MATH 4073 **Homework 2** **Due Mon, 09/12/16**

Problem 1. Find the limits and the convergence rates as $n \to \infty$ of the following sequences (recall Definition 1.18 and Example 2 on pages 37 and 38 of Sec. 1.3):

(a) $\lim\limits_{n\to\infty} \left(\frac{1}{n^3} - \sin \frac{1}{n^3} \right)$;

(b) $\lim\limits_{n\to\infty} \frac{n^3}{n^3+7}$;

(c) $\lim\limits_{n\to\infty} \left[(n+5)^{1/3} - n^{1/3} \right]$;

(d) $\lim\limits_{n\to\infty} \left[\ln\left(n^3+5\right) - \ln\left(n^3\right) \right]$.

Hints: (a) Use the Taylor expansion $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.
(b) You can rewrite the expression as $\frac{n^3}{n^3+7} = \frac{1}{1+\frac{7}{n^3}} = \frac{1}{1-\left(-\frac{7}{n^3}\right)}$, and use the formula for the sum of a geometric series, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, valid for $|x| < 1$.
(c) Write $(n+5)^{1/3} - n^{1/3} = n^{1/3}\left[\left(1+\frac{5}{n}\right)^{1/3} - 1\right]$, and use that the Taylor expansion of $(1+x)^\alpha$ around $x = 0$ when α is not equal to a positive integer is given by

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots ,$$

where $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.
(d) Use the basic property of logarithms to merge the two logarithms into one, and then use the Taylor expansion $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$, for $|x| < 1$.

Problem 2. In the limit $h \to 0$, find the real numbers P, Q, and R, and the integers p, q, and r in the following relations (recall Definition 1.19 and Example 3 on page 38 of Sec. 1.3):

(a) $\frac{e^h - \cos h}{h} = P + O(h^p)$;

(b) $\cos(\sin h) = Q + O\left(h^q\right)$;

(c) $\ln\sqrt{3+h} = R + O(h^r)$.

Hint: See the Hints on rate of convergence of functions at the end of this homework.

Problem 3. Consider the equation

$$f(x) = x - \cos x = 0 \ . \tag{1}$$

(a) Prove that the equation (1) has a solution in the interval $[0, \frac{\pi}{2}]$. Please specify which theorem you used to come to this conclusion.

(b) Prove that the solution of (1) in the interval $[0, \frac{\pi}{2}]$ is unique.

(c) Use the MATLAB code `bisection.m` (from the class web-site) and look at the instructions how to run it, to find the root of (1) in $[0, \frac{\pi}{2}]$. Use tolerance 10^{-12} and run the code verbosely, so that you can see the results at each step. Please attach your printout! (To save your MATLAB session, use the command `diary` as it is used in the example MATLAB session.)

(d) If E_n is the error in the nth step of the bisection method, then one can write $E_n = O(\beta_n)$ for some (simple) sequence $\{\beta_n\}$. What is β_n for the bisection method? Explain briefly why your answer is obvious.

Problem 4. In this problem we will find the value of the number

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}}},$$

and will study how a certain sequence converges to it. Note that a^{b^c} means $a^{(b^c)}$, not $(a^b)^c$ (the latter is simply a^{bc})!

(a) Consider the function $g(x) := \sqrt{2}^{\,x}$, defined for all $x \in \mathbb{R}$; a part of the graph of g is plotted below, together with the diagonal $\{y = x\}$. Use derivatives to show that g is an increasing and concave up. (To differentiate g, note that $\sqrt{2}^{\,x} = e^{\frac{\ln 2}{2}x}$.)

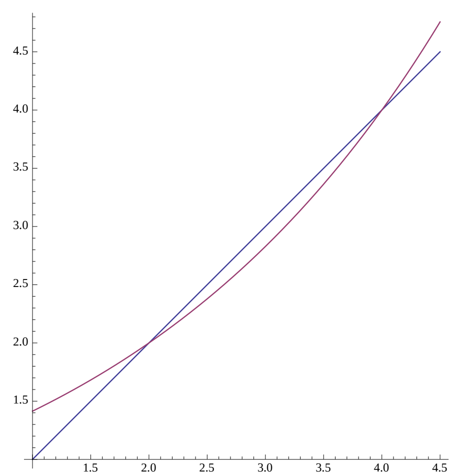

Figure 1: Plots of the diagonal and the graph of $g(x) = \sqrt{2}^{\,x}$.

(b) One can easily check that 2 and 4 are solutions of the equation $\sqrt{2}^{\,x} = x$ (you do *not* need to do this). Use what you found in (a) to convince me that this equation has no other solutions except 2 and 4.

(c) One can define a sequence recursively, starting from some value p_0 and then iterating according to the rule $p_n = g(p_{n-1}) = \sqrt{2}^{\,p_{n-1}}$, $n \geq 1$. Clearly, the numbers 2 and 4 are fixed points of this iteration. Draw (big and clear) cobweb diagrams to illustrate graphically that one of these two fixed points is attracting, while the other is repelling.

(d) Let $p_0 := 1$, $p_n := \sqrt{2}^{\,p_{n-1}}$ for $n \geq 1$. Download the MATLAB code `fixedpoint.m` from the class web-site and compute the iterates p_n starting from $p_0 = 3$, and iterating until the difference $|p_n - p_{n-1}|$ becomes smaller than 10^{-13}. To do that, open MATLAB, type

```
format long
```

and press RETURN (only to make MATLAB display more digits of the numbers), then type

```
fixedpoint( inline('2^(x/2)'), 3.0, 1e-13, 1000, 1)
```

and press RETURN again. Does the sequence $\{p_n\}_{n=0}^{\infty}$ seem to converge? To what value? Just tell me what you observe, there is no need to attach a printout for this problem.

(e) Let p_* be the limit of the sequence $\{p_n\}_{n=0}^{\infty}$ computed in part (d). The general theory predicts that, if the sequence $\{p_n\}_{n=0}^{\infty}$ converges and $|g'(p_*)| < 1$, then the sequence $\{p_n - p_*\}_{n=0}^{\infty}$ satisfies

$$\lim_{n\to\infty} \frac{p_{n+1} - p_*}{p_n - p_*} = g'(p_*) \ .$$

From your numerical results in part (c), take the values p_{16} and p_{15} and compute the ratio $\frac{p_{16} - p_*}{p_{15} - p_*}$ (I want to see the specific values of p_{16} and p_{15}). Compare it with the exact value of $g'(p_*)$. Discuss briefly what you observe. (Of course, if you take larger values of n in the calculation of $\frac{p_{n+1} - p_*}{p_n - p_*}$, your result will be closer to $g'(p_*)$.)

Hints on rate of convergence of functions. In Problem 3 of this homework you are supposed to find several limits of functions and the rates of convergence. The whole point of finding the order of convergence of a function to its limit is to get a rough idea how the function approaches its limiting value as the argument of the function approaches zero. Here are some examples that will hopefully make things clearer. The main tool in this kind of problems is the Taylor expansion.

Example 1. We know that the Taylor expansion of the function $\cos x$ around 0 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \ .$$

Clearly, when $x \to 0$, all terms except the first tend to zero, so that

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right) = 1$$

As x becomes very close to zero, the term $\frac{x^4}{4!}$ is much smaller than $\frac{x^2}{2!}$ – indeed,

$$\frac{x^4/4!}{x^2/2!} = \text{const} \cdot x^2 \ , \qquad \text{so that} \qquad \lim_{x\to 0} \frac{x^4/4!}{x^2/2!} = \lim_{x\to 0} \left(\text{const} \cdot x^2\right) = 0 \ .$$

This means that, as $x \to 0$, the term $\frac{x^4}{4!}$ becomes negligible in comparison with $\frac{x^2}{2!}$. Note that I did not even compute the constant (just wrote "const"), because the only thing that is important for me here are the powers of x. Similarly, the terms proportional to x^6, x^8, etc., are negligible in comparison with $\frac{x^2}{2!}$. Therefore we obtain

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = 1 - \frac{x^2}{2!} + (\text{terms that go to 0 faster than the term with } x^2) \ ,$$

which allows us to write

$$\cos x = 1 + O(x^2) \ .$$

Note that we do not care about the constant that multiplies x^2 (which in this particular case is equal to $-\frac{1}{2!}$ but, again, that is not important).

Example 2. To find the limit and the rate of convergence of $\frac{e^x + \cos x - 2 - x}{x^3}$ as $x \to 0$, we use the Taylor expansion of $\cos x$ around 0 (see above) and the Taylor expansion of e^x around 0:

$$e^x = 1 + \tfrac{x^1}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots \ ,$$

so that

$$\begin{aligned}
\frac{e^x + \cos x - 2 - x}{x^3} &= \tfrac{1}{x^3}\left(1 + \tfrac{x^1}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots + 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots - 2 - x\right) \\
&= \tfrac{1}{x^3}\left(\tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots\right) \\
&= \tfrac{1}{3!} + 2\tfrac{x}{4!} + \tfrac{x^2}{5!} + (\text{terms with even higher powers of } x) \\
&= \tfrac{1}{6} + O(x) \ .
\end{aligned}$$

Using the above calculations, we see right away that $e^x + \cos x - 2 - x = O(x^3)$ (why?).

Example 3. Using the Taylor expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ ,$$

we obtain

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots \ ,$$

and, therefore,

$$\frac{\ln(1+x^3)}{x^3} = 1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots = 1 + O(x^3) \ .$$