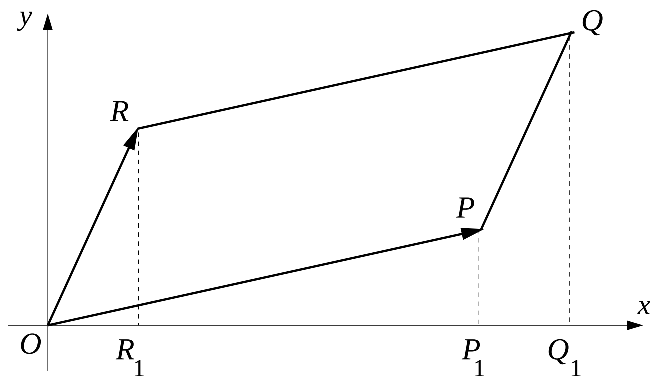
Additional FFT Problem 1.

Consider the vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$. Draw them in the plane starting at the origin O, and introduce the notations $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OR} = \mathbf{b}$, as shown in the figure. Let $\overrightarrow{RQ} = \overrightarrow{OP}$, $\overrightarrow{PQ} = \overrightarrow{OR}$. In this problem you will show that the area A_{OPQR}, of the



parallelogram $OPQR$ is equal to the absolute value of the determinant of the 2×2 matrix whose rows are the components of the vectors $\mathbf{a}$ and $\mathbf{b}$.

(a) Find the area A_{OR_1R} of the triangle OR_1R.

(b) Find the area $A_{R_1Q_1QR}$ of the trapezoid R_1Q_1QR.

(c) Find the area A_{OP_1P} of the triangle OP_1P.

(d) Find the area $A_{P_1Q_1QP}$ of the trapezoid P_1Q_1QP.

(e) Use your results in (a)-(d) to find the area of the parallelogram $OPQR$, and show that the claim about it written above is correct.

(f) Now think of the figure above as the (x, y)-plane of $\mathbb{R}^3$ (so that the z-axis is pointing towards you). Find the magnitude of the cross-product $\mathbf{a} \times \mathbf{b}$. Clearly, now you have to think of $\mathbf{a}$ and $\mathbf{b}$ as 3-dimensional vectors with zero third component. What do you notice?

Additional FFT Problem 2.

Consider the change of variables from coordinates (x, y) to coordinates (u, v) in $\mathbb{R}^2$ given

$$\begin{aligned} x &= a_1 u + b_1 v \\ y &= a_2 u + b_2 v \end{aligned} .$$

In the (u, v) plane, consider the rectangle $\widetilde{R}$ with vertices at the points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

(a) If (x, y) are variables related to (u, v) as in the equation, then where does each of the four vertices of the rectangle $\widetilde{R}$ go when it is mapped to the (x, y) plane?

(b) Since the change of coordinates is linear, a straight line is mapped to a straight line, so that the rectangle $\widetilde{R}$ in the (u, v) plane is mapped to a parallelogram in the (x, y) plane. What is the ratio of the area of this parallelogram to the area of the rectangle $\widetilde{R}$?

Hint: The result of Problem 1 will be very useful.

(c) Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ of the change of variables. Compare it with the ratio of the areas found in part (b).