilde{u} = 0 & \text{in } B^+_5(0) \setminus B^+_{1/5}(0) \\ \frac{\partial_{\tilde{g}} \tilde{u}}{\partial \nu} = R^{n-1}(\xi_\alpha \bar{v}_\alpha^{q-1}(Rx) - \alpha\mu_\alpha \bar{v}_\alpha(Rx)) & \text{on } \left\{(x',0) : \frac{1}{5} < |x'| < 5\right\}. \end{cases}$$

On $\left\{(x',0) : \frac{1}{5} < |x'| < 5\right\}$, we derive from (3.4) that

$$\left|\frac{\partial_{\tilde{g}} \tilde{u}}{\partial \nu}\right| = \left|R^{n-1}\big(\xi_\alpha \bar{v}_\alpha^{q-1}(Rx) - \alpha\mu_\alpha \bar{v}_\alpha(Rx)\big)\right| \le CR^{-1} + C\alpha\mu_\alpha R \le C\,.$$

It follows from standard elliptic estimates that for some $0 < \beta < 1$,

$$||\tilde{u}\|_{C^\beta(\overline{B^+_4 \setminus B^+_{1/4}})} \le C\,. \tag{3.5}$$

Rewriting the boundary condition of $\tilde{u}$ as

$$\frac{\partial_{\tilde{g}} \tilde{u}}{\partial \nu} = \xi_\alpha R^{-1} \tilde{u}^{q-1} - \alpha\mu_\alpha R \tilde{u}$$

and noticing (see (3.5))

$$\|\xi_\alpha R^{-1}\tilde{u}^{q-1} - \alpha\mu_\alpha R\tilde{u}\|_{C^\beta(\overline{B_4^+\setminus B_{1/4}^+})} \leq C\,,$$

we have, by standard elliptic estimates, that

$$\|\nabla_{\tilde{g}}\tilde{u}\|_{C^\beta(\overline{B_3^+\setminus B_{1/3}^+})} \leq C\,. \tag{3.6}$$

Therefore

$$|\nabla_{h_\alpha}\bar{v}_\alpha(x_0)| = R^{1-n}\left|\nabla_{\tilde{g}}\tilde{u}\left(\frac{x_0}{R}\right)\right| \leq C|x_0|^{1-n} \leq \frac{C}{(1+|x_0|^{n-1})}\,,$$

which gives us the gradient estimate.

Also, from (3.5) and (3.6), we know for some $0 < \beta' < \beta$,

$$\|\xi_\alpha R^{-1}\tilde{u}^{q-1} - \alpha\mu_\alpha R\tilde{u}\|_{C^{1,\beta'}(\overline{B_3^+\setminus B_{1/3}^+})} \leq C\,.$$

Thus by standard elliptic estimates

$$\|\nabla^2_{\tilde{g}}\tilde{u}\|_{C^{\beta'}(\overline{B_2^+\setminus B_{1/2}^+})} \leq C\,,$$

which gives us

$$|\nabla^2_{h_\alpha}\bar{v}_\alpha(x_0)| = R^{-n}\left|\nabla^2_{\tilde{g}}\tilde{u}\left(\frac{x_0}{R}\right)\right| \leq C|x_0|^{-n} \leq \frac{C}{(1+|x_0|^{n})}\,.$$

We have, in view of (3.2), established Proposition 3.1. ∎

For $n = 3$, we need to obtain an appropriate lower bound of $\bar{v}_\alpha$. Clearly one can also obtain lower bounds for $n \geq 4$ by the same method, but since we do not need it for the application in this paper, we omit it.

PROPOSITION 3.2 *For $n = 3$ and α large enough,*

$$\bar{v}_\alpha(x) \geq \frac{1}{C(1+|x|)} \quad \forall x \in \overline{B^+_{R_\alpha^{1/4}}(0)}\,,$$

where $C > 0$ is some constant independent of α.

PROOF: In view of (3.3), we only need to prove the above estimate for $|x| > 20$. In the following, α is always assumed to be suitably large. Let $\bar{x} = (0, \dots, 0, 1)$ and

$$G_\alpha(x) = \frac{1}{|x-\bar{x}|} - \frac{1}{R_\alpha^{1/2}|x-\bar{x}|^{1/2}} \quad \text{in } B_{R_\alpha^{1/3}}(\bar{x}) \setminus B_2(\bar{x})\,.$$

It is easy to see that

$$\frac{1}{2|x-\bar{x}|} \le G_\alpha(x) \le \frac{2}{|x-\bar{x}|} \quad \text{in } B_{R_\alpha^{1/3}}(\bar{x}) \setminus B_2(\bar{x})\,.$$

From (3.2) we have that in $B^+_{R_\alpha^{1/3}}(\bar{x}) \setminus B_2(\bar{x})$,

$$\Delta_{h_\alpha}\left(-\frac{1}{R_\alpha^{1/2}|x-\bar{x}|^{1/2}}\right) \ge \frac{1}{CR_\alpha^{1/2}|x-\bar{x}|^{5/2}}\,,$$
$$\left|\Delta_{h_\alpha}\left(\frac{1}{|x-\bar{x}|}\right)\right| \le \frac{C\mu_\alpha}{|x-\bar{x}|^2}\,,$$

where $B^+_{R_\alpha^{1/3}}(\bar{x}) = \left\{x \in \mathbb{R}^n_+ : |x-\bar{x}| < R_\alpha^{1/3}\right\}$. It follows that $\Delta_{h_\alpha} G_\alpha \ge 0$.

Also, it follows from (3.2) that for all $x = (x', 0),\ 1 < |x'| < R_\alpha^{1/3}$,

$$\frac{\partial_{h_\alpha}}{\partial\nu}\left(\frac{1}{|x-\bar{x}|}\right) \le -\frac{1}{C|x-\bar{x}|^3}\,,$$
$$\left|\frac{\partial_{h_\alpha}}{\partial\nu}\left(\frac{1}{R_\alpha^{1/2}|x-\bar{x}|^{1/2}}\right)\right| \le \frac{C}{R_\alpha^{1/4}|x-\bar{x}|^3}\,.$$

Therefore, using (3.4), we have, for all $x = (x', 0),\ 1 < |x'| < R_\alpha^{1/3}$,

$$-\alpha\mu_\alpha\bar{v}_\alpha - \frac{\partial_{h_\alpha}}{\partial\nu}(G_\alpha) \ge -\frac{C}{R_\alpha(1+|x|)} + \frac{1}{C|x-\bar{x}|^3} > 0\,.$$

We will use the maximum principle on $A = \left\{x \in \mathbb{R}^n_+ : 10 < |x-\bar{x}| < R_\alpha^{1/3}\right\}$. Let $\Sigma_1 = \partial A \cap \{x_n = 0\}$, $\Sigma_2 = \partial A \cap \{|x-\bar{x}| = 10\}$, and $\Sigma_3 = \partial A \cap \left\{|x-\bar{x}| = R_\alpha^{1/3}\right\}$. Choose $0 < \tau < 1$ small enough such that $\tau G_\alpha \le \bar{v}_\alpha$ on Σ_2. Define $H_\alpha = \tau G_\alpha - \max_{\Sigma_3}(\tau G_\alpha)$; then

$$\begin{cases} \Delta_{h_\alpha}(\bar{v}_\alpha - H_\alpha) \le 0 & \text{in } A \\ \bar{v}_\alpha - H_\alpha \ge 0 & \text{on } \Sigma_2 \cup \Sigma_3 \\ \frac{\partial_{h_\alpha}(\bar{v}_\alpha - H_\alpha)}{\partial\nu} > 0\,, & \text{on } \Sigma_1\,. \end{cases}$$

It follows from the maximum principle that

$$\bar{v}_\alpha \geq H_\alpha \quad \text{in } A\,.$$

Consequently, for all $x \in B^+_{R_\alpha^{1/4}}(0) \setminus B^+_{10}(\bar{x})$,

$$\bar{v}_\alpha(x) \geq H_\alpha(x) \geq \frac{C\tau}{|x-\bar{x}|} - \frac{C\tau}{R_\alpha^{1/3}} \geq \frac{C\tau}{2|x-\bar{x}|}\,.$$

Proposition 3.2 is established. ∎

For convenience, throughout the rest of this section we set $\Gamma_1 = \partial B^+_{R_\alpha}(0) \cap \{(x',0) : x' \in \mathbb{R}^{n-1}\}$ and $\Gamma_2 = \partial B^+_{R_\alpha}(0) \cap \{(x', x_n) : x_n > 0\}$. We always use dV for the volume element of the standard Euclidean metric, dS for the surface element of the standard Euclidean metric, ν for the unit outer normal vector of the corresponding surface with respect to the specified metrics, and $\cdot$ for the inner product under the standard Euclidean metric. The balance checking via the Pohozaev identity will be performed in $B^+_{R_\alpha}(0)$.

The following identity can easily be verified (see [22]):

$$2\Delta\bar{v}_\alpha(\nabla\bar{v}_\alpha \cdot x) = \operatorname{div}[2(\nabla\bar{v}_\alpha \cdot x)\nabla\bar{v}_\alpha - |\nabla\bar{v}_\alpha|^2 x] + (n-2)|\nabla\bar{v}_\alpha|^2\,.$$

It follows that

$$\begin{aligned}(3.7)\quad \int_{B^+_{R_\alpha}} \Delta\bar{v}_\alpha(\nabla\bar{v}_\alpha \cdot x)\,dV - \frac{n-2}{2}\int_{B^+_{R_\alpha}} |\nabla\bar{v}_\alpha|^2\,dV \\ = \frac{1}{2}\int_{B^+_{R_\alpha}} \operatorname{div}[2(\nabla\bar{v}_\alpha \cdot x)\nabla\bar{v}_\alpha - |\nabla\bar{v}_\alpha|^2 x]\,dV\,.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\int_{B^+_{R_\alpha}} \operatorname{div}[2(\nabla\bar{v}_\alpha \cdot x)\nabla\bar{v}_\alpha - |\nabla\bar{v}_\alpha|^2 x]\,dV \\ = \frac{1}{2}\int_{\partial B^+_{R_\alpha}} [2(\nabla\bar{v}_\alpha \cdot x)(\nabla\bar{v}_\alpha \cdot \nu) - |\nabla\bar{v}_\alpha|^2 (x \cdot \nu)]\,dS\,.\end{aligned}$$

It is easy to check that

$$(\nabla\bar{v}_\alpha \cdot x)(\nabla\bar{v}_\alpha \cdot \nu) = \begin{cases} \left(\frac{\partial\bar{v}_\alpha}{\partial\nu}\right)^2 |x| & \text{on } \Gamma_2 \\ \left(\sum_{i=1}^{n-1} x_i \frac{\partial\bar{v}_\alpha}{\partial x_i}\right)\frac{\partial\bar{v}_\alpha}{\partial\nu} & \text{on } \Gamma_1\,,\end{cases}$$

and

$$x \cdot \nu = \begin{cases} |x| & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_1 . \end{cases}$$

Therefore

$$\begin{aligned} &\frac{1}{2}\int_{B^+_{R_\alpha}} \operatorname{div}[2(\nabla \bar v_\alpha \cdot x)\nabla \bar v_\alpha - |\nabla \bar v_\alpha|^2 x]\, dV \\ &= \int_{\Gamma_1} \left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i}\right)\frac{\partial \bar v_\alpha}{\partial \nu} + \int_{\Gamma_2}\left[|x|\left(\frac{\partial \bar v_\alpha}{\partial \nu}\right)^2 - \frac{|x|}{2}|\nabla \bar v_\alpha|^2\right] dS \\ &= \int_{\Gamma_1} \left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i}\right)\frac{\partial \bar v_\alpha}{\partial \nu} + \int_{\Gamma_2}\frac{|x|}{2}\left[\left(\frac{\partial \bar v_\alpha}{\partial \nu}\right)^2 - |\partial_{\tan} \bar v_\alpha|^2\right] dS\, , \end{aligned} \tag{3.8}$$

where $\partial_{\tan}$ denotes the tangential differentiation on Γ_2.

On the other hand,

$$\int_{B^+_{R_\alpha}} |\nabla \bar v_\alpha|^2 = -\int_{B^+_{R_\alpha}} \bar v_\alpha \Delta \bar v_\alpha + \int_{\partial B^+_{R_\alpha}} \bar v_\alpha \frac{\partial \bar v_\alpha}{\partial \nu}\, .$$

Therefore,

$$\begin{aligned} &\int_{B^+_{R_\alpha}} \Delta \bar v_\alpha (\nabla \bar v_\alpha \cdot x) dV - \frac{n-2}{2}\int_{B^+_{R_\alpha}} |\nabla \bar v_\alpha|^2\, dV \\ &\quad = \int_{B^+_{R_\alpha}} \Delta \bar v_\alpha (\nabla \bar v_\alpha \cdot x) dV + \frac{n-2}{2}\int_{B^+_{R_\alpha}} \Delta \bar v_\alpha \bar v_\alpha\, dV \\ &\qquad - \frac{n-2}{2}\int_{\partial B^+_{R_\alpha}} \bar v_\alpha \frac{\partial \bar v_\alpha}{\partial \nu}\, . \end{aligned}$$

Combining (3.7), (3.8), and the above, we have

$$\begin{aligned} &\int_{B^+_{R_\alpha}} \Delta \bar v_\alpha (\nabla \bar v_\alpha \cdot x) dV + \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha \Delta \bar v_\alpha\, dV \\ &\qquad\qquad = J(R_\alpha, \bar v_\alpha) + I(R_\alpha, \bar v_\alpha)\, , \end{aligned} \tag{3.9}$$

where

$$J(R_\alpha, \bar v_\alpha) = \frac{1}{2}\int_{\Gamma_2}\left\{\left|\frac{\partial \bar v_\alpha}{\partial \nu}\right|^2 |x| - |\partial_{\tan}\bar v_\alpha|^2 |x| + (n-2)\frac{\partial \bar v_\alpha}{\partial \nu}\bar v_\alpha\right\} dS\, ,$$

$$I(R_\alpha, \bar v_\alpha) = \frac{1}{2}\int_{\Gamma_1}\left\{2\left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i}\right)\frac{\partial \bar v_\alpha}{\partial \nu} + (n-2)\frac{\partial \bar v_\alpha}{\partial \nu}\bar v_\alpha\right\} dS\, .$$

Replacing $\Delta \bar v_\alpha$ in (3.9) by

$$\Delta \bar v_\alpha = \Delta_{h_\alpha} \bar v_\alpha - (h_\alpha^{ij} - \delta^{ij})\partial_{ij}\bar v_\alpha + h_\alpha^{ij}\Gamma_{ij}^k \partial_k \bar v_\alpha\,,$$

we have

$$\begin{aligned}
&- \int_{B^+_{R_\alpha}} (x^i\partial_i \bar v_\alpha)\Delta_{h_\alpha}\bar v_\alpha\, dV - \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha \Delta_{h_\alpha}\bar v_\alpha\, dV\\
&+ \int_{B^+_{R_\alpha}} (x^k\partial_k \bar v_\alpha)(h_\alpha^{ij} - \delta^{ij})\partial_{ij}\bar v_\alpha\, dV - \int_{B^+_{R_\alpha}} (x^l\partial_l \bar v_\alpha)(h_\alpha^{ij}\Gamma_{ij}^k\partial_k \bar v_\alpha) dV\\
&+ \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha (h_\alpha^{ij} - \delta^{ij})\partial_{ij}\bar v_\alpha\, dV - \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha (h_\alpha^{ij}\Gamma_{ij}^k)\partial_k \bar v_\alpha\, dV\\
&\qquad\qquad = -J(R_\alpha, \bar v_\alpha) - I(R_\alpha, \bar v_\alpha)\,,
\end{aligned}$$

where $x^i\partial_i \bar v_\alpha = \sum_{i=1}^n x_i\partial_i \bar v_\alpha$, and so on, here and in the discussion below. So far we have not used the equation of $\bar v_\alpha$. Now we use equation (3.1) satisfied by $\bar v_\alpha$ and obtain

$$A(h_\alpha, \bar v_\alpha) = -J(R_\alpha, \bar v_\alpha) - I(R_\alpha, \bar v_\alpha) \tag{3.10}$$

where

$$\begin{aligned}
&A(h_\alpha, \bar v_\alpha)\\
&\;= \int_{B^+_{R_\alpha}} (x^k\partial_k \bar v_\alpha)(h_\alpha^{ij} - \delta^{ij})\partial_{ij}\bar v_\alpha\, dV - \int_{B^+_{R_\alpha}} (x^l\partial_l \bar v_\alpha)(h_\alpha^{ij}\Gamma_{ij}^k\partial_k \bar v_\alpha) dV\\
&+ \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha (h_\alpha^{ij} - \delta^{ij})\partial_{ij}\bar v_\alpha\, dV - \frac{n-2}{2}\int_{B^+_{R_\alpha}} \bar v_\alpha (h_\alpha^{ij}\Gamma_{ij}^k)\partial_k \bar v_\alpha\, dV\,.
\end{aligned}$$

Using (3.2), we have

$$\begin{aligned}
A(h_\alpha, \bar v_\alpha) &= O\Big(\int_{B^+_{R_\alpha}} \mu_\alpha |x|^2 |\nabla \bar v_\alpha|\,|\nabla^2 \bar v_\alpha|\, dV\Big)\\
&\quad + O\Big(\int_{B^+_{R_\alpha}} \mu_\alpha |x|\,|\nabla \bar v_\alpha|^2\, dV\Big)\\
&\quad + O\Big(\int_{B^+_{R_\alpha}} \mu_\alpha |x|\,\bar v_\alpha\,|\nabla^2 \bar v_\alpha|\, dV\Big)\\
&\quad + O\Big(\int_{B^+_{R_\alpha}} \mu_\alpha \bar v_\alpha |\nabla \bar v_\alpha|\, dV\Big)
\end{aligned} \tag{3.11}$$

We simplify $I(R_\alpha, \bar v_\alpha)$ by using the equation of $\bar v_\alpha$ (3.1). It is easy to see from (3.2) that

$$\frac{\partial_{h_\alpha} \bar v_\alpha}{\partial \nu} = \frac{\partial \bar v}{\partial \nu} + O(\mu_\alpha |x'|\, |\nabla \bar v_\alpha|) \quad \text{on } \Gamma_1 .$$

It follows that

$$\begin{aligned} (3.12)\quad & 2I(R_\alpha, \bar v_\alpha) \\ &= \int_{\Gamma_1} \left\{ 2 \left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i} \right) \frac{\partial_{h_\alpha} \bar v_\alpha}{\partial \nu} + (n-2) \frac{\partial_{h_\alpha} \bar v_\alpha}{\partial \nu} \bar v_\alpha \right\} dS \\ &\qquad + O\left(\int_{\Gamma_1} [\mu_\alpha\, |x'|^2\, |\nabla \bar v_\alpha|^2 + \mu_\alpha\, |x'|\, \bar v_\alpha\, |\nabla \bar v_\alpha|] dS \right) . \end{aligned}$$

Using the boundary condition of $\bar v_\alpha$ in (3.1), we have

$$\begin{aligned} &\int_{\Gamma_1} \left\{ 2 \left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i} \right) \frac{\partial_{h_\alpha} \bar v_\alpha}{\partial \nu} + (n-2) \frac{\partial_{h_\alpha} \bar v_\alpha}{\partial \nu} \bar v_\alpha \right\} dS \\ &= \int_{\Gamma_1} \left\{ 2(\xi_\alpha \bar v_\alpha^{q-1} - \alpha \mu_\alpha \bar v_\alpha) \left(\sum_{i=1}^{n-1} x_i \frac{\partial \bar v_\alpha}{\partial x_i} \right) \right. \\ &\quad \left. + (n-2)(\xi_\alpha \bar v_\alpha^{q-1} - \alpha \mu_\alpha \bar v_\alpha) \bar v_\alpha \right\} dS \\ &= -\frac{2(n-1)}{q} \int_{\Gamma_1} \xi_\alpha \bar v_\alpha^q \, dS + (n-1)\alpha\mu_\alpha \int_{\Gamma_1} \bar v_\alpha^2 \, dS \\ &\quad + \frac{2}{q} \int_{\partial \Gamma_1} \xi_\alpha \bar v_\alpha^q |x| dS - \int_{\partial \Gamma_1} \alpha \mu_\alpha \bar v_\alpha^2 |x| dS \\ &\quad + (n-2) \int_{\Gamma_1} \xi_\alpha \bar v_\alpha^q \, dS - (n-2)\alpha\mu_\alpha \int_{\Gamma_1} \bar v_\alpha^2 \, dS \\ &= \alpha\mu_\alpha \int_{\Gamma_1} \bar v_\alpha^2 \, dS + \frac{2}{q} \int_{\partial\Gamma_1} \xi_\alpha \bar v_\alpha^q |x| dS - \int_{\partial\Gamma_1} \alpha\mu_\alpha \bar v_\alpha^2 |x| dS . \end{aligned}$$

Thus

$$\begin{aligned} & I(R_\alpha, \bar v_\alpha) \\ (3.13)\qquad &= \frac{\alpha\mu_\alpha}{2} \int_{\Gamma_1} \bar v_\alpha^2 \, dS + O\Big(\int_{\partial\Gamma_1} (\bar v_\alpha^q + \alpha\mu_\alpha \bar v_\alpha^2) |x| dS \Big) \\ &\quad + O\left(\int_{\Gamma_1} \left[\mu_\alpha\, |x'|^2\, |\nabla \bar v_\alpha|^2 + \mu_\alpha\, |x'|\, \bar v_\alpha\, |\nabla \bar v_\alpha| \right] dS \right) . \end{aligned}$$

Clearly

$$J(R_\alpha, \bar v_\alpha) = O\biggl(\int_{\Gamma_2} (|x|\,|\nabla \bar v_\alpha|^2 + \bar v_\alpha\,|\nabla \bar v_\alpha|)dS\biggr)\,. \tag{3.14}$$

In view of all the above estimates, we rewrite (3.10) as the following Pohozaev-type identity:

$$\begin{aligned}
&\alpha\mu_\alpha \int_{\Gamma_1} \bar v_\alpha^2\, dS \\
&\quad = O\biggl(\int_{B^+_{R_\alpha}} \mu_\alpha\,|x|^2\,|\nabla \bar v_\alpha|\,|\nabla^2 \bar v_\alpha|\,dV\biggr) + O\biggl(\int_{B^+_{R_\alpha}} \mu_\alpha\,|x|\,|\nabla \bar v_\alpha|^2\,dV\biggr) \\
&\qquad + O\biggl(\int_{B^+_{R_\alpha}} \mu_\alpha\,|x|\,\bar v_\alpha\,|\nabla^2 \bar v_\alpha|\,dV\biggr) + O\biggl(\int_{B^+_{R_\alpha}} \mu_\alpha \bar v_\alpha |\nabla \bar v_\alpha|\,dV\biggr) \\
&\qquad + O\biggl(\int_{\Gamma_2} (|x|\,|\nabla \bar v_\alpha|^2 + \bar v_\alpha |\nabla \bar v_\alpha|)dS\biggr) \\
&\qquad + O\biggl(\int_{\partial\Gamma_1} (\bar v_\alpha^q + \alpha\mu_\alpha \bar v_\alpha^2)|x| dS\biggr) \\
&\qquad + O\biggl(\int_{\Gamma_1} [\mu_\alpha\,|x'|^2\,|\nabla \bar v_\alpha|^2 + \mu_\alpha\,|x'|\,\bar v_\alpha\,|\nabla \bar v_\alpha|]dS\biggr)\,.
\end{aligned} \tag{3.15}$$

We will derive a contradiction from (3.15) by showing that the left-hand side is much larger than the right-hand side for α large.

LEMMA 3.3 *For $n \geq 3$, there exists some constant $C > 0$ independent of α such that $\int_{\Gamma_1} \bar v_\alpha^2\, dS > 1/C$ for all $\alpha \geq 1$. Moreover, for $n = 3$, $\int_{\Gamma_1} \bar v_\alpha^2\, dS \geq (\log R_\alpha)/C$ for all $\alpha \geq 1$.*

PROOF: We only need to prove the lemma for large α. It follows easily from (3.3) that

$$\int_{\Gamma_1} \bar v_\alpha^2\, dS \geq \frac{1}{C}\,.$$

For $n = 3$, it follows from Proposition 3.2 that

$$\int_{\Gamma_1} \bar v_\alpha^2\, dS \geq \frac{1}{C} \int_{\partial\mathbb{R}^3_+ \cap B_{R_\alpha^{1/4}}} \left(\frac{1}{1+|x|}\right)^2 dS \geq \frac{\log R_\alpha}{C}\,.$$

∎

LEMMA 3.4 *The following estimates hold:*

$$\int_{\partial\Gamma_1} \bar{v}_\alpha^q \, |x| \, dS \le C R_\alpha^{1-n} = C(\alpha\mu_\alpha)^{n-1} \,,$$

$$\int_{\partial\Gamma_1} \alpha\mu_\alpha \bar{v}_\alpha^2 \, |x| \, dS \le \alpha\mu_\alpha R_\alpha^{3-n} \,,$$

$$\int_{\Gamma_1} (\mu_\alpha \, |x'|^2 \, |\nabla \bar{v}_\alpha|^2 + \mu_\alpha \, |x'| \, \bar{v}_\alpha \, |\nabla \bar{v}_\alpha|) dS \le \begin{cases} C\mu_\alpha \log R_\alpha & n = 3 \\ C\mu_\alpha & n \ge 4 \,, \end{cases}$$

$$\int_{\Gamma_2} (|x| \, |\nabla \bar{v}_\alpha|^2 + \bar{v}_\alpha \, |\nabla \bar{v}_\alpha|) dS \le C(\alpha\mu_\alpha)^{n-2} \,,$$

$$\int_{B^+_{R_\alpha}} (\mu_\alpha |x|^2 \, |\nabla \bar{v}_\alpha| \, |\nabla^2 \bar{v}_\alpha| + \mu_\alpha |x| \, |\nabla \bar{v}_\alpha|^2) dV \le \begin{cases} C\mu_\alpha \log R_\alpha & n = 3 \\ C\mu_\alpha & n \ge 4 \,, \end{cases}$$

$$\int_{B^+_{R_\alpha}} (\mu_\alpha |x| \, \bar{v}_\alpha \, |\nabla^2 \bar{v}_\alpha| + \mu_\alpha \bar{v}_\alpha |\nabla \bar{v}_\alpha|) dV \le \begin{cases} C\mu_\alpha \log R_\alpha & n = 3 \\ C\mu_\alpha & n \ge 4 \,, \end{cases}$$

PROOF: These estimates follow easily from (3.4), Proposition 3.1, and some elementary calculations. ∎

PROOF OF THEOREM 0.1: We draw a contradiction from (3.15) by using Lemmas 3.3 and 3.4 because the left-hand side is clearly much larger than the right-hand side in (3.15) as α tends to infinity. ∎

Appendix A

Let (M, g) be a smooth, compact Riemannian manifold of dimension n ($n \ge 3$) with boundary. In this appendix we present some weighted Sobolev embedding inequalities that should be well-known. We include a proof for completeness.

THEOREM A.1 *There exists some constant $C = C(M, g)$ such that for all $x_0 \in \overline{M}$, $\mu > 0$, $u \in H^1(M), u(x) = 0 \; \forall d(x_0, x) < \mu$, we have*

$$\left(\int_M \frac{|u(x)|^p}{d(x_0, x)^{2n}} \, dv_g \right)^{2/p} \le C \int_M \frac{|\nabla_g u(x)|^2}{d(x_0, x)^{2n-4}} \, dv_g$$

and

$$\left(\int_{\partial M} \frac{|u(x)|^q}{d(x_0, x)^{2n-2}} \, ds_g \right)^{2/q} \le C \int_M \frac{|\nabla_g u(x)|^2}{d(x_0, x)^{2n-4}} \, dv_g \,,$$

where $d(x_0, x)$ denotes the distance between x_0 and x.

Theorem A.1 in the case $x_0 \in \partial M$ follows immediately from Lemma A.2 and Lemma A.4 below. The general case can be proved in a similar way.

LEMMA A.2 *For $n \geq 3$, there exists some constant $C = C(n) > 0$ such that for all $u \in H^1(B_1^+(0))$, $u \equiv 0$ in an open neighborhood of $x = 0$, we have*

$$\left(\int_{B_1^+(0)} \frac{|u(x)|^p}{|x|^{2n}}\, dx\right)^{2/p} \leq C \int_{B_1^+(0)} \frac{|\nabla u(x)|^2}{|x|^{2n-4}}\, dx$$

and

$$\left(\int_{|x'|<1} \frac{|u(x',0)|^q}{|x'|^{2n-2}}\, dx'\right)^{2/q} \leq C \int_{B_1^+(0)} \frac{|\nabla u(x)|^2}{|x|^{2n-4}}\, dx\,,$$

where $x = (x', x_n)$, $p = 2n/(n-2)$, $q = 2(n-1)/(n-2)$, $B_1^+(0) = \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$.

PROOF: It follows from the hypothesis that for some $\mu = \mu(u) > 0$, $u(x) = 0\ \forall |x| < \mu,\ x_n > 0$. Consider

$$v(y) = u(y/|y|^2)\,, \quad |y| > 1\,,\ y_n > 0\,.$$

Clearly

$$v(y) = 0 \quad \forall |y| > 1/\mu\,,\ y_n > 0\,,$$

and, for some $C(n) > 0$,

$$\int_{B_1^+(0)} \frac{|u(x)|^p}{|x|^{2n}}\, dx = C(n) \int_{\{|y|\geq 1, y_n \geq 0\}} |v(y)|^p\, dy\,,$$
$$\int_{B_1^+(0)} \frac{|\nabla u(x)|^2}{|x|^{2n-4}}\, dx = C(n) \int_{\{|y|\geq 1, y_n \geq 0\}} |\nabla v(y)|^2\, dy\,,$$
$$\int_{|x'|<1} \frac{|u(x',0)|^q}{|x|^{2n-2}}\, dx' = C(n) \int_{|y'|>1} |v(y',0)|^q\, dy'\,.$$

It follows from standard Sobolev embedding theorems (with appropriate extensions of v to $|y| < 1$) that

$$\left(\int_{\{|y|\geq 1, y_n\geq 0\}} |v(y)|^p\, dy\right)^{2/p} + \left(\int_{|y'|>1} |v(y',0)|^q\, dy'\right)^{2/q} \leq C(n) \int_{\{|y|\geq 1, y_n \geq 0\}} |\nabla v(y)|^2\, dy\,.$$

Lemma A.2 follows immediately. ∎

The following corollary is immediate:

COROLLARY A.3 *There exist some constants $\delta = \delta(M, g) > 0$ and $C = C(M, g) > 0$ such that for all $x_0 \in \partial M$, $u \in H^1(M)$, $u \equiv 0$ in an open neighborhood of x_0 we have*

$$\left(\int_{B_\delta^+(x_0)} \frac{|u(x)|^p}{d(x_0,x)^{2n}}\, dv_g\right)^{2/p} + \left(\int_{\partial M \cap \overline{B_\delta^+(x_0)}} \frac{|u(x)|^q}{d(x_0,x)^{2n-2}}\, ds_g\right)^{2/q} \le C \int_{B_\delta^+(x_0)} \frac{|\nabla_g u(x)|^2}{d(x_0,x)^{2n-4}}\, dv_g\,,$$

where $B_\delta^+(x_0) = \{x \in M : \; d(x_0, x) < \delta\}$.

LEMMA A.4 *For $\delta > 0$, there exists $C = C(M, g, \delta) > 0$ such that for all $x_0 \in \overline{M}$, $u \in H^1(M \setminus B_{\delta/2}^+(x_0))$, we have*

$$\left(\int_{M \setminus B_\delta^+(x_0)} |u|^p\, dv_g\right)^{2/p} + \left(\int_{\partial M \setminus \overline{B_\delta^+(x_0)}} |u|^q\; ds_g\right)^{2/q} \le C\left\{\int_{M \setminus B_{\delta/2}^+(x_0)} |\nabla_g u|^2\, dv_g + \int_{B_\delta^+(x_0) \setminus B_{\delta/2}^+(x_0)} u^2\, dv_g\right\}.$$

PROOF: Suppose the contrary of Lemma A.4, namely, that for some $\delta > 0$, there exists a sequence of points $\{x_i\} \in \overline{M}$, $\{u_i\} \in H^1(M \setminus B_{\delta/2}^+(x_i))$, satisfying

$$\left(\int_{M \setminus B_\delta^+(x_i)} |u_i|^p\, dv_g\right)^{2/p} + \left(\int_{\partial M \setminus \overline{B_\delta^+(x_i)}} |u_i|^q\, ds_g\right)^{2/q} = 1 \tag{A.1}$$

and

$$\int_{M \setminus B_{\delta/2}^+(x_i)} |\nabla_g u_i|^2\, dv_g + \int_{B_\delta^+(x_i) \setminus B_{\delta/2}^+(x_i)} u_i^2\, dv_g < \frac{1}{i}\,.$$

It follows that

$$\|u_i\|_{H^1(M \setminus B_{\delta/2}^+(x_i))} \le C$$

and

$$\lim_{i \to \infty} \left\{\int_{M \setminus B_{\delta/2}^+(x_i)} |\nabla_g u_i|^2\, dv_g + \int_{B_\delta^+(x_i) \setminus B_{\delta/2}^+(x_i)} u_i^2\, dv_g\right\} = 0\,. \tag{A.2}$$

After passing to some subsequence, we have that u_i converges weakly to u in $H^1(M \setminus B^+_{\delta/2}(x_i))$. In view of (A.2), $u \equiv 0$. It follows from the compact embedding of H^1 into L^2 that

$$\int_{M\setminus B^+_{\delta/2}(x_i)} u_i^2 \to 0\,.$$

This, together with (A.2), yields

$$\|u_i\|_{H^1(M\setminus B^+_{\delta/2}(x_i))} \to 0$$

which contradicts (A.1) because of the Sobolev embedding theorems. Lemma A.4 is established. ∎

Appendix B

For $n \geq 3$, let B_1 denote the unit ball in $\mathbb{R}^n$, and let $g = g_{ij}(x)dx^i\,dx^j$ be a C^2-metric on B_1 with $g_{ij}(0) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x_k}(0) = 0$ for all $1 \leq i, j, k \leq n$. Let $K \in L^\infty(B_1)$.

PROPOSITION B.1 *There exists some constant $\tau_1 > 0$ depending only on $n, \|g_{ij}\|_{C^2(B_1)}$, and $\|K\|_{L^\infty(B_1)}$ such that for all $0 < \tau \leq \tau_1$, there exists some function $G(y) = |y|^{2-n} + E(y)$ satisfying*

$$\begin{cases} -\Delta_g G + K(y)G = n(n-2)\omega_n \delta_0 & \textit{in } B_\tau \setminus \{0\} \\ G = 0 & \textit{on } \partial B_\tau\,, \end{cases} \tag{B.1}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and E satisfies the following: For all $0 < \varepsilon < 1$, there exists some constant $C(\varepsilon)$ depending only on ε, n, $\|g_{ij}\|_{C^2(B_1)}$, and $\|K\|_{L^\infty(B_1)}$ such that

$$|y|^{n-4+\varepsilon}|E(y)| + |y|^{n-3+\varepsilon}|\nabla E(y)| \leq C(\varepsilon) \quad \forall y \in B_\tau\,,\ n \geq 4\,,$$

and

$$|y|^{\varepsilon-1}|E(y) - E(0)| + |y|^{\varepsilon}|\nabla E(y)| \leq C(\varepsilon) \quad \forall y \in B_\tau\,,\ n = 3\,.$$

REMARK B.2 *In fact, such G is unique.*

PROOF: The existence of $\tau_1 > 0$ is well-known; see, for example, [7]. Clearly (B.1) is equivalent to

$$\begin{cases} -\Delta_g E + K(y)E = O(|y|^{2-n}) & \text{in } B_\tau \\ E = -\tau^{2-n} & \text{on } \partial B_\tau\,. \end{cases} \tag{B.2}$$

Since $|y|^{2-n} \in L^r(B_\tau)$ for all $r < n/(n-2)$, it is well-known that (B.2) has a unique solution $E \in W^{2,r}(B_\tau)$. It follows from the Sobolev embedding theorems that

$$\|E\|_{L^s(B_\tau)} \le \begin{cases} C(s) & \forall s < \frac{n}{(n-4)}\,,\ n \ge 5\,, \\ C(s) & \forall\ s < \infty\,,\ n = 4\,, \end{cases}$$

and

$$\|E\|_{C^{1-\varepsilon}(B_\tau)} \le C(\varepsilon) \quad \forall 0 < \varepsilon < 1\,,\ n = 3\,. \tag{B.3}$$

For $0 < r \le \tau/5$, $x \in A_0 = \{x : \frac{1}{5} \le |x| \le 5\}$, set

$$E_1(x) = \begin{cases} r^{n-2}E(rx) & n \ge 4 \\ r^{n-2}(E(rx) - E(0)) & n = 3\,. \end{cases}$$

Then E_1 satisfies

$$-\Delta_h E_1(x) + K(rx)r^2 E_1(x) = O(r^2) \quad x \in A_0\,,$$

where $|O(r^2)| \le Cr^2$ with C independent of r and $h = h_{ij}(x)dx^i\,dx^j = g_{ij}(rx)dx^i\,dx^j$. For $n \ge 4$, for all $0 < \varepsilon < 1$, we can choose some $s_1 = s_1(\varepsilon) < n/(n-4)$ such that

$$\|E_1\|_{L^{s_1}(A_0)} \le r^{2-\varepsilon}\|E\|_{L^{s_1}(B_\tau)} \le C(\varepsilon)r^{2-\varepsilon}\,.$$

Using the equation of E_1 and applying the bootstrap method finite times (using the L^p theory of elliptic equations and the Sobolev embedding theorems, see e.g., [17]), we have

$$|E_1(x)| + |\nabla E_1(x)| \le C(\varepsilon)r^{2-\varepsilon}\,, \quad \frac{1}{2} \le |x| \le 2\,.$$

Consequently,

$$|y|^{n-2}|E(y)| + |y|^{n-1}|\nabla E(y)| \le C(\varepsilon)|y|^{2-\varepsilon}\,, \quad |y| \le \frac{\tau}{5}\,.$$

This establishes Proposition B.1 in the case $n \ge 4$. For $n = 3$, we know from (B.3), for all $0 < \varepsilon < 1$, that

$$|E_1(x)| \le C(\varepsilon)r^{2-\varepsilon} \quad \forall x \in A_0\,.$$

It follows from the equation of E_1 and standard elliptic estimates that

$$|\nabla E_1(x)| \le C(\varepsilon)r^{2-\varepsilon} \quad \forall \frac{1}{2} \le |x| \le 2\,.$$

Consequently,

$$|y|\,|E(y)-E(0)|+|y|^2\,|\nabla E(y)|\le C(\varepsilon)|y|^{2-\varepsilon}\,,\quad |y|\le\frac{\tau}{5}\,.$$

Proposition B.1 is thus established. ∎

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