MATH 4073 Homework 1 Due Fri, 09/02/16

Problem 1. Consider the function

$$f(x) = \mathrm{e}^{2x/\pi} + (1 - \mathrm{e}) \sin x \ , \tag{1}$$

where $\mathrm{e} = 2.718281828459\ldots$ is the base of the natural logarithms.

(a) Use the Mean Value Theorem to show that the derivative of f vanishes (i.e., becomes equal to zero) at least once in the interval $[0, \frac{\pi}{2}]$, without computing f' explicitly.

Hint: Find the values of $f(0)$ and $f(\frac{\pi}{2})$ and use them.

(b) In the rest of this problem you will give another solution of what you already proved in part (a), and, in addition, will show that the point where f' vanishes is unique. Start by finding the derivative f' of the function f given by (1).

(c) Apply the Intermediate Value Theorem to the equation $f'(x) = 0$ (using the concrete expression for f' derived in part (b)) to prove that the equation $f'(x) = 0$ has at least one solution in the interval $[0, \frac{\pi}{2}]$.

(d) Show that the solution of $f'(x) = 0$ whose existence was proved in part (c) is in fact unique.

Hint: If you take the derivative of the left-hand side of the equation $f'(x) = 0$, you will obtain

$$f''(x) = \frac{4}{\pi^2}\mathrm{e}^{2x/\pi} + (\mathrm{e} - 1) \sin x \ .$$

What can you say about the sign of $f''(x)$ for $x \in [0, \frac{\pi}{2}]$? What does this imply for the behavior of $f'(x)$ on this interval?

Problem 2. Applying the L'Hospital rule to find limits of ratios where both the numerator and the denominator tend to zero is sometimes long and error-prone. Use the Taylor expansions

$$\mathrm{e}^{x^2} = 1 + \frac{1}{1!}(x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots \ , \qquad \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \ ,$$

to compute the value of the limit

$$\lim_{x \to 0} \frac{\exp(x^2) - \cos x}{x^2} \ .$$

Problem 3. In this problem you will use Taylor's Theorem to approximate the value of $\sqrt{17}$.

(a) Write the second-degree Taylor polynomial,

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0)^1 + \frac{f''(x_0)}{2!}(x - x_0)^2$$

for the function $f(x) = \sqrt{x}$ around $x_0 = 16$. You have to find explicitly the numerical values of the coefficients of $P_2(x)$; there is no need to expand the factors $(x - x_0)^j$.

Hint: The answer is: $P_2(x) = 4 + \frac{1}{8}(x - 16) - \frac{1}{512}(x - 16)^2$, but I want to see your derivations.

(b) Find the numerical value of $P_2(17)$.

(c) Show that the remainder term is

$$R_2(x) = \frac{f'''(\xi(x))}{3!}(x - x_0)^3 = \frac{1}{16\,[\xi(x)]^{5/2}}\,(x - 16)^3 \;,$$

and find the maximum possible value of $|R_2(17)|$. Here $\xi(x)$ is a number between $x_0 = 16$ and $x = 17$; this number is unknown, so to find the maximum possible value of $|R_2(17)|$, you have to allow $\xi(x)$ to be *anywhere* between 16 and 17. The maximum possible value of $|R_2(17)|$ is a *rigorous* upper bound on the size of the error if you replace the exact value $f(17) = \sqrt{17}$ with its approximation, $P_2(17)$.

(d) Compute the true numerical value of the so-called *absolute error*, $|P_2(17) - \sqrt{17}|$ (the absolute error is the absolute value of the difference between the exact and the approximate values). Compare the true value of $|P_2(17) - \sqrt{17}|$ with the upper bound for the error obtained in part (c). Discuss briefly your observations.

Problem 4. Express the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(5 + \frac{2i}{n}\right)^{10}$$

as a definite integral, and compute its exact value.

Problem 5. Recall the Comparison Property for definite integrals: if $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x \tag{2}$$

(see, e.g., Stewart, *Calculus*, 7th edition, Section 4.2).

(a) If m and M are constants such that $m \le f(x) \le M$ for $a \le x \le b$, use the Comparison Property (2) to prove that

$$m(b - a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b - a) \;.$$

(b) Use your result from part (a) to prove that

$$2 \le \int_{-1}^{1} \sqrt{1 + x^6}\,\mathrm{d}x \le 2\sqrt{2} \;.$$

Problem 6.

(a) Convert the number 1101011001.001_2 to base 10.

(b) Convert the number 287_{10} to base 2.

(c) Convert the number 11010110011.001_2 to base 16.

(d) Convert the number $\mathrm{B2F}_{16}$ to base 2.