MATH 2934 – Additional problem assigned on 2/18/16

Additional problem.

One can easily show that, for any $a > 0$,

$$\frac{1}{a^3} = \frac{1}{2}\int_0^\infty x^2\,\mathrm{e}^{-ax}\,\mathrm{d}x\ ;$$

you do *not* need to show this here (but think how you would do it if you had to).

Use the formula above to prove that

$$\sum_{n=1}^\infty \frac{1}{n^3} = \frac{1}{2}\int_0^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x - 1}\ .$$

Remark: The famous *Riemann zeta-function* was defined by the German mathematician Bernhart Riemann in 1859 as

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s}$$

for every $s > 1$. It can also be expressed as

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}\,\mathrm{d}x}{\mathrm{e}^x - 1}\ ,$$

a particular case of which you proved in this problem.

Another famous representation for $\zeta(s)$ was discovered by the Swiss mathematician Leonhard Euler who in 1737 proved the identity

$$\sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p\ \text{prime}} \frac{1}{1 - p^{-s}}\ .$$

The infinite product on the right hand side extends over all prime numbers p:

$$\prod_{p\ \text{prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1-2^{-s}}\cdot\frac{1}{1-3^{-s}}\cdot\frac{1}{1-5^{-s}}\cdot\frac{1}{1-7^{-s}}\cdot\frac{1}{1-11^{-s}}\cdot\frac{1}{1-13^{-s}}\cdot\dots\ .$$

The values of $\zeta(k)$ are known when k is an even positive integer:

$$\zeta(2) = \frac{\pi^2}{6}\ ,\quad \zeta(4) = \frac{\pi^4}{90}\ ,\quad \zeta(6) = \frac{\pi^6}{945}\ ,\quad \zeta(8) = \frac{\pi^8}{9450}\ ,\quad \zeta(10) = \frac{\pi^{10}}{93555}\ ,\quad \dots\ .$$

So far, however, it is not known how (and if) $\zeta(k)$ for $k = 3, 5, 7, 9, 11, \dots$ can be expressed in terms of simpler mathematical constants (like $\zeta(k)$ for even positive integer k above). Even the fact that $\zeta(3)$ is irrational was established quite recently – in 1979 – by the French mathematician Roger Apéry.

In the Problems Plus section in Chapter 15 of Stewart's book (page 1077), the value of $\zeta(2)$ is obtained by using double integrals, and the value of $\zeta(3)$ is related to a triple integral. In only a couple of months you will be able to understand these complicated things!!!