# *The dimension of the spaces of cusp forms on Siegel upper half plane of degree two* (I)

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## Introduction

**0-1.** Let $H_n=\{Z\in M_n(\boldsymbol{C});\ Z={}^tZ,\ \mathrm{Im}\,Z>0\}$ be the Siegel upper half plane of degree $n$, and let $\Gamma$ be a lattice of $\mathrm{Sp}(n, \boldsymbol{R})$, the group of analytic automorphisms of $H_n$. Namely $\Gamma$ is a discrete subgroup of $\mathrm{Sp}(n, \boldsymbol{R})$ such that the quotient $\Gamma\backslash H_n$ has a finite volume. Let, for each positive integer $k$, $S_k(\Gamma)$ be the $\boldsymbol{C}$-space consisting of the holomorphic functions on $H_n$ satisfying the following conditions:

(i) $f(\gamma\langle Z\rangle)=\det(CZ+D)^k f(Z)$, for all $\gamma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma$,

(ii) $(\det \mathrm{Im}\,Z)^{k/2}|f(Z)|$ is bounded on $H_n$.

It is known that $S_k(\Gamma)$ is of finite dimension. Our chief interest in this paper lies on the computation of the dimension of this space by using *Selberg's trace formula*, in the case of $n=2$.

In the case $n=1$, it is a classical result that the dimension is expressed in terms of *the signature* of $\Gamma$: namely for $k\geqq 2$,

$$\text{(0.1)}\qquad \dim S_k(\Gamma)=\frac{k-1}{4\pi}\mathrm{vol}(\Gamma\backslash H_1)+\sum_{i=1}^{s}\left(\frac{k}{2n_i}-1-\left[\frac{k-2}{2n_i}\right]+\frac{1}{2}\left(1-\frac{1}{n_i}\right)\right)-\frac{t}{2}+\delta_{2,k},$$

where $(n_1, \cdots, n_s;\ t)$ is the signature of $\Gamma$, i.e., the quotient $\Gamma\backslash H_1$ has $t$ cusps and $s$ inequivalent elliptic fixed points $z_i$ $(1\leqq i\leqq s)$ whose stabilizer in $\Gamma$ mod $\pm 1$ has order $n_i$. Such expression is known also in the case of Hilbert modular cusp forms ([26], see also [11]). So we may ask

*Problem* 1. *Can the dimension of $S_k(\Gamma)$ be expressed in terms of "signature" also for $n\geqq 2$?*

There is another problem concerning the calculation of the dimension, which seems interesting in view of the indication of A. Selberg [24], that certain integrals which arise in the dimension formula are evaluated by special values of some Dirichlet series. Let, for instance, $\Gamma=\mathrm{Sp}(n, \boldsymbol{Z})$. Then by Siegel [32], the volume of the fundamental domain is expressed by the special values of Riemann zetafunction:

(0.2) $$\mathrm{vol}(\mathrm{Sp}(n, \boldsymbol{Z})\backslash H_n)=2\prod_{i=1}^{n}\frac{\Gamma(i)\zeta(2i)}{\pi^i}.$$

Also, the contributions from the unipotent elements are expressed by T. Shintani [30] by the special values of certain zetafunctions associated with the prehomogeneous vector space of quadratic forms. So we may ask

*Problem* 2. *What kind of zetafunctions will appear in our calculation of the dimension? Or, more generally, what kind of "arithmetic quantities" will appear in it?*

In fact we shall see that *all* terms in our dimension formula are connected with some special values of certain zetafunctions.

**0-2.** Let us recall the known results on our Problem 1 for $n=2$ at this time. In [5], [18], U. Christian and Y. Morita calculated the dimension for $\Gamma=\Gamma(N)$, the principal congruence subgroup of Sp(2, $\boldsymbol{Z}$) of level $N\geqq 3$. In that case $\Gamma$ is torsion free and the conjugacy classes of $\Gamma$ that make nonzero contributions to the dimension formula are only unipotent ones. Following Morita, J. Sakamoto tried to compute the elliptic contributions and succeeded partly ([22]). Also, T. Arakawa [1] computed the dimension in the case where $\Gamma=\Gamma(N)$ is the congruence subgroup related to the binary hermitian forms over indefinite division quaternion algebras over the rational number field. There remains to compute explicitly the contributions of some kinds of elliptic conjugacy classes and those mixed kinds of classes whose semisimple factors are elliptic.

In this paper we shall calculate the contributions of these remaining classes and thus obtain a general formula for the dimension of $S_k(\Gamma)$ (§5, Theorem 5-1), which we think gives an answer to Problem 1 above. As an application we shall derive from it the explicit formulae for $\Gamma=\mathrm{Sp}(2, \boldsymbol{Z})$, and $\Gamma_0(p)$. The latter is a new result and it has some importance in view of its connection with the theory of theta series. Similar formula for other arithmetic subgroups $\Gamma$ will be treated in another paper [9].

Here we must note that there are two other methods to calculate the dimension of $S_k(\Gamma)$. Firstly J. Igusa [15] determined the structure of the graded ring of automorphic forms for Sp(2, $\boldsymbol{Z}$) by using his theory of moduli space of curves of genus two, and derived the following formula:

(0.3) $$\sum_{k=1}^{\infty}\dim S_k(\mathrm{Sp}(2, \boldsymbol{Z}))t^k=\frac{1+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}-\frac{1}{(1-t^4)(1-t^6)}.$$

Secondly, T. Yamazaki [36] obtained the same result for $\Gamma(N)$, by using the formula of Riemann-Roch-Hirzebruch. Also R. Tsushima [34] has obtained recently the above formula by this method. Thus we have now three different proofs of (0.3).



**0-3.** Although our problems 1, 2 have their own significance, our primary motivation to them is as follows: Let $G_Q=\mathrm{Sp}(2, Q)$ and $G'_Q$ a $Q$-form of $U\mathrm{Sp}(2)$, the compact real form of Sp(2, $\boldsymbol{C}$). In [16], Y. Ihara studied the Dirichlet series attached to automorphic forms on $G'_A$, the adelized group of $G'_Q$. There he raised several interesting problems and among others, he gave a conjectural question that there should be a relation between automorphic forms on $G_A$ and $G'_A$. This is considered as a genus two version of the well known results of M. Eichler [6], H. Shimizu [28]. Recently T. Ibukiyama has given in [13] an exact formulation of this conjecture with a number of examples. Our intention has been to prove this conjecture by comparing the traces of Hecke operators on both spaces.

A general arithmetic formula for traces of Hecke operators on the space of forms on $G'_A$ has been given in [8], and the explicit formula for the dimension of this space in [10]. Thus the present paper may be regarded as a second step toward the above conjecture.

**0-4.** Now let us explain briefly an outline of this paper. The starting point of our calculation is the Selberg-Godement's dimension formula: for $k\geqq 5$,

(0.4) $$\dim S_k(\Gamma)=\frac{a(k)}{\sharp Z(\Gamma)}\int_{\Gamma\backslash H_2}\sum_{\gamma\in\Gamma}H_\gamma(Z)dZ,$$

where $a(k)=2^{-8}\pi^{-3}(2k-2)(2k-3)(2k-4)$, $Z(\Gamma)=$: center of $\Gamma$,

$$H_\gamma(Z)=(\det\ \mathrm{Im}\ Z)^k\det\Big(\frac{Z-\gamma\langle\bar{Z}\rangle}{2i}\Big)^{-k}\det(C\bar{Z}+D)^{-k},$$

for $\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma$, and $dZ=(\det Y)^{-3}dXdY$ $(Z=X+iY)$ is a Sp(2, $\boldsymbol{R}$)-invariant measure of $H_2$.

Roughly speaking, there are two main difficulties in our calculation of (0.4); one is the problem of analysis concerning the evaluation of the integrals appearing in it, and the other is the problem of arithmetic nature coming from the complicated structure of $\Gamma$. If the quotient $\Gamma\backslash H_2$ is compact, then we can interchange the integral and the infinite sum so that (0.4) is easily reformulated as

(0.5) $$\dim S_k(\Gamma)=\frac{a(k)}{\sharp Z(\Gamma)}\sum_{\{\gamma\}_\Gamma}\mathrm{vol}(C(\gamma\,;\Gamma)\backslash C(\gamma\,;G_R))\cdot I(\gamma),$$

$$I(\gamma)=\int_{C(\gamma;G_R)\backslash H_2}H_\gamma(Z)dZ,$$

where $C(\gamma\,;\Gamma)$, $C(\gamma\,;G_R)$ denote the centralizor of $\gamma$ in $\Gamma$, in $G_R=\mathrm{Sp}(2, \boldsymbol{R})$ respectively, and the sum is extended over the complete set of conjugacy classes $\{\gamma\}_\Gamma$ of $\Gamma$. In this case, the integral $I(\gamma)$ has been evaluated by Langlands [17], in a more general context. If, on the other hand, the quotient is not compact, (0.5) breaks in two points; firstly, $C(\gamma;\Gamma)$ is not necessarily a lattice in $C(\gamma;G_R)$ and it may happen that $\mathrm{vol}(C(\gamma;\Gamma)\backslash C(\gamma;G_R))=\infty$. Secondly, the sum of the integrals does not necessarily converge. In order to repair these points, we introduce the closed connected subgroup $C_0(\gamma;G_R)$ of $C(\gamma;G_R)$ which is characterized by the following properties:

(i) $C_0(\gamma;G_R)$ has no compact semi-direct factor,
(ii) $C_0(\gamma;\Gamma)=C_0(\gamma;G_R)\cap\Gamma$ is a lattice of $C_0(\gamma;G_R)$,
(iii) $[C(\gamma;\Gamma):C_0(\gamma;\Gamma)]<\infty$.

We then define an equivalence relation in $\Gamma$, by saying that two elements $\gamma_1,\gamma_2$ of $\Gamma$ belong to the same family if $C_0(\gamma_1;G_R)=C_0(\gamma_2;G_R)$ and they have the same semi-simple factor. As for the termwise integrability, we can justify it by multiplying certain dumping factors to the integrands. After these remarks, (0.5) is modified in the following form:

$$\dim S_k(\Gamma)=\frac{a(k)}{\#Z(\Gamma)}\sum_{[\gamma]_\Gamma}\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G_R)) \tag{0.6}$$

$$\times\lim_{s\to+0}\sum_{\delta\in[\gamma]_\Gamma/\underset{\Gamma}{\sim}}\frac{I_0(\delta;s)}{[\bar{C}(\delta;\Gamma):\bar{C}_0(\delta;\Gamma)]},$$

$$I_0(\delta;s)=\int_{C_0(\delta;G_R)\backslash H_2}H_\delta(Z)(\text{a dumping factor in } s)dZ,$$

where the first sum is extended over a complete set of $\Gamma$-conjugacy classes of the families $[\gamma]_\Gamma$, and the second sum is extended over the set of non-conjugate elements in each family $[\gamma]_\Gamma$.

There are some advantages in this reformulation. First of all the two difficulties mensioned above are separated completely; $\Gamma$ does not appear in the last integrals so that we can evaluate them after normalizing the elements $\delta$ of $[\gamma]_\Gamma$ simultaneously by $G_R$-conjugation. Secondly, it can be shown (Theorem 1-7) that there are only a finite number of conjugacy classes of families of nonhyperbolic types so that the first sum is actually a finite sum. The third point is closely connected with Problem 2. Namely it gives an answer to the following question: we must combine certain kinds of infinite number of conjugacy classes in $\Gamma$ in order to make the contribution a rational number, or at any rate, to give it a reasonable or computable expression, which is always the case for non-elliptic classes. We shall observe, after the computation of the integrals $I_0(\delta;s)$, that the sum

$$\sum_{\delta\in[\gamma]_\Gamma/\underset{\Gamma}{\sim}}\frac{I_0(\delta;s)}{[\bar{C}(\delta;\Gamma):\bar{C}_0(\delta;\Gamma)]} \tag{0.7}$$

can always be expressed by using some kinds of zetafunctions. Also for elliptic classes, the sum



$$\sum_{\{\gamma\}_\Gamma}\frac{\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G_R))}{[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)]} \tag{0.8}$$

where $\{\gamma\}_\Gamma=[\gamma]_\Gamma$ runs over the $\Gamma$-conjugacy classes contained in a $G_Q$-conjugacy class, can be given a more simplified expression (a kind of Maß formula), which is again expressed by some special values of zetafunctions.

In § 1 we shall discuss some preliminary results on the conjugacy classes in $G_R=\mathrm{Sp}(2,\boldsymbol{R})$, in $G_Q$ or a $Q$-form of $\mathrm{Sp}(2,\boldsymbol{R})$, and in $\Gamma$. The above reformulation (0.6) will be discussed in § 2, by quoting some results of Christian [4], [5], and Morita [18]. In § 3, § 4, and § 5, the computations of $I_0(\gamma;s)$ will be carried out case by case. At the end of § 5, we shall resume these results and state it as a general formula for $\dim S_k(\Gamma)$, which is the main theorem of this paper (Theorem 5-1). In § 6, we quote from Münchhausen [19], [20], Sakamoto [22] some results on conjugacy classes of $\mathrm{Sp}(2,\boldsymbol{Z})$, and applying them to our formula, we shall obtain an explicit formula for $\dim S_k(\mathrm{Sp}(2,\boldsymbol{Z}))$, from which it is easy to derive Igusa's formula (0.3). Finally in § 7, we shall obtain in the same way as in § 6, an explicit formula for $\dim S_k(\Gamma_0(p))$, where $p$ is any odd prime.

Finally, it is a great pleasure for the author to express here heartily his gratitude to Professor Y. Ihara, who suggested the author the problem by drawing attention to the theory of automorphic forms for groups of type Sp(2), to Professor H. Shimizu, who gave the author many valuable advices, to Professor Y. Morita, who showed deep interests in this problem and gave the author continuous encouragement during the preparation of this paper. The author is also grateful to Professor T. Ibukiyama, who helped him with many valuable comments and discussions. The author also thanks very much to the referee, who checked so carefully the original manuscript and corrected the errors contained in it.

*Notations*

We denote by $\boldsymbol{Z}$, $Q$, $\boldsymbol{R}$, and $\boldsymbol{C}$, the ring of rational integers, the fields of rational, real, and complex numbers, respectively. For a ring $B$, $M_n(B)$, $GL_n(B)$, $SL_n(B)$ denote the full matrix ring of degree $n$, the group of invertible elements of $M_n(B)$, the group of matrices with determinant one, respectively. Also, $SM_n(B)$ denotes the module of all symmetric matrices in $M_n(B)$. We write $i$ the complex number $\sqrt{-1}$, and $\bar{z}$ the complex conjugate of $z\in\boldsymbol{C}$. The element of $SO(2)$, the special orthogonal group of degree two, is denoted as $k(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. If $H$ is a subgroup of a group $G$, we denote by $\underset{H}{\sim}$ the equivalence relation defined by $H$-conjugation, and by $\{g\}_H$ the equivalence class represented by $g\in G$. If $G$ is an algebraic group defined over $Q$, we denote by $G_{\boldsymbol{Z}}$, $G_Q$, $G_R$ the group of $\boldsymbol{Z}$-valued, $Q$-valued, and $\boldsymbol{R}$-valued points of $G$ respectively. We write also $G_p$,

$G_A$ the $p$-adic completion, the adelization of $G$, respectively. For a subgroup $C$ of $GL_n(\boldsymbol{R})$, we put $\bar{C}=\{\pm 1\}\cdot C/\{\pm 1\}$. Some more standard notations will be used frequently.

## §1. Preliminaries

### 1-1. Classification of conjugacy classes in Sp(2, $\boldsymbol{R}$)

Let $G_{\boldsymbol{R}}=\mathrm{Sp}(2, \boldsymbol{R})$ be the real symplectic group of degree two, i.e.,

$$G_{\boldsymbol{R}}=\left\{g\in GL_4(\boldsymbol{R})\,;\ g\begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}{}^t g=\begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}\right\}.$$

The group $G_{\boldsymbol{R}}$ operates on $H_2$, the Siegel upper half plane of degree two,

$$H_2=\{Z\in SM_2(\boldsymbol{C})\,;\ \mathrm{Im}\,Z>0\},$$

by $Z\mapsto \gamma\langle Z\rangle=(AZ+B)(CZ+D)^{-1}$, $\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_{\boldsymbol{R}}$. This action of $G_{\boldsymbol{R}}$ is transitive and the stabilizor $K$ of the point $i1_2$, which is equal to $SO(4)\cap G_{\boldsymbol{R}}=\left\{\begin{pmatrix}A & B\\ -B & A\end{pmatrix};\ A+iB\in U(2)\right\}$, is a maximal compact subgroup of $G_{\boldsymbol{R}}$. In his paper [18], Morita classified the conjugacy classes that have nonempty intersections with $\Gamma(N)$, the principal congruence subgroup of $\mathrm{Sp}(2, \boldsymbol{Z})$ of level $N\geqq 3$. We begin our study by supplementing his results, listing up a complete set of representatives of conjugacy classes in $G_{\boldsymbol{R}}$. To each representative $\gamma$ of the class $\{\gamma\}_{G_{\boldsymbol{R}}}$, we attach its centralizor $C(\gamma\,;G_{\boldsymbol{R}})$ in $G_{\boldsymbol{R}}$, making emphasis on its structure as a Lie group.

THEOREM 1-1. *Each element of $G_{\boldsymbol{R}}$ is conjugate in $G_{\boldsymbol{R}}$ to one and only one of the following* 21 *representatives.*

(a) *central* $\gamma=\pm 1_4$; $C(\gamma\,;G_{\boldsymbol{R}})=G_{\boldsymbol{R}}$.

(b) *elliptic* (*i.e.*, $\gamma\neq$*central, and has a fixed point in* $H_2$)

(b-1) $$\gamma=\alpha(\mu,\nu)=\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & \cos\nu & 0 & \sin\nu\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & -\sin\nu & 0 & \cos\nu\end{pmatrix}\qquad (k(\mu)^2,\ k(\nu)^2,\ k(\mu)k(\nu)\neq 1_2,\ \mu\neq\nu)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}\cos\xi & 0 & \sin\xi & 0\\ 0 & \cos\eta & 0 & \sin\eta\\ -\sin\xi & 0 & \cos\xi & 0\\ 0 & -\sin\eta & 0 & \cos\eta\end{pmatrix};\ \xi,\ \eta\in\boldsymbol{R}\right\}\cong SO(2)\times SO(2).$$

(b-2) $$\gamma=\alpha(\mu,\mu)=\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & \cos\mu & 0 & \sin\mu\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & -\sin\mu & 0 & \cos\mu\end{pmatrix}\qquad (k(\mu)^2\neq 1_2)$$



$$C(\gamma\,;G_{\boldsymbol{R}})=K=SO(4)\cap G_{\boldsymbol{R}}\cong U(2).$$

(b-3) $$\gamma=\gamma(\mu)=\begin{pmatrix}\cos\mu & \sin\mu & 0 & 0\\ -\sin\mu & \cos\mu & 0 & 0\\ 0 & 0 & \cos\mu & \sin\mu\\ 0 & 0 & -\sin\mu & \cos\mu\end{pmatrix}\qquad (k(\mu)^2\neq 1_2)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}p & q & 0 & 0\\ -q & p & 0 & 0\\ 0 & 0 & p & q\\ 0 & 0 & -q & p\end{pmatrix}\begin{pmatrix}a & 0 & b & 0\\ 0 & a & 0 & b\\ c & 0 & d & 0\\ 0 & c & 0 & d\end{pmatrix};\ (p^2+q^2)(ad-bc)=1\right\}\cong U(1,1).$$

(b-4) $$\gamma=\alpha(\mu,0)=\pm\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & 1 & 0 & 0\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad (k(\mu)^2\neq 1_2)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & a & 0 & b\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & c & 0 & d\end{pmatrix};\ ad-bc=1\right\}\cong SO(2)\times SL_2(\boldsymbol{R}).$$

(b-5) $$\gamma=\alpha(0,\pi)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix};\ a_id_i-b_ic_i=1\ (i=1,2)\right\}\cong SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R}).$$

(c) *elliptic/hyperbolic*

$$\gamma=\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & \lambda & 0 & 0\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & 0 & 0 & \lambda^{-1}\end{pmatrix}\qquad (k(\mu)^2\neq 1_2,\ \lambda^2\neq 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & a & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix};\ \theta\in\boldsymbol{R},\ a\in\boldsymbol{R}^\times\right\}\cong SO(2)\times\boldsymbol{R}^\times$$

(d) *elliptic/parabolic*

$$\gamma=\pm\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & 1 & 0 & \lambda\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad (k(\mu)^2\neq 1_2,\ \lambda=\pm 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\pm\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & t\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix};\ \theta\in\boldsymbol{R},\ t\in\boldsymbol{R}\right\}\cong\{\pm 1\}\times SO(2)\times\boldsymbol{R}$$

(e) *hyperbolic*

(e-1) $$\gamma=h(a,\,b)=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & b^{-1}\end{pmatrix}\qquad (a^2,\ b^2,\ ab\neq 1,\ a\neq b)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}c & 0 & 0 & 0\\ 0 & d & 0 & 0\\ 0 & 0 & c^{-1} & 0\\ 0 & 0 & 0 & d^{-1}\end{pmatrix};\ c,\ d\in\boldsymbol{R}^{\times}\right\}\cong\boldsymbol{R}^{\times}\times\boldsymbol{R}^{\times}$$

(e-2) $$\gamma=h(a,\,a)=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\qquad (a^2\neq 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}V & 0\\ 0 & {}^tV^{-1}\end{pmatrix};\ V\in GL_2(\boldsymbol{R})\right\}\cong GL_2(\boldsymbol{R})$$

(e-3) $$\gamma=h(a,\,1)=\pm\begin{pmatrix}a & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad (a^2\neq 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}c & 0 & 0 & 0\\ 0 & d_1 & 0 & d_2\\ 0 & 0 & c^{-1} & 0\\ 0 & d_3 & 0 & d_4\end{pmatrix};\ c\in\boldsymbol{R}^{\times},\ d_1d_4-d_2d_3=1\right\}\cong\boldsymbol{R}^{\times}\times SL_2(\boldsymbol{R})$$

(f) *hyperbolic/parabolic*

$$\gamma=\pm\begin{pmatrix}a & 0 & 0 & 0\\ 0 & 1 & 0 & \lambda\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad (a^2\neq 1,\ \lambda=\pm 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\pm\begin{pmatrix}c & 0 & 0 & 0\\ 0 & 1 & 0 & t\\ 0 & 0 & c^{-1} & 0\\ 0 & 0 & 0 & 1\end{pmatrix};\ c\in\boldsymbol{R}^{\times},\ t\in\boldsymbol{R}\right\}\cong\{\pm 1\}\times\boldsymbol{R}^{\times}\times\boldsymbol{R}$$

(g) *hyperparabolic*

$$\gamma=\begin{pmatrix}1 & 0 & 0 & s\\ 0 & 1 & s & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & 0 & 0 & 0\\ 0 & a^{-1} & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a\end{pmatrix}\qquad (a^2\neq 1,\ s\neq 0)$$



$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}1 & 0 & 0 & t\\ 0 & 1 & t & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c & 0 & 0 & 0\\ 0 & c^{-1} & 0 & 0\\ 0 & 0 & c^{-1} & 0\\ 0 & 0 & 0 & c\end{pmatrix};\ t\in\boldsymbol{R},\ c\in\boldsymbol{R}^{\times}\right\}\cong\boldsymbol{R}\times\boldsymbol{R}^{\times}$$

(h) *d-unipotent*

$$\gamma=\pm\begin{pmatrix}1 & 0 & b & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ a & 1 & 0 & 0\\ 0 & 0 & 1 & -a\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad (a,\ b\neq 0)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\pm\begin{pmatrix}1 & 0 & s & t\\ 0 & 1 & t & u\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ c & 1 & 0 & 0\\ 0 & 0 & 1 & -c\\ 0 & 0 & 0 & 1\end{pmatrix};\ \begin{matrix}c,\ s,\ t,\ u\in\boldsymbol{R}\\ s=a^{-1}bc\\ t=(a^{-1}bc^2-bc)/2\end{matrix}\right\}\cong\{\pm 1\}\times\boldsymbol{R}^2$$

(i) *hyperelliptic*

$$\gamma=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\begin{pmatrix}\cos\mu & \sin\mu & 0 & 0\\ -\sin\mu & \cos\mu & 0 & 0\\ 0 & 0 & \cos\mu & \sin\mu\\ 0 & 0 & -\sin\mu & \cos\mu\end{pmatrix}\qquad (a^2\neq 1,\ k(\mu)^2\neq 1_2)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\begin{pmatrix}c & 0 & 0 & 0\\ 0 & c & 0 & 0\\ 0 & 0 & c^{-1} & 0\\ 0 & 0 & 0 & c^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta & 0 & 0\\ -\sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \sin\theta\end{pmatrix};\ \begin{matrix}c>0\\ \theta\in\boldsymbol{R}\end{matrix}\right\}\cong\boldsymbol{R}_{+}^{\times}\times SO(2)$$

(j) *δ-parabolic*

(j-1) $$\gamma=\delta(s_1,\,s_2)=\begin{pmatrix}1 & 0 & s_1 & 0\\ 0 & -1 & 0 & s_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\qquad ((s_1,\ s_2)=(\pm 1,\ \pm 1))$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\pm\begin{pmatrix}1 & 0 & t_1 & 0\\ 0 & \pm 1 & 0 & t_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \pm 1\end{pmatrix};\ t_1,\ t_2\in\boldsymbol{R}\right\}=\{\pm 1\}\times\{\pm 1\}\times\boldsymbol{R}^2$$

(j-2) $$\gamma=\delta(s,\,0)=\pm\begin{pmatrix}1 & 0 & s & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\qquad (s=\pm 1)$$

$$C(\gamma\,;G_{\boldsymbol{R}})=\left\{\pm\begin{pmatrix}1 & 0 & t & 0\\ 0 & d_1 & 0 & d_2\\ 0 & 0 & 1 & 0\\ 0 & d_3 & 0 & d_4\end{pmatrix};\ \begin{matrix}t\in\boldsymbol{R}\\ d_1d_4-d_2d_3=1\end{matrix}\right\}\cong\{\pm 1\}\times\boldsymbol{R}\times SL_2(\boldsymbol{R})$$

(k) *parabolic*

(k-1) $\gamma=\pm\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$$C(\gamma\,;G_{R})=\left\{\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}1&S\\0&1\end{pmatrix};\ A\in O(2),\ S\in SM_2(\boldsymbol{R})\right\}\cong O(2)\ltimes \boldsymbol{R}^3$$

(k-2) $\gamma=\pm\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$$C(\gamma\,;G_{R})=\left\{\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}\begin{pmatrix}1&S\\0&1\end{pmatrix};\ A\in O(1,1),\ S\in SM_2(\boldsymbol{R})\right\}\cong O(1,1)\ltimes \boldsymbol{R}^3$$

(k-3) $\gamma=\pm\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$$C(\gamma\,;G_{R})=\left\{\pm\begin{pmatrix}*&0&*&*\\ *&1&*&*\\ *&0&*&*\\ 0&0&0&1\end{pmatrix}\in G_{R}\right\}\cong\{\pm1\}\times(SL_2(\boldsymbol{R})\ltimes(\boldsymbol{R}\ltimes\boldsymbol{R}^2))$$

(l) *paraelliptic*

$$\gamma=\begin{pmatrix}1&0&s&0\\0&1&0&s\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\cos\mu&\sin\mu&0&0\\-\sin\mu&\cos\mu&0&0\\0&0&\cos\mu&\sin\mu\\0&0&-\sin\mu&\cos\mu\end{pmatrix}\qquad(s=\pm1,\ k(\mu)^2\neq 1_2)$$

$$C(\gamma\,;G_{R})=\left\{\begin{pmatrix}1&0&t&0\\0&1&0&t\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta&0&0\\-\sin\theta&\cos\theta&0&0\\0&0&\cos\theta&\sin\theta\\0&0&-\sin\theta&\cos\theta\end{pmatrix};\ t,\ \theta\in\boldsymbol{R}\right\}\cong\boldsymbol{R}\times SO(2).$$

We omit the proof of this theorem, since it follows by an easy but careful chase of the proof of Theorem 1 in Morita [17].

**1-2. Discrete subgroups of** $\mathrm{Sp}(2,\boldsymbol{R})$

Recall that $G_{R}=\mathrm{Sp}(2,\boldsymbol{R})$ is regarded as the group of all unitary transformations of a hermitian space of rank two over the quaternion algebra $M_2(\boldsymbol{R})$, equipped with the canonical involution $\overline{\begin{pmatrix}a&b\\c&d\end{pmatrix}}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$. Moreover, it is known that all $Q$-forms of $G_{R}$ are obtained in this way from the $Q$-forms of $M_2(\boldsymbol{R})$, or, the indefinite quaternion algebras $B$ over $Q$. Since any non-degenerate hermitian



form of rank two over $B$ is isomorphic to $F(x,y)=x_1\bar{y}_2+x_2\bar{y}_1$, we may write, without loss of generality, $G_Q$ in the following form:

$$\begin{aligned}(1.1)\qquad G_Q&=U(2,B)\\&=\{g\in M_2(B)\,;\ F(x\cdot g,\ y\cdot g)=F(x,y)\ \text{ for all } x,\ y\in B^2\}.\end{aligned}$$

If $B=M_2(Q)$, an isomorphism $\varphi$ from $U(2,B)$ to $\mathrm{Sp}(2,Q)$ is given explicitly by

$$(1.2)\qquad \varphi(g)=\begin{pmatrix}a_1&a_2&b_2&-b_1\\a_3&a_4&b_4&-b_3\\c_3&c_4&d_4&-d_3\\-c_1&-c_2&-d_2&d_1\end{pmatrix},\quad g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in U(2,B),$$

where $A=\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}$, $B=\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}$, $C=\begin{pmatrix}c_1&c_2\\c_3&c_4\end{pmatrix}$, $D=\begin{pmatrix}d_1&d_2\\d_3&d_4\end{pmatrix}$. The $Q$-rank of $G_Q$ is one or two, according as $B$ is a division algebra or not. Since the $\boldsymbol{R}$-rank of $G_{R}$ is two, the theorem of Margulis says that any lattice of $G_{R}$ is arithmetic. In particular, if the quotient $\Gamma\backslash H_2$ is not compact, we can find a $Q$-form of $G_{R}$ such that $\Gamma$ is commensurable with $G_{Z}$. In the expression (1.1) of $G_Q$, $G_Z$ is given by

$$(1.3)\qquad G_Z=U(2,B)_L=\{g\in U(2,B)\,;\ L\cdot g=L\},$$

where $L$ is a lattice in $B^2$. It is not difficult to see that they are all commensurable, with commensurator $G_Q$ itself ($B$ is fixed). Of special interests for us are those for $L=O^2$, where $O$ is a maximal order of $B$. Then we have $G_Z=U(2,O)$; if $B=M_2(Q)$ and $O=M_2(\boldsymbol{Z})$, then $\varphi(U(2,O))=\mathrm{Sp}(2,\boldsymbol{Z})$.

**1-3. A reduction of $\Gamma$-conjugacy to $G_A$-conjugacy**

The first step to the computation problem in Selberg's trace formula is the classification of conjugacy classes in the lattice $\Gamma$, for which $S_k(\Gamma)$ is defined. This is by no means an easy task, even if we restrict ourselves to the simplest case $\Gamma=U(2,O)$. However, if we consider the conjugacy classes in $G_Q$, or in $\tilde{G}_Q$, the group of all positive similitudes of the hermitian space $(B^2,F)$ which is defined by

$$\begin{aligned}(1.4)\qquad \tilde{G}_Q&=GU^+(2,B)\\&=\left\{\begin{matrix}g\in M_2(B)\,;\ F(x\cdot g,\ y\cdot g)=n(g)F(x,y),\ n(g)\in Q_+^\times\\ \text{for all } x,\ y\in B^2\end{matrix}\right\},\end{aligned}$$

then the problem will become much easier. For we can localize the problem, reducing it to the conjugacy problem in $G_p$, $\tilde{G}_p$ for each prime $p$, by virtue of the following

THEOREM 1-2. *The Hasse principle holds for conjugacy classes in $G_Q$ and $\tilde{G}_Q$; namely two elements of $G_Q$ (resp. $\tilde{G}_Q$) are conjugate in $G_Q$ (resp. $\tilde{G}_Q$) if and* *only if they are conjugate in* $G_p$ *(resp.* $\tilde{G}_p$*) for all* $p$.

The proof of this theorem is found in T. Asai [2] in the case $G_Q=\mathrm{Sp}(n, Q)$. For $\tilde{G}_Q=GU(n, B)$, it is proved in our previous paper [10] under the assumption that the hermitian form $F$ is definite. However, the proof in the general case goes quite similarly as in [2], [10], and we shall omit it here. We prefer $\tilde{G}_Q$ to $G_Q$ for some reasons, one of which is that it fits to our program to treat later the trace formula also for Hecke operators (cf. §0-3).

Now let us assume that $\Gamma$ satisfies the following conditions:

(i) There exists a $\boldsymbol{Z}$-order $R$ of $M_2(B)$ such that

$$\Gamma=R^{\times}\cap\tilde{G}_Q \tag{1.5}$$

(ii) $n(R_p^{\times}\cap\tilde{G}_p)=\boldsymbol{Z}_p^{\times}$ for all $p$.

If $\Gamma=U(2, O)$, these conditions are satisfied by taking $R=M_2(O)$. It follows then by the strong approximation theorem, that

$$\tilde{G}_A=(\prod_p(R_p^{\times}\cap\tilde{G}_p)\times\tilde{G}_R)\cdot\tilde{G}_Q. \tag{1.6}$$

We put

$$\mathfrak{U}=\prod_p(R_p^{\times}\cap\tilde{G}_p)\times\tilde{G}_R=\prod_p U_p\times U_\infty,$$

$$U_p=R_p^{\times}\cap\tilde{G}_p,\quad U_\infty=\tilde{G}_R.$$

We say, as in [8], [10], that an element $g\in\tilde{G}_Q$ or a conjugacy class $\{g\}_{\tilde{G}_Q}$ is "locally integral", if the $\tilde{G}_p$-conjugacy class of $g$ has a representative in $R_p\cap\tilde{G}_p$, for all $p$. We have

THEOREM 1-3. *A conjugacy class in* $\tilde{G}_Q$ *is locally integral if and only if it is integral (i.e., it has a representative in* $R\cap\tilde{G}_Q$*).*

PROOF. By definition, $\{g\}_{\tilde{G}_Q}$ being locally integral means that $hgh^{-1}\in R_A\cap\tilde{G}_A$ for some $h\in\tilde{G}_A$. By (1.6), we can write $h=u\gamma$ with $u\in\mathfrak{U}$, $\gamma\in\tilde{G}_Q$. Then we have

$$\gamma g\gamma^{-1}\in u^{-1}(R_A\cap\tilde{G}_A)u\cap\tilde{G}_Q=R\cap\tilde{G}_Q. \qquad \text{q.e.d.}$$

Combining Theorems 1-2 and 1-3, we can determine the $\tilde{G}_Q$-conjugacy classes that have nonempty intersections with $\Gamma$, completely by local computations. In fact, the problem of conjugacy classes in $\tilde{G}_p$, and $U_p$ for $R=M_2(O)$, has been settled in [10]. It should be noted that, in $G_Q$ or $\tilde{G}_Q$ a conjugacy class is *not* uniquely determined by its characteristic polynomial. On the contrary, in most cases there are *infinite* number of classes for each characteristic polynomial. As an illustration, we sketch the most interesting case without proof. Namely we have



THEOREM 1-4. *Let* $f(x)=(x^2+sx+n)^2\in\boldsymbol{Z}[x]$ *be such that* $s^2-4n<0$, *and let* $S_e(f)$ *be the set of elements of* $\tilde{G}_Q$ *for which* $n^{-1/2}g$ *is elliptic and* $f(g)=0$.

(i) *We have a canonical correspondence*

$$S_e(f)/\underset{\tilde{G}_Q}{\sim}\longleftrightarrow\left\{\begin{array}{l}\text{isomorphic classes of quaternion algebras}\\ D \text{ over } Q \text{ such that } B\otimes_Q K\cong D\otimes_Q K\end{array}\right\}$$

*where* $K=Q(\sqrt{s^2-4n})$. *The correspondence is given explicitly by*

$$C(g;\tilde{G}_Q)=Q[g]^{\times}\cdot D_+^{\times},\quad g\in S_e(f)\ (D_+^{\times}=\{a\in D^{\times};\ a\bar{a}>0\}),$$

*and it is one-to-one or two-to-one according as* $D$ *is indefinite or definite.*

(ii) *Let* $g\in S_e(f)$ *and* $D=D(g)$ *be as in* (i), *and assume that* $\Gamma=U(2, O)$ *for a maximal order* $O$ *of* $B$. *Then we have*

$$\{g\}_{\tilde{G}_Q}\cap\Gamma\neq\emptyset\Leftrightarrow d(D(g))\,|\,d(B)\cdot d(K),$$

*where* $d(D(g))$, $d(B)$, $d(K)$ *are the discriminant of* $D(g)$, $B$, $K$, *respectively.* (*cf.* [10] (I), *Proposition* 3, 4).

Now let us consider the gap between $\Gamma$-conjugacy and $\tilde{G}_A$-conjugacy. We take and fix an element $g$ of $R\cap\tilde{G}_Q$ and put

$$\begin{aligned} &Z(g)=\{z\in M_2(B);\ zg=gz\}\\ &C(g)=\{x^{-1}gx;\ x\in\tilde{G}_Q\}\\ &M(g, T)=\{x\in\tilde{G}_Q;\ x^{-1}gx\in T\},\end{aligned} \tag{1.7}$$

where $T$ is a subset of $R\cap\tilde{G}_Q$ such that $\Gamma T\Gamma=T$. $Z(g)$ is an algebra over $Q$. We say that two $\boldsymbol{Z}$-orders $\Lambda_1$, $\Lambda_2$ of $Z(g)$ belong to the same $\tilde{G}_Q$-type, and write $\Lambda_1\sim\Lambda_2$, if $\Lambda_2=a\Lambda_1a^{-1}$ for some $a\in Z(g)^{\times}\cap\tilde{G}_Q$. Put, for each $\boldsymbol{Z}$-order $\Lambda$ of $Z(g)$,

$$\begin{aligned} &C(g, \Lambda)=\{x^{-1}gx;\ x\in\tilde{G}_Q,\ Z(g)\cap xRx^{-1}\sim\Lambda\}\\ &M(g, T, \Lambda)=\{x\in\tilde{G}_Q;\ x^{-1}gx\in T,\ Z(g)\cap xRx^{-1}\sim\Lambda\}.\end{aligned} \tag{1.8}$$

Then we have a disjoint decomposition of $C(g)$, $M(g, T)$:

$$\begin{aligned} &C(g)=\coprod_{\Lambda/\sim}C(g, \Lambda)\\ &M(g, T)=\coprod_{\Lambda/\sim}M(g, T, \Lambda).\end{aligned} \tag{1.9}$$

LEMMA 1-1. *The map* $x^{-1}gx\mapsto x$ *induces the following bijections:*

(i) $C(g)\cap T\xrightarrow{\sim}(Z(g)^{\times}\cap\tilde{G}_Q)\backslash M(g, T)$

(ii) $C(g, \Lambda)\cap T\xrightarrow{\sim}(Z(g)^{\times}\cap\tilde{G}_Q)\backslash M(g, T, \Lambda)$

(iii) $C(g, \Lambda)\cap T/\underset{\Gamma}{\sim}\xrightarrow{\sim}(Z(g)^{\times}\cap\tilde{G}_Q)\backslash M(g, T, \Lambda)/\Gamma$ (*cf.* [8], [21], [27]).

Assume that $T$ is expressed as $T=T_A\cap\tilde{G}_Q$ for a subset $T_A$ of $R_A\cap\tilde{G}_A$ such that $\mathfrak{U}T_A\mathfrak{U}=T_A$. Put

(1.10) $M(g, T_A, \Lambda)=\{x\in\tilde{G}_A;\ x^{-1}gx\in T_A,\ Z(g)\cap xRx^{-1}\sim\Lambda\}$.

LEMMA 1-2. (i) *The inclusion map* $M(g, T, \Lambda)\hookrightarrow M(g, T_A, \Lambda)$ *induces the following surjection*

$$\phi: (Z(g)^{\times}\cap\tilde{G}_Q)\backslash M(g, T, \Lambda)/\Gamma \longrightarrow (Z(g)^{\times}\cap\tilde{G})_A\backslash M^*(g, T_A, \Lambda)/\mathfrak{U},$$

*where*

$$M^*(g, T_A, \Lambda)=\bigcup_{\Lambda'\in L_{\tilde{G}}(\Lambda)} M(g, T_A, \Lambda')$$

(*For the definition of* $L_{\tilde{G}}(\Lambda)$, *see* (2.18).).

(ii) $\phi$ *is* $h_0(\Lambda;\tilde{G})$*-to-one, where*

(1.11) $$h_0(\Lambda;\tilde{G})=\sharp((Z(g)^{\times}\cap\tilde{G}_Q)\backslash(Z(g)^{\times}\cap\tilde{G}_Q)\cdot I(\Lambda)/(\Lambda_A^{\times}\cap\tilde{G}_A))$$

$=:$ *the two-sided G-class number of* $\Lambda$,

$$I(\Lambda)=\{z\in(Z(g)^{\times}\cap\tilde{G})_A;\ z\Lambda z^{-1}=\Lambda\} \qquad (cf.\ [8]).$$

By Lemmas 1-1, 1-2, the gap between $\Gamma$-conjugacy classes and $\tilde{G}_A$-conjugacy classes can be filled by considering those $\boldsymbol{Z}$-orders $\Lambda$ in $Z(g)$ such that $M(g, T_A, \Lambda)\neq\emptyset$.

### 1-4. Structure of the centralizors

The second step to the application of Selberg's trace formula is the study of the centralizors $C(\gamma;\Gamma)$ of elements $\gamma$ of $\Gamma$. It is easily seen that $C(\gamma;\Gamma)$ is always a discrete subgroup of $C(\gamma;G_R)$. The following fact plays an important role in our computation. Note that it is an obvious analogue of Theorem 7 in Shimizu [26].

THEOREM 1-5. *Let* $\Gamma$ *be a lattice of* $G_R=\mathrm{Sp}(2, \boldsymbol{R})$ *and let* $\gamma$ *be an element of* $\Gamma$. *Then* $C(\gamma;\Gamma)$ *is a lattice of* $C(\gamma;G_R)$ *except for the following case*: $\gamma$ *is conjugate in* $G_Q=\mathrm{Sp}(2, Q)$ *to an element of the form* $\begin{pmatrix}1&S\\0&1\end{pmatrix}$, $S\in SM_2(Q)$, *with* $-\det S\in(Q^{\times})^2$.

PROOF. If the quotient $\Gamma\backslash H_2$ is compact, the assertion is well known and easy to prove, for more general topological groups. Let us assume, therefore, that the quotient is not compact so that $\Gamma$ is commensurable with $G_Z=U(2, O)$. Then it is known that $\Gamma$ is contained in $\tilde{G}_Q$, modulo the center of $\tilde{G}_R$. Therefore $C(\gamma;G_R)$ has a natural $Q$-form, and it suffices to prove that $C(\gamma;G_Z)=C(\gamma;G)_Z$ is a lattice, with one exception. It is easy to see, by Theorem 1-1 and the following Lemma 1-3, that $C(\gamma;G)_Z$ is a lattice unless $C(\gamma;G_R)$ contains, as semi-direct factor, a group isomorphic to (a) $\boldsymbol{R}^{\times}$ ($\gamma$ is of type "hyper"), (b) $GL_2(\boldsymbol{R})$ ($\gamma$ is of type (e-2)), or (c) $O(1, 1)$ ($\gamma$ is of type (k-2)). First consider (a): then by Lemma 1-3, it suffices to prove the case where the algebra $K=Q[\gamma]$ is a totally real field of degree 4, and $K_0=Q[\gamma+\gamma^{-1}]$ is a real quadratic subfield of $K$. We have then



$$C(\gamma;G_Q)=K^{(1)}=\{x\in K^{\times};\ N_{K/K_0}(x)=1\},$$
$$C(\gamma;G)_Z=O_K^{(1)}=\{x\in O_K^{\times};\ N_{K/K_0}(x)=1\}.$$

Since $O_K^1\cap K^{(1)}=O_K^{(1)}$, $K_\infty^{(1)}/O_K^{(1)}$ is isomorphic to a subgroup of $K_\infty^1/O^1$, the assertion follows by Dirichlet's unit theorem. Next consider the case (b): $\gamma$ is then conjugate in $G_Q=U(2, B)$ to an element of the form $\gamma_0=\begin{pmatrix}a&0\\0&a\end{pmatrix}$, where $a$ is an element of a real quadratic field $K$ in $B$, such that $\mathrm{Nr}(a)=a\bar{a}=1$. Then we have

$$C(\gamma_0;G_Q)=U(2, K)=K^1\cdot SU(2, K),$$
$$C(\gamma_0;G)_Z=U(2, O_K)=O_K^1\cdot SU(2, O_K).$$

It is easy to see that $SU(2, K)$ is the norm one group of an indefinite quaternion algebra over $Q$, and $SU(2, O_K)$ is a group of units of an order of it so that $SU(2, O_K)$ is a lattice of $SU(2, K_\infty)=SL_2(\boldsymbol{R})$. Since $K_\infty^1/O^1$ is compact, this proves the assertion in case (b). Finally consider (c): $\gamma$ is then conjugate in $G_Q$ to an element of the form $\gamma_0=\begin{pmatrix}1&s\\0&1\end{pmatrix}\in U(2, B)$, with $\mathrm{tr}(s)=0$, $\mathrm{Nr}(s)<0$. $C(\gamma_0;G_R)$ is a semi-direct product of

$$V=\left\{\begin{pmatrix}1&t\\0&1\end{pmatrix};\ t\in B_\infty,\ \mathrm{tr}(t)=0\right\}\cong SM_2(\boldsymbol{R}),$$

and

$$W=\left\{\begin{pmatrix}a&0\\0&\bar{a}^{-1}\end{pmatrix};\ as\bar{a}=s,\ a\in B^{\times}\right\}\cong O(1, 1).$$

It is clear that $V\cap U(2, O)$ is a lattice of $V$. On the other hand, $W\cap U(2, O)$ is a lattice of $W$ if and only if $Q[s]$ is a real quadratic field; in fact $W\cap U(2, O)$ contains as index finite subgroup, the group of norm one units of the ring of integers of $Q[s]$. We see that this is not the case if and only if $G_Q=\mathrm{Sp}(2, Q)$ (i.e. $B=M_2(Q)$) and $Q[s]=Q\oplus Q$, so that $-\mathrm{Nr}(s)\in(Q^{\times})^2$. This completes the proof. q.e.d.

To state the Lemma 1-3, we consider, instead of $G_Q=U(2, B)$, an isomorphic group

(1.12) $$G_Q^*=U(2, B)^*=\{g\in M_2(B);\ g^t\bar{g}=1_2\}.$$

There is an isomorphism $\phi^*: G_Q\simeq G_Q^*$ such that $\phi^*(U(2, O))=U(2, O)^*$ (cf. [10], Lemma 11). Let $f(x)\in Q[x]$ be the characteristic polynomial of $g\in G_Q^*$, and suppose that it splits over $Q$ into a product $f(x)=f_1(x)f_2(x)$ such that $x^2f_i(x^{-1})=f_i(x)$ $(i=1, 2)$.

LEMMA 1-3. *Notations being as above, assume that* $f_1(x)\neq f_2(x)$. *Then* $g$ *is conjugate in* $G_{\mathbf{Q}}^*$ *to an element of the form*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\in G_{\mathbf{Q}}^*, \quad a, b\in B^1,$$

*with* $f_1(a)=f_2(b)=0$. (*Cf.* [10], *Lemma* 2.)

The following theorem is an easy consequence of Theorem 1-5 and the proof of it.

THEOREM 1-6. *Let* $\Gamma$ *be a lattice of* $G_{\mathbf{R}}$ *and* $\gamma$ *be an element of* $\Gamma$. *Then there exists a connected closed subgroup* $C_0(\gamma; G_{\mathbf{R}})$ *of* $C(\gamma; G_{\mathbf{R}})$ *which is characterized by the following conditions*:

(i) $C_0(\gamma; G_{\mathbf{R}})$ *has no compact semi-direct factor.*
(ii) $C_0(\gamma; G_{\mathbf{R}})\cap\Gamma=C_0(\gamma; \Gamma)$ *is a lattice of* $C_0(\gamma; G_{\mathbf{R}})$.
(iii) $[C(\gamma; \Gamma): C_0(\gamma; \Gamma)]<\infty$.

*It contains* $\gamma_u$, *the unipotent factor of* $\gamma$.

In fact we have only to drop the compact semi-direct factors of $C(\gamma; G_{\mathbf{R}})$ to obtain $C_0(\gamma; G_{\mathbf{R}})$, if $C(\gamma; \Gamma)$ is a lattice of $C(\gamma; G_{\mathbf{R}})$. In the exceptional case, it is given by

$$C_0(\gamma; G_{\mathbf{R}})=\left\{\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}; T\in SM_2(\mathbf{R})\right\}, \quad \text{if } \gamma=\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}. \tag{1.13}$$

REMARK 1-1. It should be noted that the definition of $C_0(\gamma; G_{\mathbf{R}})$ is *not* compatible with $G_{\mathbf{R}}$-conjugacy: we can not conclude $C_0(\gamma_1; G_{\mathbf{R}})=g^{-1}C_0(\gamma_2; G_{\mathbf{R}})g$, from $\gamma_1=g^{-1}\gamma_2 g$. However, this is true if $g$ is an element of $G_{\mathbf{Q}}$. Note also that the Lie group $C_0(\gamma; G_{\mathbf{R}})$ is unimodular, and it has a natural $Q$-form $C_0(\gamma; G_{\mathbf{Q}})=C_0(\gamma; G_{\mathbf{R}})\cap G_{\mathbf{Q}}$.

**1-5.** Finally the third step to our problem on conjugacy classes concerning the application of Selberg's trace formula is as follows: For certain types of elements, there are infinite number of classes in $\Gamma$ that have nonzero contributions, so that we must combine them suitably to get a rational number, or at any rate, to give it a computable expression. For parabolic conjugacy classes, this problem is settled by [5], [18], [1], and generalized by T. Shintani [30] to arbitrary degree $n$. In the light of their results, we make the following

DEFINITION 1-1. Two elements $\gamma_1, \gamma_2$ of $\Gamma$ are said to belong to the same *family*, if (i) their semi-simple factors coincide, and (ii) $C_0(\gamma_1; G_{\mathbf{R}})=C_0(\gamma_2; G_{\mathbf{R}})$.

We denote by $[\gamma]_\Gamma$ the family (the equivalence class) represented by $\gamma$. Note that this definition of $[\gamma]_\Gamma$ is compatible with $\tilde{G}_{\mathbf{Q}}$-conjugacy, and $\Gamma$-conjugacy.



If $\gamma_1, \gamma_2$ belong to the same family, then $C_0(\gamma_1; \Gamma)=C_0(\gamma_2; \Gamma)$.

THEOREM 1-7. *Let* $\Gamma$ *be a lattice of* $G_{\mathbf{R}}$. *Then there are only a finite number of* $\Gamma$*-conjugacy classes of families that belong to one of the types*: (a) *central*, (b) *elliptic*, (d) *elliptic/parabolic*, (j) $\delta$*-parabolic*, (k) *parabolic, and* (l) *paraelliptic.*

In fact, the assertion for parabolic types is an easy consequence of the arithmeticity of $\Gamma$ and the finiteness of the number of cusps. For elliptic and other types, it can be proved by using Theorems 1-2, 1-3, and local computations given in [10].

REMARK 1-2. If $\Gamma$ is a congruence subgroup of Sp(2, $\mathbf{Z}$), Theorem 1-7 is a direct consequence of I. Münchhausen [19] (cf. §6, Theorem 6-1).

## §2. Cusp forms and Selberg's trace formula

In this section, we recall briefly the Selberg-Godement's dimension formula for $S_k(\Gamma)$ (cf. [25]), and a reformulation of it due to Christian [5], Morita [18], and Arakawa [1], which will be the starting point of our calculation.

**2-1.** Let $\Gamma$ be a lattice of $G_{\mathbf{R}}=\mathrm{Sp}(2, \mathbf{R})$, and let $(\chi, V)$ be a finite dimensional representation of $\Gamma$ such that $[\Gamma: \mathrm{Ker}\,\chi]<\infty$. We assume that $\chi$ is unitary. For a positive integer $k$, we denote by $S_k(\Gamma, \chi)$ the complex vector space of cusp forms of type $(k, \chi, \Gamma)$. Namely $S_k(\Gamma, \chi)$ consists of the holomorphic $V$-valued functions $f(Z)$ on $H_2$ which satisfy the following conditions:

(i) $f(\gamma\langle Z\rangle)=\det(CZ+D)^k\chi(\gamma)f(Z)$ for all $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma$,

(ii) $(\det \mathrm{Im}\, Z)^{k/2}\|f(Z)\|$ is bounded on $H_2$.

Note that, if $\Gamma\backslash H_2$ is compact, (ii) follows from (i). If $\chi$=trivial, $V=C$, we denote $S_k(\Gamma, \chi)$ simply by $S_k(\Gamma)$. In any case, it is known that $S_k(\Gamma, \chi)$ is a finite dimensional Hilbert space with respect to the Petersson metric. In [25], R. Godement studied the kernel function of it and expressed the dimension as an integral of an infinite series:

THEOREM 2-1. (R. Godement [25]) *Suppose* $k\geqq 5$. *Put*

$$K_\Gamma(Z_1, Z_2)=\frac{a(k)}{\sharp Z(\Gamma)}\sum_{\gamma\in\Gamma}\det\left(\frac{Z_1-\gamma\langle\bar{Z}_2\rangle}{2i}\right)^{-k}\det(C\bar{Z}_2+D)^{-k}\chi(\gamma), \tag{2-1}$$

*where*

$$a(k)=2^{-8}\pi^{-3}(2k-2)(2k-3)(2k-4),$$
$$Z(\Gamma)=: \textit{the center of } \Gamma.$$

(i) *The series* $K_\Gamma(Z_1, Z_2)$ *converges absolutely and uniformly on any compact set* *in* $H_2\times H_2$, *and* $(\det \operatorname{Im} Z_1)^{k/2}(\det \operatorname{Im} Z_2)^{k/2}\times K_\Gamma(Z_1, Z_2)$ *is bounded on* $H_2\times H_2$.

(ii) $K_\Gamma(Z_2, Z_1)=K_\Gamma(Z_1, Z_2)^*$, *where* $K^*$ *denotes the adjoint operator of* $K$.

(iii) *For any* $Z_2\in H_2$ *and* $v\in V$, *the function* $Z_1\to K_\Gamma(Z_1, Z_2)v$ *belongs to* $S_k(\Gamma, \chi)$.

(iv) *For any* $f(Z)\in S_k(\Gamma, \chi)$, *we have*

$$f(Z)=\int_{\Gamma\backslash H_2}(\det \operatorname{Im} Z')^k K_\Gamma(Z, Z')f(Z')dZ',$$

*where* $dZ=(\det Y)^{-3}dXdY$ $(Z=X+iY)$ *is a* $G_R$*-invariant measure on* $H_2$.

(v) *We have*

$$\dim S_k(\Gamma, \chi)=\int_{\Gamma\backslash H_2}(\det \operatorname{Im} Z)^k \operatorname{tr} K_\Gamma(Z, Z)dZ\,. \tag{2.2}$$

**2-2.** Following Morita [18], we put for each $\gamma\in G_R$ and $Z\in H_2$,

$$H_\gamma(Z)=(\det \operatorname{Im} Z)^k\det\left(\frac{Z-\gamma\langle\bar{Z}\rangle}{2i}\right)^{-k}\det(C\bar{Z}+D)^{-k}\operatorname{tr}\chi(\gamma)\,. \tag{2.3}$$

A direct computation shows that one has, for any $g\in G_R$,

$$H_\gamma(g\langle Z\rangle)=H_{g^{-1}\gamma g}(Z)\,,\quad\text{and}\quad H_{\gamma^{-1}}(Z)=\overline{H_\gamma(Z)}\,. \tag{2.4}$$

Then (2.2) is written as

$$\dim S_k(\Gamma, \chi)=\frac{a(k)}{\#Z(\Gamma)}\int_{\Gamma\backslash H_2}\sum_{\gamma\in\Gamma}H_\gamma(Z)dZ\,. \tag{2.5}$$

If the quotient $\Gamma\backslash H_2$ is compact, the infinite series in the integrand converges absolutely and uniformly on $\Gamma\backslash H_2$, so that we can interchange the integral and the infinite sum; namely the right hand side of (2.5) is equal to

$$\frac{a(k)}{\#Z(\Gamma)}\sum_{\gamma\in\Gamma}\int_{\Gamma\backslash H_2}H_\gamma(Z)dZ\,. \tag{2.6}$$

By (2.4) and

$$C(\gamma\,;\,\Gamma)\backslash H_2=\coprod_{\delta\in\{\gamma\}_\Gamma}\delta\langle\Gamma\backslash H_2\rangle\quad\text{(disjoint)}, \tag{2.7}$$

we can first sum up the integrals, over each conjugacy classes $\{\gamma\}_\Gamma$, so that we have, as is well known,

$$\begin{aligned}\dim S_k(\Gamma, \chi)&=\frac{a(k)}{\#Z(\Gamma)}\sum_{\{\gamma\}_\Gamma}\int_{C(\gamma;\,\Gamma)\backslash H_2}H_\gamma(Z)dZ\\&=\frac{a(k)}{\#Z(\Gamma)}\sum_{\{\gamma\}_\Gamma}\operatorname{vol}(C(\gamma\,;\,\Gamma)\backslash C(\gamma\,;\,G_R))\int_{C(\gamma;\,G_R)\backslash H_2}H_\gamma(\hat{Z})d\hat{Z}\,,\end{aligned} \tag{2.8}$$

where $d\hat{Z}$ is a quotient measure on $C(\gamma\,;\,G_R)\backslash H_2$ induced from $dZ$. Here we should note that, under the above condition on $\Gamma$, every element $\gamma$ of $\Gamma$ is semi-simple and $C(\gamma\,;\,\Gamma)$ is a lattice of $C(\gamma\,;\,G_R)$, i.e., $\operatorname{vol}(C(\gamma\,;\,\Gamma)\backslash C(\gamma\,;\,G_R))<\infty$. We should note also that, under the same assumption, the evaluation of the integrals in (2.8) has been established by Langlands [17], in a more general context.



**2-3.** Now let us assume that $\Gamma\backslash H_2$ is *not* compact. In this case (2.8) does not hold. In fact, it is known that the infinite sum of integrals extended over certain kinds of unipotent conjugacy classes do not converge ([1], [5], [18]). We shall see in §4 similar results for elliptic/parabolic, paraelliptic, and $\delta$-parabolic classes. We shall sketch briefly how one can overcome this difficulty, following [1], [5], [18]. Thus we assume that $G_Q$ has $Q$-rank one or two and $\Gamma$ is commensurable with $G_Z$. In the first case $G_Q$ has a parabolic $Q$-subgroup $P_0$, unique up to $G_Q$-conjugation, such that $(P_0)_R$ is isomorphic to the one given in (2.9) below. On the other hand, if $G_Q$ has $Q$-rank two, it has, up to $G_Q$-conjugation, three $Q$-parabolic subgroups $P_0$, $P_1$ and $P_2$:

$$\begin{aligned}P_0&=\left\{\begin{pmatrix}*&*&*&*\\ *&*&*&*\\ 0&0&*&*\\ 0&0&*&*\end{pmatrix}\in G_Q\right\},\\ P_1&=\left\{\begin{pmatrix}*&0&*&*\\ *&*&*&*\\ *&0&*&*\\ 0&0&0&*\end{pmatrix}\in G_Q\right\},\end{aligned} \tag{2.9}$$

$P_2=P_0\cap P_1$ : a Borel subgroup of $G_Q$.

In the following, all statements for $P_1$ should be omitted if $\Gamma$ belongs to the first case. Let $\{1=g_1, g_2, \cdots, g_r\}$ be a complete set of representatives of the double cosets in $P_2\backslash G_Q/\Gamma$. By a reduction theory of a connected semi-simple algebraic group over $Q$, there exists a Siegel domain $F$ such that $\bigcup_{i=1}^{r}g_i^{-1}\langle F\rangle$ is a fundamental set in $H_2$ for $\Gamma$ ([3]). In particular, it contains a fundamental domain of $\Gamma$, and has a finite volume. Note also that the set $\mathfrak{X}_i$ consisting of the real part $X_i$ of the point $Z_i=X_i+iY_i$ of $g_i^{-1}\langle F\rangle$ is compact for each $i$. Put, for each subset $S$ of $\Gamma$,

$$\theta_i(S, k\,;\,Y)=\int_{\mathfrak{X}_i}\sum_{\gamma\in S}\|H_\gamma(Z)\|dX\,, \tag{2.10}$$

where $Z=X+iY$ is a point of $g_i^{-1}\langle F\rangle$. Then the following estimates for $\theta_i(S, k\,;\,Y)$ are due to Christian [4].

THEOREM 2-2. (Christian) (i) *There exist positive half integers* $a_1$, $a_2$ *with* $a_1+a_2\leqq 3$, $a_2\leqq 2$, *such that*

$$\theta_i(S, k\,;\,Y)=O(y_1^{a_1}y_2^{a_2}),\quad Y=\begin{pmatrix}y_1&y_{12}\\ y_{12}&y_2\end{pmatrix}, \tag{2.11}$$

*holds for* $Y\in \mathrm{Im}(g_i^{-1}\langle F\rangle)$ *and* $k>4$.
(ii) *If* $S\cap\Gamma_{0,i}=\emptyset$ *(resp.* $S\cap\Gamma_{1,i}=\emptyset$*), then* $a_1$, $a_2$ *can be so chosen that* $a_1+a_2<3$ *(resp.* $a_2<2$*), where we put*

$$\Gamma_{0,i}=g_i^{-1}P_0g_i\cap\Gamma,\quad \Gamma_{1,i}=g_i^{-1}P_1g_i\cap\Gamma. \tag{2.12}$$

In fact, this is just a restatement of a special case $n=2$ of Satz 1 in [4], with $\Gamma$ instead of $\mathrm{Sp}(n, \boldsymbol{Z})$. The $Q$-rank one case is more or less easier. Note that the condition $Z=X+iY\in g_i^{-1}\langle F\rangle$ implies that $Y$ is reduced with respect to the isomorphic image of the reductive part $(\Gamma_{0,i})_M$ of $\Gamma_{0,i}$, under the isomorphism $(g_i^{-1}P_{0R}g_i)_M\cong GL_2(\boldsymbol{R})$. Therefore we have also a restatement of Satz 2 of [4]:

LEMMA 2-1. *For a positive number* $a_1$, $a_2$ *such that* $a_1+a_2<3$, $a_2<2$, *we have for each* $i$ $(1\leqq i\leqq r)$,

$$\int_{g_i^{-1}\langle F\rangle}y_1^{a_1}y_2^{a_2}dZ<\infty. \tag{2.13}$$

Let us now divide the set $\Gamma$ into disjoint union of $S_0$, $S_1$, $S_2$ and $S_3$; namely (i) $S_3$ consists of elements that are not conjugate to any one of $\Gamma_{0,i}$, $\Gamma_{1,i}$ $(1\leqq i\leqq r)$, (ii) $S_0$ consists of elements that are conjugate to an element of $\Gamma_{0,i}$, but are not conjugate to any one of $\Gamma_{1,i}$, (iii) $S_1$ consists of elements that are conjugate to an element of $\Gamma_{1,i}$, but are not conjugate to any one of $\Gamma_{2,i}=g_i^{-1}P_2g_i\cap\Gamma$, and (iv) $S_2$ consists of the remaining elements of $\Gamma$. Then it is easy to see by Lebesque's theorem (cf. Theorem 2-1, (i)), that

$$\int_{\Gamma\backslash H_2}\sum_{\gamma\in\Gamma}H_\gamma(Z)dZ=\sum_{j=0}^{3}I_j, \tag{2.14}$$

$$I_j=\lim_{s\to+0}\int_{\Gamma\backslash H_2}\sum_{\gamma\in S_j}H_\gamma(Z;s)dZ,$$

where $H_\gamma(Z;s)=H_\gamma(Z)\times$(a dumping factor in $s$), which will be explained below. For the elements of $S_3$, it follows from Theorem 2-2, (ii) and Lemma 2-1, that we need no dumping factors, so that we put simply $H_\gamma(Z;s)=H_\gamma(Z)$. And we have

$$I_3=\sum_{\gamma\in S_3}\int_{\Gamma\backslash H_2}H_\gamma(Z)dZ \tag{2.15}$$

$$=\sum_{\{\gamma\}_\Gamma\in S_3/\tilde{\Gamma}}\mathrm{vol}(C(\gamma;\Gamma)\backslash C(\gamma;G_R))\int_{C(\gamma;G_R)\backslash H_2}H_\gamma(\hat{Z})d\hat{Z}.$$

For the elements of $S_j$ $(j=0, 1)$, we write

$$\gamma=h^{-1}g_i^{-1}\gamma_ig_ih\quad (h\in\Gamma,\ \gamma_i\in\Gamma_{j,i}),$$

and put

$$H_\gamma(Z;s)=\begin{cases}H_\gamma(Z)(\det \mathrm{Im}\, g_ih\langle Z\rangle)^{-s} & \cdots j=0\\ H_\gamma(Z)(\det \mathrm{Im}\, g_ih\langle Z\rangle)^{-s}y_1^{*s} & \cdots j=1.\end{cases}$$



where

$$\mathrm{Im}\, g_ih\langle Z\rangle=\begin{pmatrix}y_1^* & y_{12}^*\\ y_{12}^* & y_2^*\end{pmatrix}.$$

Then we have, by using the formula (i) of Lemma 2-2 below,

$$I_j=\lim_{s\to+0}\sum_{\gamma\in S_j}\int_{\Gamma\backslash H_2}H_\gamma(Z;s)dZ \tag{2.16}$$

$$=\lim_{s\to+0}\sum_{\{\gamma\}_\Gamma\in S_j/\tilde{\Gamma}}\int_{C(\gamma;\Gamma)\backslash H_2}H_\gamma(Z;s)dZ.$$

For the elements of $S_2$, we note first that their eigenvalues are all $\pm1$, so that they are either (a) central, (b) elliptic of order two, (h) $d$-unipotent, (j) $\delta$-parabolic, or (k) parabolic. We put $H_\gamma(Z;s)=H_\gamma(Z)$ if $\gamma$ belongs to (a) or (b); $H_\gamma(Z;s)=H_\gamma(Z)\times(\det \mathrm{Im}\, g_ih\langle Z\rangle)^{-s}$ if $\gamma$ belongs to (h), (j), or it belongs to $(k)$ and it is not conjugate in $G_Q$ to $\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}$, $-\det S\in(Q^\times)^2$, where $g_i$, $h$ are as above. In this exceptional case in $(k)$, we put $H_\gamma(Z;s)=H_\gamma(Z)\times(\det \mathrm{Im}\, g_ih\langle Z\rangle)^{-s}y_1^{*s}$. Then we see from Lemma 7-3, 7-4, that these definitions of $H_\gamma(Z;s)$ do not depend on the choice of $g_i$, $h$. Moreover, we can easily prove the following

LEMMA 2-2. *We have*

(i) $H_{\delta^{-1}\gamma\delta}(Z;s)=H_\gamma(\delta\langle Z\rangle;s)$ *for any* $\delta\in\Gamma$.
(ii) $H_\gamma(g\langle Z\rangle;s)=H_\gamma(Z;s)$ *for any* $g\in C_0(\gamma;G_R)$.

**2-3.** We are now to have the following reformulation of Theorem 2-1, which asserts the termwise integrability, modulo dumping factors.

THEOREM 2-3. *For* $k\geqq5$, *we have*

$$\dim S_k(\Gamma)=\frac{a(k)}{\sharp Z(\Gamma)}\sum_{[\gamma]_\Gamma}\mathrm{vol}(C_0(\gamma;\Gamma)\backslash C_0(\gamma;G_R)) \tag{2.17}$$

$$\times\lim_{s\to+0}\sum_{\delta\in[\gamma]_\Gamma/\tilde{\Gamma}}\frac{I_0(\delta;s)}{[\bar{C}(\delta;\Gamma):\bar{C}_0(\delta;\Gamma)]},$$

$$I_0(\delta;s)=\int_{C_0(\delta;G_R)\backslash H_2}H_\delta(\hat{Z};s)d\hat{Z},$$

*where the first sum is extended over the set of* $\Gamma$*-conjugacy classes of the families* $[\gamma]_\Gamma$.

PROOF. The assertion follows from Theorem 1-6, Lemma 2-1, 2-2, (2.8), (2.14), (2.15) and (2.16), for the subsets $S_0$, $S_1$ and $S_3$. Here we note that in (2.17), the limit and the first sum have been exchanged by Theorem 2-2 and Lemma 2-1, applying Lebesgue's theorem. For the subset $S_2$, it will be proved in the course of calculations of the integrals in §3, §4 and §5. q.e.d.

REMARK 2-1. We may call $I_j$ the contribution from the set $S_j$. In fact, it can be proved by estimating the sum of $H_\gamma(Z)$ over each $S_j$, that the equality

$$I_j=\int_{\Gamma\backslash H_2}\sum_{\gamma\in S_j}H_\gamma(Z)dZ$$

holds.

REMARK 2-2. We shall see in § 3, that also for elliptic elements contained in $S_0\cup S_1$, the assertion for the termwise integrability holds without dumping factors. Here it should be noted that in (2.17), no data of $\Gamma$ appear in the last integral $I_0(\delta; s)$, so that they can be evaluated after normalizing the elements $\delta$ in the family $[\gamma]_\Gamma$ *simultaneously* by $G_R$-conjugation.

**2-5. Elliptic contributions**

For elliptic conjugacy classes, the right hand side of (2.17) can be given a more suitable expression for the explicit computation. Thus we assume that $\Gamma$ satisfies the condition (1.5). Then we have

THEOREM 2-4. *The elliptic contribution in (2.17) is equal to*

$$\dim S_k(\Gamma,\chi)|_e=a(k)\sum_{\{g\}\tilde{G}_Q}I_0(g)\sum_{L_{\tilde{G}}(\Lambda)}M_{\tilde{G}}(\Lambda)\prod_p c_p(g, R_p, \Lambda_p),\tag{2.18}$$

*where the notations are as follows:*

(1) *The first sum is extended over the conjugacy classes in $\tilde{G}_Q$ of the elements with finite orders, which are "locally integral" (cf. Theorem 1-3).*

(2) *$L_{\tilde{G}}(\Lambda)$ runs over the "$\tilde{G}$-genera" of $\boldsymbol{Z}$-orders in $Z(g)$: the $\tilde{G}$-genus $L_{\tilde{G}}(\Lambda)$ containing $\Lambda$ consists of all $\boldsymbol{Z}$-orders in $Z(g)$ which are conjugate in $Z(g)_p^\times\cap\tilde{G}_p$ with $\Lambda_p$ for all $p$.*

(3) *$M_{\tilde{G}}(\Lambda)$ is the "$\tilde{G}$-Maß" of $\Lambda$, which is defined as follows: Decompose the adelized group $(Z(g)^\times\cap\tilde{G})_A$ into disjoint union*

$$(Z(g)^\times\cap\tilde{G})_A=\coprod_{k=1}^{h}(Z(g)^\times\cap\tilde{G}_Q)y_k(\Lambda_A^\times\cap\tilde{G}_A),\quad \Lambda_A=\Lambda\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_A,$$

*and put $\Lambda_k=y_k\Lambda y_k^{-1}=\bigcap_p(y_{kp}\Lambda_p y_{kp}^{-1}\cap Z(g))$. Then we define*

$$M_{\tilde{G}}(\Lambda)=\mathrm{vol}(\Lambda_0^\times\cap C_0(g; G_R)\backslash C_0(g; G_R))\sum_{k=1}^{h}\frac{1}{[\Lambda_k^\times\cap\tilde{G}_Q:\Lambda_0^\times\cap C_0(g; G_Q)]},\tag{2.19}$$

*where $\Lambda_0$ is a fixed $\boldsymbol{Z}$-order of $Z(g)$.*

(4) $$c_p(g, R_p, \Lambda_p)=\sharp((Z(g)^\times\cap\tilde{G})_p\backslash M_p(g, R_p, \Lambda_p)/U_p),$$

*where $M_p(g, R_p, \Lambda_p)$ is defined in the same way as in (1.8).*

(5) $$I_0(g)=\int_{C_0(g; G_R)\backslash H_2}H_g(\hat{Z})d\hat{Z}.$$



PROOF. If we note that the family $[\gamma]_\Gamma$ reduces to a single element $\{\gamma\}$ if $\gamma$ is elliptic, the assertion follows from Lemma 1-1, 1-2, and Remark 2-2, along the same way as in [8]. q. e. d.

REMARK 2-3. If $R_p=M_2(O_p)$ (cf. § 1-3), the computation of the local factors $c_p(g, R_p, \Lambda_p)$ has been carried out in [10]. Also $M_{\tilde{G}}(\Lambda)$ can be evaluated by using the theory of Tamagawa numbers of semi-simple algebraic groups over $Q$. We shall use the above theorem and these facts in [9] to compute the dimension $S_k(\Gamma)$ explicitly for $\Gamma=U(2, O)$ in the case of $Q$-rank one.

**2-6.** Finally, we quote from [1], [5] and [18] the following result:

THEOREM I-0. *In the formula (2.17), the integral $I_0(\gamma; s)$ vanishes unless $\gamma$ belongs to either one of the types listed in Theorem 1-7.*

Therefore, by Theorem 1-7, the first sum in (2.17) is *finite*. In the following of this paper, we shall consider the integrals $I_0(\gamma; s)$ for the elements of types listed in Theorem 1-7. *Also, we shall restrict ourselves to the case where the representation $\chi$ is trivial.*

## § 3. Computation of $I_0(\gamma; s)$ for elliptic elements

**3-1.** Let us first consider the case where $\gamma$ has an isolated fixed point in $H_2$, or equivalently, $C(\gamma; G_R)$ is compact. We put according to the Definition 1-1, $C_0(\gamma; G_R)=\{1\}$. By Theorem 1-1, $\gamma$ is then conjugate in $G_R$ to an element of the form $\alpha(\mu, \nu)$, with $k(\mu)^2$, $k(\nu)^2$, $k(\mu)k(\nu)\neq 1_2$. The following result is contained, though not so explicitly, in Langlands [17] (see also Harish-Chandra [7]).

THEOREM I-1. *Assume that $\gamma$ has an isolated fixed point, and conjugate in $G_R$ to $\alpha(\mu, \nu)$. Put $\varepsilon_1=e^{i\mu}$, $\varepsilon_2=e^{i\nu}$. Then we have*

$$I_0(\gamma; s)=a(k)^{-1}\frac{(\varepsilon_1\varepsilon_2)^{-k}}{(1-\varepsilon_1^{-2})(1-\varepsilon_1^{-1}\varepsilon_2^{-1})(1-\varepsilon_2^{-2})}.\tag{3.1}$$

**3-2.** Now suppose that the set of fixed points of $\gamma$ has positive dimension. By Theorem 1-1, $\gamma$ is then conjugate in $G_R$ to one of the following elements:

$$\gamma(\mu)=\begin{pmatrix}\cos\mu & \sin\mu & & \\ -\sin\mu & \cos\mu & & \\ & & \cos\mu & \sin\mu \\ & & -\sin\mu & \cos\mu\end{pmatrix},\qquad \sin\mu\neq 0,$$

$$\beta(\mu)=\alpha(\mu, 0)=\pm\begin{pmatrix}\cos\mu & 0 & \sin\mu & 0\\ 0 & 1 & 0 & 0\\ -\sin\mu & 0 & \cos\mu & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\qquad \sin\mu\neq 0,$$

$$\delta=\alpha(0,\pi)=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}.$$

Put

$$C_0(\beta;G_R)=\left\{\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix};\ ad-bc=1\right\}. \tag{3.2}$$

LEMMA 3-1. (i) *As a fundamental domain of* $C_0(\beta;G_R)$ *in* $H_2$, *we can take the following set*

$$F_0(\beta)=\left\{\begin{pmatrix}z_1&t\\t&i\end{pmatrix}\in H_2;\ \operatorname{Im}z_1>0,\ t\geqq 0\right\}. \tag{3.3}$$

(ii) *No element* $g\neq 1$ *of* $C_0(\beta;G_R)$ *has a fixed point in the interior* $F_0(\beta)^0$ *of* $F_0(\beta)$, *and the map* $(g,Z)\mapsto g\langle Z\rangle$ *from* $C_0(\beta;G_R)\times F_0(\beta)$ *to* $H_2$ *induces a diffeomorphism from* $C_0(\beta;G_R)\times F_0(\beta)^0$ *to* $H_2-\left\{\begin{pmatrix}z_1&0\\0&z_2\end{pmatrix}\right\}$.

PROOF. We have, for $g=\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix}\in C_0(\beta;G_R)$ and $Z=\begin{pmatrix}z_1&z_{12}\\z_{12}&z_2\end{pmatrix}\in H_2$,

$$g\langle Z\rangle=\begin{pmatrix}z_1-\dfrac{cz_{12}^2}{cz_2+d}&\dfrac{z_{12}}{cz_2+d}\\[2mm]\dfrac{z_{12}}{cz_2+d}&\dfrac{az_2+b}{cz_2+d}\end{pmatrix}. \tag{3.4}$$

Therefore we can send any point $Z\in H_2$ to a point of the form $\begin{pmatrix}z_1^*&z_{12}^*\\z_{12}^*&i\end{pmatrix}$, by putting $\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}\sqrt{y_2}&x_2\sqrt{y_2}^{-1}\\0&\sqrt{y_2}^{-1}\end{pmatrix}^{-1}$ $(z_2=x_2+iy_2)$. Then by putting $\begin{pmatrix}a&b\\c&d\end{pmatrix}=k(\theta)$ in $g$ and applying again (3.4) to $\begin{pmatrix}z_1^*&z_{12}^*\\z_{12}^*&i\end{pmatrix}$, we can send this point to $F_0(\beta)$. Now it is clear that no two points in $F_0(\beta)$ are transposed by $C_0(\beta;G_R)$. This proves (i), and (ii) follows easily from these facts. q. e. d.

Next consider the $\delta$ above and put

$$C_0(\delta;G_R)=C(\delta;G_R),$$

$$C_1(\delta;G_R)=\left\{\begin{pmatrix}a_1&0&b_1&0\\0&a_2&0&b_2\\0&0&a_1^{-1}&0\\0&c_2&0&d_2\end{pmatrix}\in G_R;\ a_1>0\right\}. \tag{3.5}$$



LEMMA 3-2. (i) *As a fundamental domain of* $C_1(\delta;G_R)$ *in* $H_2$, *we can take the following set*

$$F_0(\delta)=\left\{\begin{pmatrix}i&t\\t&i\end{pmatrix}\in H_2;\ t\geqq 0\right\}. \tag{3.6}$$

(ii) *No element* $g\neq 1$ *of* $C_1(\delta;G_R)$ *has a fixed point in the interior* $F_0(\delta)^0$ *of* $F_0(\delta)$, *and the map* $(g,\hat{Z})\mapsto g\langle\hat{Z}\rangle$ *from* $C_1(\delta;G_R)\times F_0(\delta)$ *to* $H_2$ *induces a diffeomorphism from* $C_1(\delta;G_R)\times F_0(\delta)^0$ *to* $H_2-\left\{\begin{pmatrix}z_1&0\\0&z_2\end{pmatrix}\right\}$.

(iii) $F_0(\delta)$ *is also a fundamental domain of* $C_0(\delta;G_R)$. *The stabilizor of* $\begin{pmatrix}i&t\\t&i\end{pmatrix}$ *in* $C_0(\delta;G_R)$ *is as follows*:

$$\begin{cases}\left\{\begin{pmatrix}\cos\theta&0&(1+t^2)^{1/2}\sin\theta&0\\0&\cos\theta&0&-(1+t^2)^{1/2}\sin\theta\\-(1+t^2)^{-1/2}\sin\theta&0&\cos\theta&0\\0&(1+t^2)^{-1/2}\sin\theta&0&\cos\theta\end{pmatrix}\right\}\cdots \textit{if } t>0,\\[4mm]\left\{\begin{pmatrix}\cos\theta_1&0&\sin\theta_1&0\\0&\cos\theta_2&0&\sin\theta_2\\-\sin\theta_1&0&\cos\theta_1&0\\0&-\sin\theta_2&0&\cos\theta_2\end{pmatrix}\right\}\cdots\cdots\cdots\cdots \textit{if } t=0.\end{cases} \tag{3.7}$$

PROOF. Since $C_0(\beta;G_R)\subset C_1(\delta;G_R)$, any point of $H_2$ can be transformed to a point $Z=\begin{pmatrix}z_1&t\\t&i\end{pmatrix}\in F_0(\beta)$. Writing $z_1=x_1+iy_1$, we put, in the expression (3.5) of $g\in C_1(\delta;G_R)$,

$$\begin{pmatrix}a_1&b_1\\0&a_1^{-1}\end{pmatrix}=\begin{pmatrix}\sqrt{y_1}&x_1\sqrt{y_1}^{-1}\\0&\sqrt{y_1}^{-1}\end{pmatrix}^{-1},\ \begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Then we have

$$g\langle Z\rangle=\begin{pmatrix}i&t\sqrt{y_1}^{-1}\\t\sqrt{y_1}^{-1}&i\end{pmatrix}\in F_0(\delta).$$

It is easy to see that no two points of $F_0(\delta)$ are transposed by $C_0(\delta;G_R)$, which proves (i) and the first assertion of (iii). (ii) is an easy consequence of (i). Finally, the stabilizor of $\begin{pmatrix}i&t\\t&i\end{pmatrix}$ in $C_0(\delta;G_R)$ is equal to

$$\begin{pmatrix}1&0&0&t\\0&1&t&0\\0&0&1&0\\0&0&0&1\end{pmatrix}K\begin{pmatrix}1&0&0&t\\0&1&t&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}\cap C_0(\delta;G_R).$$

A direct calculation shows that this group is as asserted in (3.7). q. e. d.

Finally consider $\gamma=\gamma(\theta)$. Put

$$
(3.8)\qquad
\begin{aligned}
C_0(\gamma\,;\,G_{\boldsymbol R})&=\left\{\begin{pmatrix} a & 0 & b & 0\\ 0 & a & 0 & b\\ c & 0 & d & 0\\ 0 & c & 0 & d\end{pmatrix};\ ad-bc=1\right\},\\
C_1(\gamma\,;\,G_{\boldsymbol R})&=\left\{\begin{pmatrix} a & 0 & b & 0\\ 0 & a & 0 & b\\ 0 & 0 & a^{-1} & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix};\ a>0\right\}.
\end{aligned}
$$

LEMMA 3-3. (i) *As a fundamental domain of* $C_1(\gamma\,;\,G_{\boldsymbol R})$ *in* $H_2$, *we can take the set*

$$
F_1(\gamma)=\coprod^*_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\boldsymbol R)}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & 1 & 0\\ 0 & c & 0 & d\end{pmatrix}\langle F_0(\delta)\rangle\,,
$$

*where* $\coprod^*$ *means the disjointness except for the set of measure* 0, *corresponding to the orbit of* $\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}\in F_0(\delta)$ (*i.e.*, $t=0$).

(ii) *A fundamental domain of* $C_0(\gamma\,;\,G_{\boldsymbol R})$ *in* $H_2$ *is given by*

$$
F_0(\gamma)=\coprod_{\substack{b\in\boldsymbol R\\ a>0}}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & a^{-1}\end{pmatrix}\langle F_0(\delta)\rangle\,,
$$

*and the map* $(g,\,Z)\mapsto g\langle Z\rangle$ *from* $C_0(\gamma\,;\,G_{\boldsymbol R})\times F_0(\gamma)$ *to* $H_2$ *induces a diffeomorphism from* $C_0(\gamma\,;\,G_{\boldsymbol R})/\{\pm 1\}\times F_0(\gamma)^0$ *to* $H_2-\left\{\begin{pmatrix} z_1 & 0\\ 0 & z_2\end{pmatrix}\right\}$.

PROOF. (i) is a direct consequence of Lemma 3-2. Let us consider the condition in which the two points

$$
\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a_1 & 0 & b_1\\ 0 & 0 & 1 & 0\\ 0 & c_1 & 0 & d_1\end{pmatrix}\left\langle\begin{pmatrix} i & t_1\\ t_1 & i\end{pmatrix}\right\rangle,\quad
\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a_2 & 0 & b_2\\ 0 & 0 & 1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix}\left\langle\begin{pmatrix} i & t_2\\ t_2 & i\end{pmatrix}\right\rangle
$$

are in the same orbit of $C_0(\gamma\,;\,G_{\boldsymbol R})$. By Lemma 3-2, (iii), we have then $t_1=t_2$. Suppose $t_1=t_2=t>0$. Again by Lemma 3-2, (iii), we see that the condition is equivalent to the equality

$$
\begin{aligned}
&\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}^{-1}\begin{pmatrix}\cos\theta & (1+t^2)^{1/2}\sin\theta\\ -(1+t^2)^{-1/2}\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}\\
&=\begin{pmatrix}\cos\theta & (1+t^2)^{1/2}\sin\theta\\ -(1+t^2)^{-1/2}\sin\theta & \cos\theta\end{pmatrix}^{-1},\qquad(\text{for some }\theta\in\boldsymbol R).
\end{aligned}
$$



If we write

$$
A_i=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}(1+t^2)^{-1/4} & 0\\ 0 & (1+t^2)^{1/4}\end{pmatrix}\begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}\begin{pmatrix}(1+t^2)^{1/4} & 0\\ 0 & (1+t^2)^{-1/4}\end{pmatrix},
$$

($i=1,\,2$), then it is equivalent to saying that $A_1$ and $A_2$ are $SO(2)$-conjugate. So the problem is reduced to classify the set $\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}SL_2(\boldsymbol R)$ by $SO(2)$-conjugation. Since each element of this set has real eigenvalues $s$, $-s^{-1}$ ($s>0$), we can write it in the form $X^{-1}\begin{pmatrix} s & 0\\ 0 & -s^{-1}\end{pmatrix}X$, $X\in SL_2(\boldsymbol R)$. For each fixed $s>0$, we have a bijection

$$
\left\{X^{-1}\begin{pmatrix} s & 0\\ 0 & -s^{-1}\end{pmatrix}X;\ X\in SL_2(\boldsymbol R)\right\}\Big/\underset{SO(2)}{\sim}\xrightarrow{\ \sim\ }D\backslash SL_2(\boldsymbol R)/SO(2)\,,
$$

$$
X^{-1}\begin{pmatrix} s & 0\\ 0 & -s^{-1}\end{pmatrix}X\longmapsto DXSO(2)
$$

where $D=\left\{\begin{pmatrix} v & 0\\ 0 & v^{-1}\end{pmatrix};\ v>0\right\}$. Our assertion follows from these facts. q. e. d.

**3-3.** Suppose $g$ is an element of $C_0(\delta\,;\,G_{\boldsymbol R})=C(\delta\,;\,G_{\boldsymbol R})$ of the form in (b-5), Theorem 1-1. For $Z=\begin{pmatrix} z_1 & z_{12}\\ z_{12} & z_2\end{pmatrix}\in H_2$, one has

$$
(3.9)\qquad
\begin{aligned}
g\langle Z\rangle=&\frac{1}{(c_1z_1+d_1)(c_2z_2+d_2)-c_1c_2z_{12}^2}\\
&\times\begin{pmatrix}(a_1z_1+b_1)(c_2z_2+d_2)-a_1c_2z_{12}^2 & z_{12}\\ z_{12} & (c_1z_1+d_1)(a_2z_2+b_2)-c_1a_2z_{12}^2\end{pmatrix}.
\end{aligned}
$$

If, in particular, $c_1=0$ and $Z=\begin{pmatrix} i & t\\ t & i\end{pmatrix}\in F_0(\delta)$, one has

$$
(3.10)\qquad
\begin{aligned}
&g\langle Z\rangle=X+iY\,,\\
&X=\frac{a_1}{c_2^2+d_2^2}\begin{pmatrix}-a_1c_2d_2t^2+b_1(c_2^2+d_2^2) & td_2\\ td_2 & a_1^{-1}(a_2c_2+b_2d_2)\end{pmatrix}\\
&Y=\frac{a_1}{c_2^2+d_2^2}\begin{pmatrix} a_1(d_2^2+c_2^2+t^2c_2^2) & -tc_2\\ -tc_2 & a_1^{-1}\end{pmatrix}.
\end{aligned}
$$

If we write, by the Iwasawa decomposition of $SL_2(\boldsymbol R)$,

$$
(3.11)\qquad\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}=\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix} v & 0\\ 0 & v^{-1}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\qquad(u,\,\theta\in\boldsymbol R,\ v>0),
$$

then (3.10) is expressed as

$$
(3.12)\qquad
\begin{aligned}
X&=\begin{pmatrix} a_1^2t^2\sin\theta\cos\theta+a_1b_1 & a_1tv\cos\theta\\ a_1tv\cos\theta & u\end{pmatrix}\\
Y&=\begin{pmatrix} a_1^2(1+t^2\sin^2\theta) & a_1tv\sin\theta\\ a_1tv\sin\theta & v^2\end{pmatrix}.
\end{aligned}
$$

A direct calculation shows that one has

$$\frac{\partial(x_1, x_{12}, x_2, y_1, y_{12}, y_2)}{\partial(a_1, b_1, u, v, \theta, t)}=4a_1^4v^3t,$$

from which it follows that, for $g\in C_1(\delta; G_R)$,

(3.13) $$dZ=(\det Y)^{-3}dXdY=tdt(2a_1^{-2}da_1db_1)(2v^{-3}dudvd\theta),$$

where $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ is as in (3.11). Note that $2a_1^{-2}da_1db_1$ (resp. $2v^{-3}dudvd\theta$) is a left invariant measure of $\left\{\begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix}\right\}$ (resp. $SL_2(\boldsymbol{R})$).

PROPOSITION 3-1. *For any measurable function $f(Z)$ on $H_2$, we have the following integral formula:*

(3.14) $$\int_{H_2} f(Z)dZ=\frac{1}{2\pi}\int_0^\infty\int_{C_0(\delta; G_R)} t(1+t^2)^{1/2}f\left(g\left\langle\begin{pmatrix} i & t \\ t & i \end{pmatrix}\right\rangle\right)dgdt,$$

*where $dg$ is an invariant measure on $C_0(\delta; G_R)\cong SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ defined by $dg=d\alpha_1d\alpha_2$, $d\alpha_i=2v_i^{-3}du_idv_id\theta_i$ $(i=1, 2)$.*

PROOF. By (3.13), we have

(3.15) $$\int_{H_2} f(Z)dZ=\int_0^\infty\int_{C_1(\delta; G_R)} tf\left(g\left\langle\begin{pmatrix} i & t \\ t & i \end{pmatrix}\right\rangle\right)dgdt \qquad (g\in C_1(\delta; G_R)).$$

For $t>0$, denote by $g_t(\theta)$ the element of $C_0(\delta; G_R)$ given in (3.7). Then by Lemma 3-2, (iii), the integral above is equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty\int_{C_1(\delta; G_R)} tf\left(gg_t(\theta)\left\langle\begin{pmatrix} i & t \\ t & i \end{pmatrix}\right\rangle\right)d\theta dgdt.$$

For each $t$, one has a disjoint decomposition

$$C_0(\delta; G_R)=\coprod_{0\leq\theta<2\pi} C_1(\delta; G_R)g_t(\theta),$$

so that each element $h$ of $C_0(\delta; G_R)$ is expressed as $h=gg_t(\theta)$, $g\in C_1(\delta; G_R)$ uniquely (if $t>0$). Identifying $C_0(\delta; G_R)$ with $SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$, we write $h=gg_t(\theta)=(\alpha_1^*k_t(\theta), \alpha_2k_t(\theta)^{-1})$, where

$$k_t(\theta)=\begin{pmatrix} \cos\theta & (1+t^2)^{1/2}\sin\theta \\ -(1+t^2)^{-1/2}\sin\theta & \cos\theta \end{pmatrix}.$$

Then the Haar measure of $C_0(\delta; G_R)\cong SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ is expressed as

$$\begin{aligned} dh&=d(\alpha_1^*k_t(\theta))d(\alpha_2k_t(\theta)^{-1}) \\ &=d(\alpha_1^*k_t(\theta))d\alpha_2 \\ &=(1+t^2)^{-1/2}d\alpha_1^*d\theta d\alpha_2, \end{aligned}$$

where $d\alpha_1^*=2a^{-2}dadb$ for $\alpha_1^*=\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.



The assertion follows from this. q.e.d.

3-4. For $\hat{Z}=\begin{pmatrix} i & t \\ t & i \end{pmatrix}\in F_0(\delta)$, we have

$$H_\delta(\hat{Z}; s)=H_\delta(\hat{Z})=(-1)^k(1+t^2)^{-k}.$$

Therefore by (2.17), and Lemma 3-2, we have

$$\begin{aligned} I_0(\delta; s)&=\frac{1}{2\pi}\int_0^\infty\frac{(-1)^kt}{(1+t^2)^{k-1/2}}dt \\ &=\frac{(-1)^k}{2\pi}\cdot\frac{\Gamma(k-3/2)}{2\Gamma(k-1/2)}. \end{aligned}$$

Let $\gamma$ be an element of $\Gamma$, and assume that it is conjugate to $\delta$ in $G_R$. We have $\gamma=g^{-1}\delta g$ for some $g\in G_R$, and we can put $C_0(\gamma; G_R)=g^{-1}C_0(\delta; G_R)g$. Then it is easy to see that $I_0(\gamma; s)=I_0(\delta; s)$. Moreover, by the same way as above, we have

$$\int_{C_0(\gamma; G_R)\backslash H_2}|H_\gamma(\hat{Z}; s)|d\hat{Z}=\frac{1}{2\pi}\int_0^\infty\frac{tdt}{(1+t^2)^{k-1/2}}<+\infty.$$

Therefore, by Lebesgue's theorem we have

$$\int_{\Gamma\backslash H_2}\sum_{\gamma'\in\Gamma}H_{\gamma'}(Z; s)dZ=\int_{C(\gamma; \Gamma)\backslash H_2}H_\gamma(Z; s)dZ,$$

which proves the assertion in Theorem 2-3 on the termwise integrability in (2.17), for the subset $\{\gamma\}_\Gamma$ of $\Gamma$. We summarize the result as

THEOREM I-2. *Let $\gamma\in G_R$ be conjugate to $\delta$. Then we have*

(3.16) $$I_0(\gamma; s)=\frac{(-1)^k}{2\pi(2k-3)}=a(k)^{-1}\frac{(-1)^k(2k-2)(2k-4)}{2^9\pi^4}.$$

3-5. Let us next consider $\beta=\beta(\theta)$. By Lemma 3-1, every point $Z\in H_2$ is expressed (uniquely for $t>0$) as $Z=g\left\langle\begin{pmatrix} x_1+iy_1 & t \\ t & i \end{pmatrix}\right\rangle$, where $g$ is an element of $C_0(\beta; G_R)$ as in (3.2). If one writes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the expression of $g$ as in (3.11), with $\varphi$ instead of $\theta$, then one has $Z=X+iY$,

$$X=\begin{pmatrix} x_1+t^2\sin\varphi\cos\varphi & vt\cos\varphi \\ vt\cos\varphi & u \end{pmatrix}$$

$$Y=\begin{pmatrix} y_1+t^2\sin^2\varphi & vt\sin\varphi \\ vt\sin\varphi & v^2 \end{pmatrix}.$$

Then we have

$$\frac{\partial(x_1, x_{12}, x_2, y_1, y_{12}, y_2)}{\partial(x_1, t, u, y_1, v, \varphi)}=2v^3t,$$

so that the invariant measure $dZ$ is expressed as

$$dZ=(tdt)(y_1^{-3}dx_1dy_1)d\alpha,$$

where $d\alpha=2v^{-3}dudvd\varphi$ is the Haar measure of $SL_2(\boldsymbol{R})$. Thus we have proved

PROPOSITION 3-2. *For any measurable function $f(Z)$ on $H_2$, we have the following integral formula*

$$\int_{H_2} f(Z)dZ=\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{SL_2(R)} ty^{-3} f\left(g_\alpha\left\langle\begin{pmatrix} x+iy & t \\ t & i\end{pmatrix}\right\rangle\right)d\alpha dydxdt,$$

*where $g_\alpha$ is an element of $C_0(\beta\,;G_R)$ with $\alpha=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$.*

Following lemma is proved by a direct computation.

LEMMA 3-4. *We have, for $\hat{Z}=\begin{pmatrix} x+iy & t \\ t & i\end{pmatrix}\in F_0(\beta)$ $(\beta=\beta(\theta))$,*

$$H_\beta(\hat{Z})=(-2iy)^k[(1+x^2+y^2+t^2y)\sin\theta+\{(1-\cos\theta)t^2-2y\cos\theta\}i]^{-k}.$$

By this lemma and Proposition 3-2, we see that there exists a constant $c>0$, which depends only on $\theta$, such that

$$\int_{C_0(\beta;G_R)\backslash H_2}|H_\beta(\hat{Z}\,;s)|d\hat{Z}$$

$$\leqq c\int_{-\infty}^\infty\int_0^\infty\int_0^\infty\frac{ty^{k-3}}{(1+x^2+y^2+t^2y)^k}dtdydx<+\infty.$$

This proves the assertion of Theorem 2-3 for each subset $\{\gamma\}_\Gamma$, where $\gamma$ is conjugate in $G_R$ to some $\beta(\theta)$.

To evaluate the integral $I_0(\beta\,;s)$, we need the following formula which is proved by an elementary residue calculus:

LEMMA 3-5. *Let $b,c\in\boldsymbol{C}$ be constants with $c\neq0$, such that $(x+b)^2+c=0$ has no real roots, and let $k$ be a half integer with $k\geqq1$. Then one has*

$$\int_{-\infty}^\infty\frac{dx}{\{(x+b)^2+c\}^k}=\frac{\Gamma(k-1/2)\Gamma(1/2)}{c^{k-1/2}\Gamma(k)},\tag{3.17}$$

*for a suitable choice of the branch of half-integral powers (e.g., if $k\in\boldsymbol{Z}$ and* $\mathrm{Re}(c)>0$, *then one takes the branch* $\mathrm{Re}(c^{1/2})>0$).

Using this lemma, we have

$$I_0(\beta\,;s)=\int_{F_0(\beta)}H_\beta(\hat{Z})d\hat{Z}$$

$$=\frac{(-2i)^k}{(\sin\theta)^k}\int_{-\infty}^\infty\int_0^\infty\int_0^\infty\frac{ty^{k-3}dtdydx}{\left[x^2+y^2+1+t^2y+\dfrac{1}{\sin\theta}\{(1-\cos\theta)t^2-2y\cos\theta\}i\right]^k}$$



$$=\frac{(-2i)^k}{(\sin\theta)^k}\cdot\frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)}$$

$$\times\int_0^\infty\int_0^\infty\frac{ty^{k-3}dtdy}{\left[y^2+t^2y+1+\dfrac{1}{\sin\theta}\{(1-\cos\theta)t^2-2y\cos\theta\}i\right]^{k-1/2}}.$$

Noting the equality

$$y^2-2iy\cot\theta+1=\left(y+\frac{1-\cos\theta}{\sin\theta}i\right)\left(y-\frac{1+\cos\theta}{\sin\theta}i\right),$$

we see that the double integral above is equal to

$$\frac{1}{2}\int_0^\infty\frac{y^{k-3}}{\left(y+\dfrac{1-\cos\theta}{\sin\theta}i\right)^{k-1/2}}\left\{\int_0^\infty\frac{dx}{\left(x+y-\dfrac{1+\cos\theta}{\sin\theta}i\right)^{k-1/2}}\right\}dy$$

$$=\frac{1}{2k-3}\int_0^\infty\frac{y^{k-3}dy}{\left(y+\dfrac{1-\cos\theta}{\sin\theta}i\right)^{k-1/2}\left(y-\dfrac{1+\cos\theta}{\sin\theta}i\right)^{k-3/2}}$$

$$=\frac{(\sin\theta)^k}{2k-3}\int_0^\infty\left\{\frac{x^{k-2}dx}{(x^2-2ix\cos\theta+\sin^2\theta)^{k-1/2}}-\frac{(1+\cos\theta)ix^{k-3}dx}{(x^2-2ix\cos\theta+\sin^2\theta)^{k-1/2}}\right\}.$$

**3-6.** Suppose that $k$ is an integer with $k\geqq2$, and consider the integrals appeared above. Put, for each integer $p$ such that $0\leqq p\leqq 2k-3$,

$$B_k(p\,;\theta)=\int_0^\infty\frac{x^p dx}{(x^2-2ix\cos\theta+\sin^2\theta)^{k-1/2}}\qquad(\sin\theta\neq0).\tag{3.18}$$

It is easy to show the convergence of these integrals under the above conditions. Changing the variable $x$ by $x^{-1}\sin^2\theta$, we see that they have the following symmetry:

$$B_k(k-2+p\,;\theta)=(\sin\theta)^{2p-1}B_k(k-1-p\,;\theta)\qquad(0\leqq p\leqq k-1).\tag{3.19}$$

LEMMA 3-6. *$B_k(p\,;\theta)$ satisfy the following recurrence formulae:*

$$B_k(p\,;\theta)=i\cos\theta\, B_k(p-1\,;\theta)+\frac{p-1}{2k-3}B_{k-1}(p-2\,;\theta),\tag{3.20}$$

*for $k\geqq3$, $2\leqq p\leqq 2k-3$,*

$$(2k-p-2)B_k(p\,;\theta)\tag{3.21}$$

$$=i(2k-2p-1)\cos\theta\, B_k(p-1\,;\theta)+(p-1)\sin^2\theta\, B_k(p-2\,;\theta),$$

*for $k\geqq2$, $2\leqq p\leqq 2k-3$.*

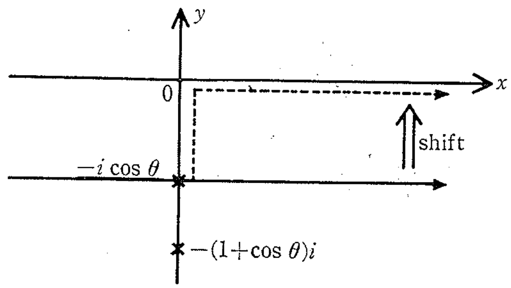
PROOF. The first formula is derived from (3.18) by integrating partially. We have, by definition

$$B_k(p\,;\theta)=\int_0^\infty\frac{x^{p-2}\{(x^2-2ix\cos\theta+\sin^2\theta)+2ix\cos\theta-\sin^2\theta\}}{(x^2-2ix\cos\theta+\sin^2\theta)^{k-1/2}}dx$$

$$=B_{k-1}(p-2;\theta)+2i\cos\theta\, B_k(p-1;\theta)-\sin^2\theta\, B_k(p-2;\theta).$$

The second formula follows from this and (3.20). q.e.d.

For small values of $k$, we can evaluate $B_k(p;\theta)$ by shifting the path of integration as follows:




(Fig. 1)

Then we have

$$B_k(p;\theta)=\int_0^\infty\frac{(t+i\cos\theta)^p}{(t^2+1)^{k-1/2}}dt-\int_0^{\cos\theta}\frac{(-i)^{p+1}(t-\cos\theta)^p}{(1-t^2)^{k-1/2}}dt,$$

(3.22)
$$\begin{aligned}
&B_2(0;\theta)=(\sin\theta+i\cos\theta)/\sin\theta\\
&B_2(1;\theta)=(\sin\theta+i\cos\theta)\\
&B_3(0;\theta)=\{2\sin^3\theta+i\cos\theta(1+2\sin^2\theta)\}/3\sin^3\theta\\
&B_3(1;\theta)=(-\cos 2\theta+i\sin 2\theta)/3\sin\theta\\
&B_3(2;\theta)=(-\cos 2\theta+i\sin 2\theta)/3\\
&B_3(3;\theta)=\{2\sin^3\theta+i\cos\theta(1+2\sin^2\theta)\}/3.
\end{aligned}$$

Now it is easy to prove the following formula by induction:

PROPOSITION 3-3. *For $k\geqq 2$, we have*

(3.23)
$$B_k(k-1;\theta)=\frac{\Gamma(1/2)\Gamma(k-1)}{\Gamma(k-1/2)}\cdot\frac{e^{-(k-1)\theta i}}{(-2i)^{k-1}}.$$

Applying it to the calculation in the preceding paragraph, we get the following

THEOREM I-3. *Suppose $\gamma\in G_R$ is conjugate to $\beta(\theta)$ with $\sin\theta\neq 0$, and $\gamma=g^{-1}\beta(\theta)g$, $g\in G_R$. Then we can put $C_0(\gamma;G_R)=g^{-1}C_0(\beta;G_R)g$, and we have*



(3.24)
$$\begin{aligned}
&I_0(\gamma;s)=\frac{1}{a(k)}\cdot\frac{i}{2^5\pi^2\sin\theta(1-\cos\theta)}\{(k-1)e^{-(k-2)\theta i}-(k-2)e^{-(k-1)\theta i}\},\\
&I_0(\gamma;s)+I_0(\gamma^{-1};s)=\frac{1}{a(k)}\cdot\frac{1}{2^4\pi^2\sin\theta(1-\cos\theta)}\\
&\qquad\times\{(k-1)\sin(k-2)\theta-(k-2)\sin(k-1)\theta\}.
\end{aligned}$$

**3-7.** Now let us consider $\gamma=\gamma(\theta)$. Let $C_0(\gamma;G_R)$ be as in (3.8). We have

$$C_0(\delta;G_R)=C_0(\gamma;G_R)\ltimes C_0(\beta;G_R)\quad\text{(semi-direct product).}$$

It follows from Proposition 3-1 and Lemma 3-3 that the integral formula

(3.25)
$$\int_{H_2}f(Z)dZ=\frac{1}{2\pi}\int_{C_0(\gamma;G_R)}\int_{F_1(\gamma)}f(g\langle\hat{Z}\rangle)d\hat{Z}dg$$

holds for any measurable function $f(Z)$ on $H_2$, where $dg$ is the invariant measure of $C_0(\gamma;G_R)\cong SL_2(\boldsymbol{R})$ as in (3.13), and

(3.26)
$$d\hat{Z}=t(1+t^2)^{1/2}dt(2v^{-3}du\,dv\,d\varphi),\quad\text{for}$$

$$\begin{aligned}
\hat{Z}&=g_\alpha\left\langle\begin{pmatrix}i&t\\t&i\end{pmatrix}\right\rangle\\
&=\begin{pmatrix}t^2\sin\varphi\cos\varphi+(1+t^2\sin^2\varphi)i & vt(\cos\varphi+i\sin\varphi)\\ vt(\cos\varphi+i\sin\varphi) & u+iv^2\end{pmatrix},
\end{aligned}$$

($g_\alpha$ is as in Proposition 3-2).

The following lemma is obtained by a direct computation.

LEMMA 3-7. *Suppose $\hat{Z}\in F_1(\gamma)$ is as in (3.26). We have*

$$\begin{aligned}
H_\gamma(\hat{Z};s)=(2v)^{2k}[&\{(u-t^2\sin\varphi\cos\varphi)\sin\theta-2ivt\sin\varphi\cos\theta\}^2\\
&+\{(1+t^2\sin^2\varphi-v^2)\sin\theta-2ivt\cos\varphi\cos\theta\}^2+4v^2(1+t^2)]^{-k}.
\end{aligned}$$

By this lemma we have

$$|H_\gamma(\hat{Z};s)|\leqq\left(\frac{2v}{\sin\theta}\right)^{2k}[(u-t^2\sin\varphi\cos\varphi)^2+(1+t^2\sin^2\varphi-v^2)^2+4v^2(1+t^2)]^{-k},$$

and it follows that

$$\begin{aligned}
&\int_{F_1(\gamma)}|H_\gamma(\hat{Z};s)|d\hat{Z}\\
&\leqq c_1\int_{-\infty}^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}\frac{t(1+t^2)^{1/2}v^{2k-3}d\varphi\,dv\,dt\,du}{[u^2+(1+t^2\sin^2\varphi-v^2)^2+4v^2(1+t^2)]^k}\\
&\leqq c_2\int_0^\infty\int_0^\infty\int_0^{2\pi}\frac{(1+s)^{1/2}x^{k-2}d\varphi\,dx\,ds}{[\{x-(1+s\sin^2\varphi)\}^2+4x(1+s)]^{k-1/2}}\\
&\leqq c_3\int_0^\infty\int_0^\infty\int_0^{2\pi}\frac{(1+s)^{1/2}x^{k-2}d\varphi\,dx\,ds}{[(x+1)^2+2sx]^{k-1/2}}
\end{aligned}$$

$$\leqq c_4\int_0^{\infty}\int_0^{\infty}\frac{s^{1/2}x^{-3/2}dxds}{\left(s+\dfrac{x^2+1}{2x}\right)^{k-1/2}}$$

$$\leqq c_5\int_0^{\infty}\frac{x^{k-7/2}}{(x^2+1)^{k-2}}dx<+\infty,$$

where $c_i$ are positive constants which depend only on $k$. Therefore we have

$$\int_{C_0(\gamma;G_R)\backslash H_2}|H_\gamma(\hat{Z};s)|d\hat{Z}<+\infty,$$

so that the assertion of Theorem 2-3 for each subset $\{\gamma\}_\Gamma$ is proved, if $\gamma$ is conjugate in $G_R$ to some $\gamma(\theta)$.

Now by comparing (3.25) with (2.17), we have

$$I_0(\gamma(\theta);s)=\frac{1}{2\pi}\int_{F_1(\gamma)}H_\gamma(\hat{Z};s)d\hat{Z}$$

$$=\frac{2^{2k-1}}{\pi(\sin\theta)^{2k}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{2t(1+t^2)^{1/2}v^{2k-3}d\varphi dtdvdu}{\left[\begin{array}{l}\{(u-t^2\sin\varphi\cos\varphi)-2ivt\sin\varphi\cot\theta\}^2\\ +\dfrac{4v^2}{\sin^2\theta}(1+t^2)+\{(1+t^2\sin^2\varphi-v^2)-2ivt\cos\varphi\cot\theta\}^2\end{array}\right]^k}.$$

Here, putting $x=u-t^2\sin\varphi\cos\varphi-2ivt\sin\varphi\cot\theta$, we can shift the path of integration with respect to $x$ to the real line. Then we can apply Lemma 3-5:

$$I_0(\gamma(\theta);s)=\frac{2^{2k-1}}{\pi(\sin\theta)^{2k}}\cdot\frac{\Gamma(1/2)\Gamma(k-1/2)}{\Gamma(k)}$$

$$\times\int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{2t(1+t^2)^{1/2}v^{2k-3}d\varphi dtdv}{\left[\{(1+t^2\sin^2\varphi-v^2)-2ivt\cos\varphi\cot\theta\}^2+\dfrac{4v^2}{\sin^2\theta}(1+t^2)\right]^{k-1/2}}.$$

Putting $x=t\cos\varphi$, $y=t\sin\varphi$, we see that the last integral is equal to

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2(1+x^2+y^2)^{1/2}v^{2k-3}dydxdv}{\left[(1+y^2-v^2-2ivx\cot\theta)^2+\dfrac{4v^2}{\sin^2\theta}(1+y^2+x^2)\right]^{k-1/2}}.$$

Again by putting $y=\tan\psi$ $(0\leqq\psi\leqq\pi/2)$, $x_1=x\cos\psi$, $v_1=v\cos\psi$, we see that it is equal to

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi/2}\frac{4(\cos\psi)^{2k-4}v_1^{2k-3}(x_1^2+1)^{1/2}d\psi dx_1dv_1}{\left[(v_1^2+2iv_1x_1\cot\theta-1)^2+\dfrac{4v_1^2(x_1^2+1)}{\sin^2\theta}\right]^{k-1/2}}$$

$$=\frac{\Gamma(1/2)\Gamma(k-3/2)}{2\Gamma(k-1)}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{4v^{2k-3}(x^2+1)^{1/2}dxdv}{\left[(v^2+2ivx\cot\theta-1)^2+\dfrac{4v^2}{\sin^2\theta}(x^2+1)\right]^{k-1/2}}.$$

Noting the identity



$$(v^2+2ivx\cot\theta-1)^2+4v^2(x^2+1)/\sin^2\theta$$

$$=\left\{v+i\frac{1+\cos\theta}{\sin\theta}(x+\sqrt{x^2+1})\right\}\left\{v-i\frac{1-\cos\theta}{\sin\theta}(x-\sqrt{x^2+1})\right\}$$

$$\times\left\{v+i\frac{1+\cos\theta}{\sin\theta}(x-\sqrt{x^2+1})\right\}\left\{v-i\frac{1-\cos\theta}{\sin\theta}(x+\sqrt{x^2+1})\right\},$$

we put $t_1=v(x+\sqrt{x^2+1})$, $t_2=v(\sqrt{x^2+1}-x)$. Then we see that the last integral is equal to

$$\frac{1}{2}\int_0^{\infty}\int_0^{\infty}\frac{t_1^{k-3}t_2^{k-3}(t_1+t_2)^2dt_1dt_2}{[(t_1+i\tan\theta/2)(t_1-i\tan\theta/2)(t_2+i\tan\theta/2)(t_2-i\tan\theta/2)]^{k-1/2}}.$$

Thus we have

$$I_0(\gamma(\theta);s)=\frac{2^{2k-3}\Gamma(k-1/2)\Gamma(k-3/2)}{\Gamma(k)\Gamma(k-1)}$$

$$\times\{B_k(k-1;\theta)\overline{B_k(k-3;\theta)}+2B_k(k-2;\theta)\overline{B_k(k-2;\theta)}$$

$$+B_k(k-3;\theta)\overline{B_k(k-1;\theta)}\},$$

where $B_k(p;\theta)$ is defined by (3.18). From the results in the preceding paragraph, it is now easy to show

$$B_k(k-1;\theta)\overline{B_k(k-3;\theta)}+2B_k(k-2;\theta)\overline{B_k(k-2;\theta)}+B_k(k-3;\theta)\overline{B_k(k-1;\theta)}$$

$$=\frac{\Gamma(k-1)\Gamma(k-2)(2k-3)\pi}{2^{2k-3}\sin^2\theta\,\Gamma(k-1/2)^2}.$$

Therefore we get the following

THEOREM I-4. *Suppose that $\gamma\in G_R$ is conjugate to $\gamma(\theta)$ with $\sin\theta\neq0$. Then we have*

(3.27) $$I_0(\gamma(\theta);s)=\frac{2^3\pi}{\sin^2\theta(2k-2)(2k-4)}=a(k)^{-1}\frac{2k-3}{2^5\pi^2\sin^2\theta}.$$

## §4. Computation of $I_0(\gamma;s)$ for elliptic/parabolic, parabolic, and $\delta$-parabolic elements

**4-1.** Let us first consider the elliptic/parabolic element

$$\beta=\beta(\theta,\lambda)=\begin{pmatrix}\cos\theta&0&\sin\theta&0\\0&1&0&\lambda\\-\sin\theta&0&\cos\theta&0\\0&0&0&1\end{pmatrix}\qquad(\sin\theta,\ \lambda\neq0).$$

Put

(4.1) $$C_0(\beta;G_R)=\left\{\pm\begin{pmatrix}1&0&0&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix};\ u\in\boldsymbol{R}\right\}.$$

Then it is easy to see that a fundamental domain of $C_0(\hat{\beta}; G_R)$ in $H_2$ is given by

$$F_0(\hat{\beta})=\left\{\begin{pmatrix} x_1+iy_1 & x_{12}+iy_{12} \\ x_{12}+iy_{12} & iy_2 \end{pmatrix}; \begin{array}{l} x_1,\ x_{12}\in \boldsymbol{R}, \\ y_1,\ y_2,\ y_1y_2-y_{12}^2>0 \end{array}\right\}.$$

A direct computation shows the following

LEMMA 4-1. *For* $\hat{Z}\in F_0(\hat{\beta})$,

$$H_{\hat{\beta}}(\hat{Z}; s)=H_{\hat{\beta}}(\hat{Z})y_1^s(y_1y_2-y_{12}^2)^{-s}$$
$$=2^{2k}y_1^s(y_1y_2-y_{12}^2)^{k-s}[(2iy_2-\lambda)\sin\theta\{x_1-2ix_{12}y_{12}(2iy-\lambda)^{-1}\}^2$$
$$+2x_{12}^2\{2y_{12}^2\sin\theta(2iy_2-\lambda)^{-1}-1+\cos\theta+iy_1\sin\theta\}+y_1^2(2iy_2-\lambda)\sin\theta$$
$$-2iy_1(2iy_2-\lambda)\cos\theta+(2iy_2-\lambda)\sin\theta-2iy_1y_{12}^2\sin\theta-2y_{12}^2(1+\cos\theta)]^{-k}.$$

Then we have, by (4.1) and the definition of $I_0(\hat{\beta}; s)$,

$$I_0(\hat{\beta}; s)=\int_0^\infty\int_{-\infty}^\infty\int_{Y>0}H_{\hat{\beta}}(\hat{Z}; s)(y_1y_2-y_{12}^2)^{-3}dy_1dy_{12}dy_2dx_1dx_{12}.$$

Applying Lemma 3-5 twice, we get

$$I_0(\hat{\beta}; s)=\frac{2^{2k}}{(2\sin\theta)^{1/2}}\cdot\frac{\Gamma(1/2)\Gamma(k-1/2)}{\Gamma(k)}\cdot\frac{\Gamma(1/2)\Gamma(k-1)}{\Gamma(k-1/2)}$$
$$\times\int_{Y>0}\frac{y_1^s(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{\left[\begin{array}{l}\{2y_{12}^2\sin\theta+A(-1+\cos\theta+iy_1\sin\theta)\}^{1/2}\\ \quad\times[A(y_1^2+1)\sin\theta-2iAy_1\cos\theta-2iy_1y_{12}^2\sin\theta-2y_{12}^2(1+\cos\theta)]^{k-1}\end{array}\right]}$$
$$=\frac{2^{2k-1/2}\pi}{(k-1)(\sin\theta)^k}$$
$$\times\int_{Y>0}\frac{y_1^s(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{\left[\begin{array}{l}\left[2y_{12}^2+A\left(\dfrac{-1+\cos\theta}{\sin\theta}+iy_1\right)\right]^{1/2}\\ \quad\times\left[A(y_1^2-2iy_1\cot\theta+1)-2iy_1y_{12}^2-2y_{12}^2\dfrac{1+\cos\theta}{\sin\theta}\right]^{k-1}\end{array}\right]},$$

where we put, for simplicity, $A=2iy_2-\lambda$.

Noting the identities

$$A(y_1^2-2iy_1\cot\theta+1)-2iy_1y_{12}^2-2y_{12}^2\frac{1+\cos\theta}{\sin\theta}$$
$$=i\left(y_1-i\frac{1+\cos\theta}{\sin\theta}\right)\left[2(y_1y_2-y_{12}^2)+i\lambda y_1+i\frac{1-\cos\theta}{\sin\theta}(2y_2+i\lambda)\right],$$
$$2y_{12}^2+A\left(\frac{-1+\cos\theta}{\sin\theta}+iy_1\right)=-\left[2(y_1y_2-y_{12}^2)+i\lambda y_1+i\frac{1-\cos\theta}{\sin\theta}(2y_2+i\lambda)\right],$$

we have



$$I_0(\beta; s)=\frac{2^{k+1}\pi}{i^k(k-1)(\sin\theta)^k}$$
$$\times\int_0^\infty\int_0^\infty\int_0^{\sqrt{y_1y_2}}\frac{y_1^s(y_1y_2-y_{12}^2)^{k-3-s}dy_{12}dy_1dy_2}{\left(y_1-i\dfrac{1+\cos\theta}{\sin\theta}\right)^{k-1}\left[(y_1y_2-y_{12}^2)+\dfrac{iy_1\lambda}{2}+i\dfrac{1-\cos\theta}{\sin\theta}\left(y_2+\dfrac{i\lambda}{2}\right)\right]^{k-1/2}}.$$

Put $y_{12}^2=t^2(1+t^2)^{-1}y_1y_2$ $(t>0)$. Then the integral above is equal to

$$\frac{2^{k+1}\pi}{i^k(k-1)(\sin\theta)^k}\int_0^\infty\int_0^\infty$$
$$\times\int_0^\infty\frac{y_1^s(y_1y_2)^{k-5/2-s}(1+t^2)^{1+s}dy_1dy_2dt}{\left(y_1-i\dfrac{1+\cos\theta}{\sin\theta}\right)^{k-1}\left[\left\{y_1+(1+t^2)\dfrac{1-\cos\theta}{\sin\theta}\right\}y_2+\dfrac{i\lambda}{2}(1+t^2)\left(y_1+i\dfrac{1-\cos\theta}{\sin\theta}\right)\right]^{k-1/2}}$$
$$=\frac{2^{k+1}\pi}{i^k(k-1)(\sin\theta)^k}\int_0^\infty\int_0^\infty\frac{(t^2+1)^{1+s}y_1^{k-5/2}}{\left(y_1-i\dfrac{1+\cos\theta}{\sin\theta}\right)^{k-1}\left[y_1+i(1+t^2)\dfrac{1-\cos\theta}{\sin\theta}\right]^{k-1/2}}$$
$$\times\left\{\int_0^\infty\frac{y_2^{k-5/2-s}dy_2}{\left[y_2+\dfrac{i\lambda(1+t^2)\left(y_1+i\dfrac{1-\cos\theta}{\sin\theta}\right)}{2\left\{y_1+i(1+t^2)\dfrac{1-\cos\theta}{\sin\theta}\right\}}\right]^{k-1/2}}\right\}dy_1dt$$
$$=\frac{2^{k+1}\pi}{i^k(k-1)(\sin\theta)^k}\cdot\frac{2^{1+s}\Gamma(1+s)\Gamma(k-3/2-s)}{\Gamma(k-1/2)}\cdot\frac{e^{-(\mathrm{sgn}\,\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}}$$
$$\times\int_0^\infty\int_0^\infty\frac{y_1^{k-5/2}dy_1dt}{\left(y_1-i\dfrac{1+\cos\theta}{\sin\theta}\right)^{k-1}\left(y_1+i(1+t^2)\dfrac{1-\cos\theta}{\sin\theta}\right)^{k-3/2-s}\left(y_1+i\dfrac{1-\cos\theta}{\sin\theta}\right)^{1+s}}.$$

Here we have used the following formula which is well-known and is easy to prove:

LEMMA 4-2. *Let* $k$, $s$ *be real numbers such that* $k>1$, $0<s<k$. *Then for any* $\lambda\in \boldsymbol{C}-(-\infty, 0]$, *we have*

(4.2) $$\int_0^\infty\frac{x^{k-1-s}dx}{(x+\lambda)^k}=\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}e^{-s\log\lambda},$$

*where we take, in* $\log\lambda$, *the principal branch such that* $-\pi<\arg(\log\lambda)<\pi$.

Now by using Lemma 3-5 again and then integrating in $t$, we get

$$I_0(\beta; s)=\frac{2^{k+1+s}\pi\Gamma(1/2)\Gamma(s+1)\Gamma(k-2-s)}{i^{k+1/2}(k-1)\Gamma(k-1/2)(1-\cos\theta)^{1/2}}\cdot\frac{e^{-(\mathrm{sgn}\,\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}}$$
$$\times\int_0^\infty\frac{y_1^{k-5/2}dy_1}{(y_1^2-2iy_1\cos\theta+\sin^2\theta)^{k-1}}.$$

**4-2.** Here we consider the integrals

$$C_k(p\,;\,\theta)=\int_0^\infty \frac{x^{p-1/2}dx}{(x^2-2ix\cos\theta+\sin^2\theta)^{k-1}} \qquad (\sin\theta\neq 0), \tag{4.3}$$

where $k$, $p$ are integers such that $k\geqq 2$, $0\leqq p\leqq 2k-3$. Changing the variable $x$ by $\sin^2\theta/x$, we see that they satisfy the following symmetry:

$$C_k(k-2+p\,;\,\theta)=(\sin\theta)^{2p-1}C_k(k-1-p\,;\,\theta) \quad (0\leqq p\leqq k-1). \tag{4.4}$$

Also, in the same way as in Lemma 3-6, we can prove

LEMMA 4-3. *The following recurrence formulae hold*:

$$C_k(p\,;\,\theta)=i\cos\theta\, C_k(p-1\,;\,\theta)+\frac{p-3/2}{2k-4}C_{k-1}(p-2\,;\,\theta), \tag{4.5}$$

*for* $k\geqq 3$, $2\leqq p\leqq 2k-3$.

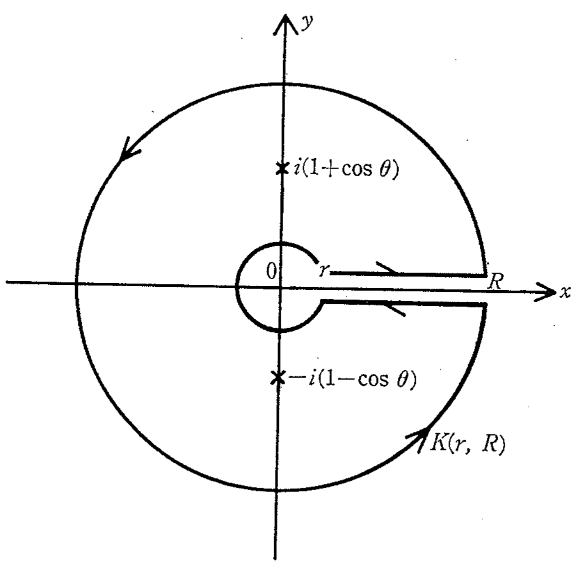
$$\begin{aligned}&(2k-p-5/2)C_k(p\,;\,\theta)\\ &=i\cos\theta(2k-2p-1)C_k(p-1\,;\,\theta)+\sin^2\theta(p-3/2)C_k(p-2\,;\,\theta),\end{aligned} \tag{4.6}$$

*for* $k\geqq 2$, $2\leqq p\leqq 2k-3$.

By these formulae, the computation of $C_k(p\,;\,\theta)$ reduces to that of $C_k(k-1\,;\,\theta)$.

PROPOSITION 4-1. *For* $k\geqq 2$, *we have*

$$C_k(k-1\,;\,\theta)=\frac{\Gamma(1/2)\Gamma(k-3/2)}{\Gamma(k-1)}\cdot\frac{e^{(2k-3)\pi i/4}e^{-(k-3/2)\theta i}}{2^{k-3/2}}. \tag{4.7}$$




(Fig. 2)



PROOF. Put $C_k=C_k(k-1\,;\,\theta)$. By Lemma 4-3, it is easy to show

$$C_k=\frac{k-5/2}{2k-4}ie^{-\theta i}C_{k-1}.$$

Therefore it suffices to prove the assertion for $k=2$:

$$C_2=C_2(1\,;\,\theta)=\int_0^\infty\frac{x^{1/2}dx}{x^2-2ix\cos\theta+\sin^2\theta}.$$

Considering the integral along the contour $K(r,\,R)$ as in Fig. 2, we have

$$C_2=\frac{\pi}{\sqrt{2}}e^{\pi i/4}e^{-\theta i/2},$$

from which the assertion follows. q. e. d.

**4-3.** Applying Proposition 4-1 to the calculation in § 4-1, we get

$$\begin{aligned}&I_0(\hat\beta(\theta,\,\lambda)\,;\,s)\\ &=-\Big(\frac{1}{a(k)}\cdot\frac{1}{2^3\pi\sin\theta\sin\frac{\theta}{2}}+o(s)\Big)e^{-i[(k-3/2)\theta+(\operatorname{sgn}\lambda)\pi(s+1)/2]}\cdot\frac{1}{|\lambda|^{s+1}},\end{aligned} \tag{4.8}$$

where $o(s)$ is a function of $s$ independent of $\lambda$ such that $\lim_{s\to+0}o(s)=0$, and $\operatorname{sgn}(x)=x/|x|$ for $x\in \boldsymbol{R}^\times$.

Now we consider the totality of $I_0(\gamma\,;\,s)$ corresponding to a family $[\gamma]_\Gamma$. By definition, such a family consists of those $\gamma'\in C(\gamma\,;\,\Gamma)$ that are conjugate in $G_{\boldsymbol R}$ to $\hat\beta(\theta,\,\lambda)$ with fixed $\theta$. It is easy to see that there exists $a$ with $0\leqq a<1$ such that

$$[\gamma]_\Gamma=g^{-1}\left\{\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & a+n\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix};\ n\in\boldsymbol{Z},\ a+n\neq 0\right\}g, \tag{4.9}$$

for some $g\in G_{\boldsymbol R}$. For each $\gamma'\in[\gamma]_\Gamma$, we can put $C_0(\gamma'\,;\,G_{\boldsymbol R})=g^{-1}C_0(\hat\beta(\theta,\,\lambda)\,;\,G_{\boldsymbol R})g$. Then $C(\gamma'\,;\,\Gamma)=C(\gamma\,;\,\Gamma)$ for all $\gamma'\in[\gamma]_\Gamma$. By (4.8), we have

$$\begin{aligned}\sum_{\gamma'\in[\gamma]_\Gamma}I_0(\gamma'\,;\,s)&=-\Big(\frac{1}{a(k)}\cdot\frac{1}{2^3\pi\sin\theta\sin\frac{\theta}{2}}+o(s)\Big)e^{-(k-3/2)\theta i}\\ &\quad\times\sum_{\substack{n\in\boldsymbol{Z}\\ a+n\neq 0}}\frac{e^{-\operatorname{sgn}(a+n)\pi i(s+1)/2}}{|a+n|^{s+1}}\\ &=-\Big(\frac{1}{a(k)}\cdot\frac{1}{2^3\pi\sin\theta\sin\frac{\theta}{2}}+o(s)\Big)e^{-(k-3/2)\theta i}\\ &\quad\times\{e^{-\pi i(s+1)/2}\zeta(s+1,\,a)+e^{\pi i(s+1)/2}\zeta(s+1,\,1-a)\},\end{aligned}$$

where $\zeta(s+1,\,a)$ is the Hurwitz's zeta function which is defined by

$$\zeta(s+1, a)=\sum_{\substack{n=0\\ a+n\neq 0}}^{\infty}\frac{1}{(a+n)^{s+1}} \qquad (s>0).$$

It is well-known that $\zeta(s+1, a)$ has the following expansion at $s=0$:

$$(4.10)\qquad \zeta(s+1, a)=\begin{cases}\dfrac{1}{s}-\dfrac{\Gamma'(a)}{\Gamma(a)}+o(s) & \text{if } a\neq 0\\[2mm] \dfrac{1}{s}+C+o(s) & \text{if } a=0.\end{cases}$$

It follows that

$$(4.11)\qquad e^{-\pi i(s+1)/2}\zeta(s+1, a)+e^{\pi i(s+1)/2}\zeta(s+1, 1-a)=-\pi(1+i\cot^*\pi a)+o(s),$$

where (and also in the following) we write

$$(4.12)\qquad \cot^*\theta=\begin{cases}\cot\theta & (\theta\neq n\pi)\\ 0 & (\theta=n\pi),\end{cases}$$

and we have used the formula

$$\frac{\Gamma'(1-a)}{\Gamma(1-a)}-\frac{\Gamma'(a)}{\Gamma(a)}=\pi\cot\pi a \qquad (a\neq 0).$$

Summing up, we have obtained the following

THEOREM I-5. (i) *Assume that $\gamma\in G_R$ is elliptic/parabolic, and it is conjugate to $\beta(\theta, \lambda)$. Then $I_0(\gamma; s)$ is given by (4.8).*

(ii) *If $\gamma\in\Gamma$, and the family $[\gamma]_\Gamma$ is expressed as in (4.9), the totality of $I_0(\gamma'; s)$ for $\gamma'\in[\gamma]_\Gamma$ is expressed as*

$$-\left\{\frac{1}{a(k)}\cdot\frac{e^{-(k-s/2)\theta i}}{2^3\pi\sin\theta\sin\frac{\theta}{2}}+o(s)\right\}\{e^{-\pi i(s+1)/2}\zeta(s+1, a)+e^{\pi i(s+1)/2}\zeta(s+1, 1-a)\}.$$

(iii) *We have*

$$(4.13)\qquad \lim_{s\to+0}\sum_{\gamma'\in[\gamma\pm1]_\Gamma} I_0(\gamma'; s)$$
$$=\frac{1}{a(k)}\cdot\frac{1}{2^2\sin\theta\sin\frac{\theta}{2}}\{\cos(k-3/2)\theta+\cot^*\pi a\sin(k-3/2)\theta\}.$$

**4-4.** Let us next consider the paraelliptic element

$$\hat\gamma=\hat\gamma(\theta, \lambda)=\begin{pmatrix}\cos\theta & \sin\theta & \lambda\cos\theta & \lambda\sin\theta\\ -\sin\theta & \cos\theta & -\lambda\sin\theta & \lambda\cos\theta\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \cos\theta\end{pmatrix} \qquad (\lambda,\ \sin\theta\neq 0).$$

Put



$$(4.14)\qquad C_0(\hat\gamma; G_R)=\left\{\pm\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix};\ u\in\boldsymbol{R}\right\}.$$

Then as a fundamental domain of $C_0(\hat\gamma; G_R)$ in $H_2$, we can take the set $F_0(\hat\beta)$, which is given in § 4-1. Next lemma follows by a direct calculation.

LEMMA 4-4. *For $\hat Z\in F_0(\hat\beta)$, we have*

$$H_{\hat\gamma}(\hat Z; s)=H_{\hat\gamma}(\hat Z)(y_1y_2-y_{12}^2)^{-s}$$
$$=\frac{2^{2k}(y_1y_2-y_{12}^2)^{k-s}}{(\sin\theta)^{2k}}$$
$$\times\left[(x_1+2iy_{12}\cot\theta)^2+4\left\{x_{12}-\frac{i}{2}(y_1-y_2)\cot\theta\right\}^2+\frac{1}{\sin^2\theta}(y_1+y_2+i\lambda)^2\right]^{-k}.$$

Then by using Lemma 3-5 twice, we have

$$I_0(\hat\gamma; s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{Y>0}H_{\hat\gamma}(\hat Z; s)(y_1y_2-y_{12}^2)^{-s}dy_1dy_2dy_{12}dx_1dx_{12}$$
$$=\frac{2^{2k}\Gamma(1/2)\Gamma(k-1/2)}{(\sin\theta)^{2k}\Gamma(k)}$$
$$\times\int_{-\infty}^{\infty}\int_{Y>0}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_2dy_{12}dx_{12}}{\left[4\left\{x_{12}^2-\frac{i}{2}(y_1-y_2)\cot\theta\right\}^2+\frac{1}{\sin^2\theta}(y_1+y_2+i\lambda)^2\right]^{k-1/2}}$$
$$=\frac{2^{2k}\Gamma(1/2)\Gamma(k-1/2)\Gamma(1/2)\Gamma(k-1)}{(\sin\theta)^{2k}\Gamma(k)2^{2k-1}\Gamma(k-1/2)}\int_{Y>0}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_2dy_{12}}{\left[\frac{1}{4}(y_1+y_2+i\lambda)^2(\sin\theta)^{-2}\right]^{k-1}}$$
$$=\frac{2^{2k-1}\Gamma(1/2)^3\Gamma(k-2-s)}{(\sin\theta)^2(k-1)\Gamma(k-s-3/2)}\int_0^\infty\int_0^\infty\frac{(y_1y_2)^{k-s-5/2}dy_1dy_2}{(y_1+y_2+i\lambda)^{2k-2}}.$$

Here we apply Lemma 4-2 twice, and see that the last integral is equal to

$$\int_0^\infty y_1^{k-s-5/2}\left\{\int_0^\infty\frac{y_2^{2k-2-1-(k+s-1/2)}dy_2}{[y_2+(y_1+i\lambda)]^{2k-2}}\right\}dy_1$$
$$=\frac{\Gamma(k+s-1/2)\Gamma(k-s-3/2)\Gamma(1+2s)\Gamma(k-s-3/2)}{\Gamma(2k-2)\Gamma(k+s-1/2)}\cdot\frac{e^{-(\mathrm{sgn}\,\lambda)\pi i(1+2s)/2}}{|\lambda|^{1+2s}}.$$

Using the duplication formula of $\Gamma$-function

$$\Gamma(k-3/2)\Gamma(k-2)=2^{5-2k}\Gamma(1/2)\Gamma(2k-4),$$

we thus get

$$I_0(\hat\gamma(\theta, \lambda); s)=\left\{\frac{2^2\pi^2}{\sin^2\theta(k-1)(k-3/2)(k-2)}+o(s)\right\}\frac{e^{-(\mathrm{sgn}\,\lambda)\pi i(1+2s)/2}}{|\lambda|^{1+2s}},$$

where $o(s)$ is independent of $\lambda$.

Now consider the sum of $I_0(\gamma'; s)$ for all $\gamma' \in \Gamma$ in the family $[\gamma]_\Gamma$, where $\gamma$ is assumed to be conjugate in $G_R$ to $\hat{\gamma}(\theta, \lambda)$. By definition, $[\gamma]_\Gamma$ consists of those elements $\gamma'$ of $C(\gamma; \Gamma)$ which are paraelliptic and whose semi-simple factors are equal to that of $\gamma$. It is easy to see that there exist $g \in G_R$ and $a \in \boldsymbol{R}$ with $0 \leqq a < 1$, such that

$$(4.15) \qquad [\gamma]_\Gamma = g^{-1}\left\{\begin{pmatrix} \cos\theta & \sin\theta & \lambda\cos\theta & \lambda\sin\theta \\ -\sin\theta & \cos\theta & -\lambda\sin\theta & \lambda\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix};\ \begin{matrix} \lambda \in a+\boldsymbol{Z} \\ \lambda \neq 0 \end{matrix}\right\} g\,.$$

Put, for any $\gamma' \in [\gamma]_\Gamma$, $C_0(\gamma'; G_R) = g^{-1}C_0(\hat{\gamma}(\theta, \lambda); G_R)g$. Then we have $C_0(\gamma'; \Gamma) = C_0(\gamma; \Gamma)$ for all $\gamma' \in [\gamma]_\Gamma$, and we can prove the following theorem in the same way as in Theorem I-5.

THEOREM I-6. (i) *Assume that $\gamma \in G_R$ is paraelliptic, and it is conjugate in $G_R$ to $\hat{\gamma}(\theta, \lambda)$. Then we have*

$$I_0(\gamma; s) = \left\{\frac{1}{a(k)} \cdot \frac{1}{2^3\pi\sin^2\theta} + o(s)\right\}\frac{e^{-(\operatorname{sgn}\lambda)\pi i(1+2s)/2}}{|\lambda|^{1+2s}}.$$

(ii) *If $\gamma \in \Gamma$, then $[\gamma]_\Gamma$ is expressed as in* (4.15), *and the totality of $I_0(\gamma'; s)$ for $\gamma' \in [\gamma]_\Gamma$ is expressed as*

$$\left\{\frac{2^5\pi^2}{\sin^2\theta(2k-2)(2k-3)(2k-4)} + o(s)\right\}$$
$$\times\{e^{-\pi i(s+1/2)}\zeta(2s+1,\ a) + e^{\pi i(s+1/2)}\zeta(2s+1,\ 1-a)\}.$$

(iii) *We have*

$$(4.16) \qquad \lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma} I_0(\gamma'; s) = -\frac{1}{a(k)} \cdot \frac{1}{2^3\sin^2\theta}(1+i\cot^*\pi a)\,.$$

**4-5.** Consider the $\delta$-parabolic element $\hat{\delta} = \hat{\delta}(\lambda_1, \lambda_2)$ where $(\lambda_1, \lambda_2) \neq (0, 0)$. Suppose first that $\lambda_1, \lambda_2 \neq 0$. Put

$$(4.17) \qquad C_0(\hat{\delta}; G_R) = \left\{\pm\begin{pmatrix} 1 & 0 & t_1 & 0 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ t_1, t_2 \in \boldsymbol{R}\right\}.$$

Then a fundamental domain of $C_0(\hat{\delta}; G_R)$ in $H_2$ can be taken as

$$F_0(\hat{\delta}) = \left\{\begin{pmatrix} iy_1 & x_{12}+iy_{12} \\ x_{12}+iy_{12} & iy_2 \end{pmatrix};\ y_1, y_2, y_1y_2-y_{12}^2 > 0\right\},$$

and it is easy to see that, for $\hat{Z} \in F_0(\hat{\delta})$, we have

$$H_{\hat{\delta}}(\hat{Z}; s) = H_{\hat{\delta}}(\hat{Z})(y_1y_2-y_{12}^2)^{-s}$$
$$= (-1)^k(y_1y_2-y_{12}^2)^{k-s}[x_{12}^2+(y_1+i\lambda_1/2)(y_2-i\lambda_2/2)]^{-k}.$$



Therefore we can proceed in the same way as in § 4-4:

$$I_0(\hat{\delta}; s) = (-1)^k\int_{-\infty}^{\infty}\int_{Y>0}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_2dy_{12}dx_{12}}{[x_{12}^2+(y_1+i\lambda_1/2)(y_2-i\lambda_2/2)]^k}$$
$$= \frac{(-1)^k\Gamma(1/2)\Gamma(k-1/2)}{\Gamma(k)}\int_{Y>0}\frac{(y_1y_2-y_{12}^2)^{k-3-s}dy_1dy_{12}dy_2}{(y_1+i\lambda_1/2)^{k-1/2}(y_2-i\lambda_2/2)^{k-1/2}}$$
$$= \frac{(-1)^k\pi\Gamma(k-1/2)\Gamma(k-2-s)}{\Gamma(k)\Gamma(k-s-3/2)}\int_0^\infty\frac{y_1^{k-s-5/2}dy_1}{(y_1+i\lambda_1/2)^{k-1/2}}\int_0^\infty\frac{y_2^{k-s-5/2}dy_2}{(y_2-i\lambda_2/2)^{k-1/2}}$$
$$= \left\{\frac{2^2\pi(-1)^k\Gamma(k-2)\Gamma(k-3/2)}{\Gamma(k)\Gamma(k-1/2)} + o(s)\right\}\frac{e^{-(\operatorname{sgn}\lambda_1)\pi i(1+s)/2}}{|\lambda_1|^{1+s}} \cdot \frac{e^{(\operatorname{sgn}\lambda_2)\pi i(1+s)/2}}{|\lambda_2|^{1+s}},$$

where $o(s)$ is a function, independent of $\lambda_1$, $\lambda_2$, such that $\lim_{s\to+0} o(s) = 0$.

Suppose that $\gamma$ is an element of $\Gamma$ and that it is conjugate in $G_R$ to $\hat{\delta}(\lambda_1, \lambda_2)$ above. This is possible only if $G_Q$ is of $Q$-rank two, so that $\Gamma$ is commensurable with $\mathrm{Sp}(2, \boldsymbol{Z})$. By definition, the family $[\gamma]_\Gamma$ consists of $\gamma' \in C(\gamma; \Gamma)$ that are $\delta$-parabolic of type $(\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \neq 0$. We can assume, without loss of generality, that there exists $g \in G_R$ such that

$$(4.18) \qquad [\gamma]_\Gamma = g^{-1}\left\{\begin{pmatrix} 1 & 0 & m+c & 0 \\ 0 & -1 & 0 & am+bn+ac \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \begin{matrix} m, n \in \boldsymbol{Z} \\ m+c \neq 0 \\ am+bn+ac \neq 0 \end{matrix}\right\} g\,,$$

where $a$, $b$ are integers with $(a, b)=1$, $b>0$, and $0 \leqq c < 1$ (cf. Theorem 6-1). Put, for each $\gamma' \in [\gamma]_\Gamma$,

$$C_0(\gamma'; G_R) = g^{-1}C_0(\hat{\delta}(\lambda_1, \lambda_2); G_R)g\,.$$

Then $C_0(\gamma'; \Gamma) = C_0(\gamma; \Gamma)$ for all $\gamma'$, and we have the following

THEOREM I-7. (i) *Suppose that $\gamma \in G_R$ is $\delta$-parabolic and it is conjugate in $G_R$ to $\hat{\delta}(\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \neq 0$. Then we have*

$$I_0(\gamma; s) = \left\{\frac{1}{a(k)} \cdot \frac{(-1)^k}{2^3\pi^2} + o(s)\right\}\frac{e^{-(\operatorname{sgn}\lambda_1)\pi i(s+1)/2}}{|\lambda_1|^{s+1}} \cdot \frac{e^{(\operatorname{sgn}\lambda_2)\pi i(s+1)/2}}{|\lambda_2|^{s+1}}.$$

(ii) *If $\gamma \in \Gamma$ and the family $[\gamma]_\Gamma$ is expressed as in* (4.18), *then the totality of $I_0(\gamma'; s)$ for $\gamma' \in [\gamma]_\Gamma$ is expressed as*

$$(4.19) \qquad \left\{\frac{1}{a(k)} \cdot \frac{(-1)^k}{2^3\pi^2b^2} + o(s)\right\}$$
$$\times\sum_{j=0}^{b-1}\left\{e^{-\pi i(s+1)/2}\zeta\left(s+1,\ \frac{j+c}{b}\right) + e^{\pi i(s+1)/2}\zeta\left(s+1,\ \frac{b-j-c}{b}\right)\right\}$$
$$\times\left\{e^{\pi i(s+1)/2}\zeta\left(s+1,\ \frac{a(j+c)}{b}\right) + e^{-\pi i(s+1)/2}\zeta\left(s+1,\ \frac{b-a(j+c)}{b}\right)\right\}.$$

(iii) *We have*

(4.20)
$$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma} I_0(\gamma';s) =\frac{1}{a(k)}\cdot\frac{(-1)^k}{2^3b^2}\sum_{j=0}^{b-1}\Big(1+i\cot^*\frac{(j+c)\pi}{b}\Big)\Big(1-i\cot^*\frac{a(j+c)\pi}{b}\Big).$$

PROOF. By the above calculation and (4.18), we have

$$\sum_{\gamma'\in[\gamma]_\Gamma} I_0(\gamma';s)=\Big\{\frac{1}{a(k)}\cdot\frac{(-1)^k}{2^3\pi^2}+o(s)\Big\}\times\sum_{m,n}\frac{e^{-\mathrm{sgn}(m+c)\pi i(s+1)/2}}{|m+c|^{s+1}}\cdot\frac{e^{\mathrm{sgn}(am+bn+ac)\pi i(s+1)/2}}{|am+bn+ac|^{s+1}},$$

where in the right hand side the sum is extended over the pair of all integers $(m, n)$ such that $m+c$, $am+bn+ac\neq 0$. If we write $m=bm_0+j$, $n=n_0-am_0$ $(0\leqq j<b)$, we get a bijection for each $j$:

$$\{(m, n)\in \boldsymbol{Z}^2;\ m\equiv j \pmod b\}\xrightarrow{\sim}\{(m_0, n_0)\in\boldsymbol{Z}^2\}.$$

The assertion follows easily from this as in §4-3. q. e. d.

REMARK 4-1. In connection with Problem 2 in §0-1, it would be interesting to note that the sum appeared in (4.20) is closely connected with the *Dedekind sum*; if $c=0$, the sum is in fact equal (up to constant) to the Dedekind sum

$$S(a, b)=\frac{1}{4b}\sum_{j=1}^{b-1}\cot\frac{j\pi}{b}\cot\frac{aj\pi}{b},$$

since we have

$$\sum_{j=0}^{b-1}\cot^*\frac{j\pi}{b}=\sum_{j=0}^{b-1}\cot^*\frac{aj\pi}{b}=0$$

(cf. F. Hirzebruch and D. Zagier [12]).

**4-6.** Finally consider the $\delta$-parabolic element $\hat\delta=\hat\delta(\lambda, 0)$, $\lambda\neq 0$. Put

(4.21)
$$C_0(\hat\delta;G_{\boldsymbol R})=\left\{\pm\begin{pmatrix}1&0&u&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix};\ \begin{matrix}u\in\boldsymbol{R}\\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{R})\end{matrix}\right\}.$$

Note that we have

$$\{\pm C_1(\delta;G_{\boldsymbol R})\}=V\ltimes C_0(\hat\delta;G_{\boldsymbol R})\qquad\text{(semi-direct product)},$$

where $C_1(\delta;G_{\boldsymbol R})$ is defined in (3.5), and

$$V=\left\{g_v=\begin{pmatrix}v&0&0&0\\0&1&0&0\\0&0&v^{-1}&0\\0&0&0&1\end{pmatrix};\ v>0\right\}.$$

Accordingly the invariant measure $dg$ of $C_1(\delta;G_{\boldsymbol R})$ is decomposed as $dg=$



$dg_\alpha(2v^{-3}dv)$, $dg_\alpha$ being the invariant measure of $C_0(\hat\delta;G_{\boldsymbol R})$ defined by $dg_\alpha=du\,d\alpha$, $\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{R})$. It is easy to see from Lemma 3-1, 3-2, that a fundamental domain of $C_0(\hat\delta;G_{\boldsymbol R})$ in $H_2$ is given by

$$F_0(\hat\delta)=\left\{g_v\left\langle\begin{pmatrix}i&t\\t&i\end{pmatrix}\right\rangle=\begin{pmatrix}iv^2&vt\\vt&i\end{pmatrix};\ v>0,\ t\geqq 0\right\}.$$

A direct calculation shows that, for $\hat Z=\begin{pmatrix}iv^2&vt\\vt&i\end{pmatrix}\in F_0(\hat\delta)$,

$$H_{\hat\delta}(\hat Z;s)=H_{\hat\delta}(\hat Z)(\det\mathrm{Im}\,\hat Z)^{-s}=(-1)^k v^{2k-2s}[v^2(1+t^2)+i\lambda/2]^{-k}.$$

By applying (3.15), we have

$$I_0(\hat\delta(\lambda,0);s)=\int_{F_0(\hat\delta)}2tv^{-3}H_{\hat\delta}(\hat Z;s)dtdv=\int_0^\infty\int_0^\infty\frac{2(-1)^k t v^{2k-2s-3}dvdt}{(v^2t^2+v^2+i\lambda/2)^k}=\frac{(-1)^k}{2(k-1)}\cdot\frac{\Gamma(s+1)\Gamma(k-2-s)}{\Gamma(k-1)}\cdot\frac{2^{s+1}e^{-(\mathrm{sgn}\,\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}}$$

(cf. Lemma 4-2).

Now suppose that $\gamma\in\Gamma$, and it is conjugate in $G_{\boldsymbol R}$ to $\hat\delta(\lambda, 0)$. This occurs only if $G_{\boldsymbol Q}$ has $\boldsymbol{Q}$-rank two. The family $[\gamma]_\Gamma$ consists of those $\gamma'\in C(\gamma;\Gamma)$ which are $\delta$-parabolic of type $\hat\delta(\lambda, 0)$ and whose semi-simple factors are equal to that of $\gamma$. It is easy to see that there exists $g\in G_{\boldsymbol R}$ such that

(4.22)
$$[\gamma]_\Gamma=g^{-1}\left\{\begin{pmatrix}1&0&n&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix};\ \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\right\}g.$$

Then we put, for each $\gamma'\in[\gamma]_\Gamma$,

$$C_0(\gamma';G_{\boldsymbol R})=g^{-1}C_0(\hat\delta(\lambda,0);G_{\boldsymbol R})g,$$

and see that $C_0(\gamma';\Gamma)=C_0(\gamma;\Gamma)$. Therefore we can prove in the same way as §4-4, 4-5, the following

THEOREM I-8. (i) *Suppose that $\gamma\in G_{\boldsymbol R}$ is $\delta$-parabolic and it is conjugate in $G_{\boldsymbol R}$ to $\hat\delta(\lambda, 0)$, $\lambda\neq 0$. Then we have*

$$I_0(\gamma;s)=\Big\{\frac{1}{a(k)}\cdot\frac{(-1)^k(2k-3)}{2^6\pi^3}+o(s)\Big\}\frac{e^{-(\mathrm{sgn}\,\lambda)\pi i(s+1)/2}}{|\lambda|^{s+1}}.$$

(ii) *If $\gamma\in\Gamma$, then $[\gamma]_\Gamma$ is expressed as in (4.22), and the totality of $I_0(\gamma';s)$ for $\gamma'\in[\gamma]_\Gamma$ is expressed as*

$$\left\{\frac{1}{a(k)}\cdot\frac{(-1)^k(2k-3)}{2^5\pi^3}+o(s)\right\}\zeta(s+1)\cos\left(\frac{s+1}{2}\right)\pi\,.$$

(iii) *We have*

(4.23) $$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma} I_0(\gamma'\,;\,s)=-\frac{1}{a(k)}\cdot\frac{(-1)^k(2k-3)}{2^6\pi^2}.$$

## § 5. Contributions from parabolic conjugacy classes

Calculations in this section are essentially based on [1], [5], [18], and on some results of Siegel [31].

**5-1.** Suppose $\gamma$ is an element of $\Gamma$ which is conjugate to $\gamma(S)=\begin{pmatrix}1&S\\0&1\end{pmatrix}$, where $S$ is a non-degenerate symmetric matrix. Then there exists $g\in G_Q$ such that the family $[\gamma]_\Gamma$ is given by

(5.1) $$[\gamma]_\Gamma=g\left\{\begin{pmatrix}1&X\\0&1\end{pmatrix};\ X\in SM_2(\boldsymbol{R})\right\}g^{-1}\cap\Gamma.$$

For each family $[\gamma]_\Gamma$, we shall fix, once and for all, such $g\in G_Q$, and associate with it a lattice $L$ in the vector space $SM_2(\boldsymbol{R})$ of symmetric $2\times 2$ matrices, which is given by

(5.2) $$\begin{pmatrix}1&L\\0&1\end{pmatrix}=\begin{pmatrix}1&SM_2(\boldsymbol{R})\\0&1\end{pmatrix}\cap g^{-1}\Gamma g\,.$$

Since $g\in G_Q$, $L$ defines a natural $Q$-structure of $SM_2(\boldsymbol{R})$ such that $\det S$, $\operatorname{tr} S\in Q$ for all $S\in L$. Let $P$ be the parabolic subgroup of $G_Q$ corresponding to $[\gamma]_\Gamma$. $P$ is defined over $Q$. Let $P=P_M\cdot P_U$ be a Levi-decomposition of $P$. We have

(5.3) $$(P_M)_R\cong GL_2(\boldsymbol{R})\,,\quad (P_U)_R\cong SM_2(\boldsymbol{R})\,.$$

Denote by $(P_M)_0$ the image of $P_M$ under this isomorphism. We may assume that, under the isomorphisms (5.3), $(P_M)_0$ acts on $(P_U)_R$ as $X\mapsto AX{}^tA$. Moreover, for some technical reasons, we make the following assumption on $\Gamma$, which holds for a wide class of arithmetic subgroups:

ASSUMPTION 5-1. For each parabolic $Q$-subgroup $P$ of $G_Q$, the equality $P\cap\Gamma=(P_M\cap\Gamma)\cdot(P_U\cap\Gamma)$ holds for a suitable choice of $P_M$.

**5-2.** First suppose that, in the expression $\gamma=g\gamma(S)g^{-1}$ of $\gamma$, $S$ is definite. Put, for any $\delta\in[\gamma]_\Gamma$,

(5.4) $$C_0(\delta\,;\,G_R)=g\begin{pmatrix}1&SM_2(\boldsymbol{R})\\0&1\end{pmatrix}g^{-1}=(P_U)_R\,,$$

and take as a Haar measure of $C_0(\delta\,;\,G_R)$ the standard Euclidean measure of $SM_2(\boldsymbol{R})$ via the expression (5.4).



LEMMA 5-1. *Under the assumption* 5-1, *we have*

(i) $$g\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}\begin{pmatrix}1&X\\0&1\end{pmatrix}g^{-1}\in\Gamma\Rightarrow X\in L,\ A\in G(L)\,,$$

*where* $G(L)=\{A\in GL_2(\boldsymbol{R})\,;\ AL{}^tA=L\}$.

(ii) *For* $\delta\in[\gamma]_\Gamma$, *put* $g^{-1}\delta g=\begin{pmatrix}1&S_\delta\\0&1\end{pmatrix}$,

$$O_\Gamma(S_\delta)=\left\{A\in G(L)\cap O(S_\delta)\,;\ g\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}g^{-1}\in\Gamma\right\}.$$

*Then we have*

$$[C(\delta\,;\,\Gamma):C_0(\delta\,;\,\Gamma)]=\sharp O_\Gamma(S_\delta)\,.$$

(iii) *Two elements* $\delta_1$, $\delta_2$ *of* $[\gamma]_\Gamma$ *are conjugate in* $\Gamma$ *if and only if there exists* $A\in G(L)$ *such that* $S_{\delta_2}=AS_{\delta_1}{}^tA$ *and* $g\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}g^{-1}\in\Gamma$.

The proof is straightforward. Here we note that the group $G(L)^+=G(L)\cap GL_2^+(\boldsymbol{R})$ is a Fuchsian group of the first kind, and $(P_M\cap\Gamma)_0^+=(P_M\cap\Gamma)_0\cap GL_2^+(\boldsymbol{R})$ is a subgroup of $G(L)^+$ of finite index. It is easy to see from (5.4) that $C_0(\gamma\,;\,G_R)\backslash H_2$ is diffeomorphic to $\begin{pmatrix}1&SM_2(\boldsymbol{R})\\0&1\end{pmatrix}\backslash H_2=\{iY\,;\,Y>0\}$, and we may put $d\hat{Z}=(\det Y)^{-3}dY$. Then we have, by [18], p. 241,

$$\begin{aligned}I_0(\gamma\,;\,s)&=\int_{g^{-1}C_0(\gamma;\,G_R)g\backslash H_2}H_{g^{-1}\gamma g}(\hat{Z})(\det\operatorname{Im}\hat{Z})^{-s}d\hat{Z}\\&=\left\{\frac{2^5\pi}{(2k-2)(2k-3)(2k-4)}+o(s)\right\}\frac{e^{\pm\pi i(-3-2s)/2}}{(\det S)^{s+3/2}},\end{aligned}$$

where the sign $\pm$ in the exponent is taken according as $S\gtrless 0$. It follows from Lemma 5-1 that

(5.5) $$\begin{aligned}&\sum_{\delta\in[\gamma]_\Gamma/\widetilde{\Gamma}}\frac{I_0(\delta\,;\,s)}{[\bar{C}(\delta\,;\,\Gamma):\bar{C}_0(\delta\,;\,\Gamma)]}\\&=\left\{\frac{1}{a(k)}\cdot\frac{1}{2\pi}+o(s)\right\}\times s\sum_{S\in L^+\bmod (P_M\cap\Gamma)_0}\frac{1}{\sharp O_\Gamma(S)(\det S)^{s+3/2}}.\end{aligned}$$

In the right hand side of this formula, $S$ runs through the complete set of representatives in $L^+=\{S\in L\,;\ S>0\}$, modulo the action of $(P_M\cap\Gamma)_0$.

THEOREM I-9. (i) *Suppose* $\gamma$ *is conjugate in* $G_Q$ *to* $\gamma(S)$, *with* $S\gtrless 0$. *Then*

$$I_0(\gamma\,;\,s)=\left\{\frac{1}{a(k)}\cdot\frac{1}{2^3\pi^2}+o(s)\right\}\cdot\frac{e^{\pm\pi i(-3-2s)/2}}{(\det S)^{s+3/2}}\,.$$

(ii) *If* $\gamma\in\Gamma$, *the totality of* $I_0(\delta\,;\,s)/[\bar{C}(\delta\,;\,\Gamma):\bar{C}_0(\delta\,;\,\Gamma)]$ *for* $\delta\in[\gamma]_\Gamma/\widetilde{\Gamma}$ *is expressed as* (5.5) *above, under the assumption* 5-1.

(iii) *We have*

(5.6)
$$\lim_{s\to+0}\sum_{\delta\in[\gamma]_{\Gamma'/\tilde{\Gamma}}}\frac{I_0(\delta\,;s)}{[\bar{C}(\delta\,;\Gamma):\bar{C}_0(\delta\,;\Gamma)]}$$
$$=\frac{1}{a(k)}\cdot\frac{1}{2^2\pi}\cdot\frac{1}{[(P_M\cap\Gamma)_0:(P_M\cap\Gamma)_0^+]}\cdot\frac{\mathrm{vol}((P_M\cap\Gamma)_0^+\backslash H_1)}{\mathrm{vol}(C_0(\gamma\,;\Gamma)\backslash C_0(\gamma\,;G_R))}.$$

(Note that the expression in (5.6) is independent of the choice of $g\in G_Q$.) In fact, the last assertion follows from

PROPOSITION 5-1.[1] *Notations being as above, one has*

(5.7)
$$\lim_{s\to+0}s\sum_{S\in L^+\bmod(P_M\cap\Gamma)_0}\frac{1}{\sharp O_\Gamma(S)(\det S)^{s+3/2}}$$
$$=\frac{[G(L):(P_M\cap\Gamma)_0]}{4[G(L):G(L)^+]}\cdot\frac{\mathrm{vol}(G(L)^+\backslash H_1)}{\mathrm{vol}(L\backslash SM_2(\boldsymbol{R}))}.$$

PROOF. We prove the assertion in the case where $\Gamma$ is commensurable with $\mathrm{Sp}(2,\boldsymbol{Z})$. The $Q$-rank one case is more or less easier. Also, we assume, to simplify the notations, that $\Gamma$ is contained in $\mathrm{Sp}(2,Q)$. Then there exists $A\in GL_3(Q)$ such that
$$L=\left\{\begin{pmatrix}x_1 & x_2\\ x_2 & x_3\end{pmatrix};\ A^{-1}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\in\boldsymbol{Z}^3\right\}.$$
Put
$$\mathfrak{S}={}^tA\mathfrak{S}_0A,\quad \mathfrak{S}_0=\begin{pmatrix}0 & 0 & 1/2\\ 0 & -1 & 0\\ 1/2 & 0 & 0\end{pmatrix}.$$
Let $\rho_A$ be the homomorphism from $\{V\in GL_2(\boldsymbol{R})\,;\det V=\pm1\}$ to $O(\mathfrak{S})\cong O(1,2)$, defined by

(5.8)
$$\rho_A\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=A^{-1}\begin{pmatrix}a^2 & 2ab & b^2\\ ac & ad+bc & bd\\ c^2 & 2cd & d^2\end{pmatrix}A.$$

The image of $\rho_A$ is $SO_0(\mathfrak{S})\cup\rho_A\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)SO_0(\mathfrak{S})$. An element $U$ of $GL_3(\boldsymbol{Z})$ is called a proper unit of $\mathfrak{S}$, if ${}^tU\mathfrak{S}U=\mathfrak{S}$ and $U$ transforms each of the two connected components of the set $\{x\in\boldsymbol{R}^3\,;{}^tx\mathfrak{S}x>0\}$ onto themselves. Denote by $\Gamma(\mathfrak{S})$ the group of proper units of $\mathfrak{S}$, and by $\Gamma(\mathfrak{S},m)$ the subgroup of it consisting of $U\in\Gamma(\mathfrak{S})$ such that $Um=m$, for each $m\in\boldsymbol{Z}^3$. Also, denote by $S_m$ the element $\begin{pmatrix}x_1 & x_2\\ x_2 & x_3\end{pmatrix}$ of $L$ such that $Am={}^t(x_1,x_2,x_3)$. Then it is easy to see that
$$\Gamma(\mathfrak{S},m)=\rho_A(\{V\in G(L)\,;VS_m{}^tV=S_m\}).$$
In [31], Siegel defined a zeta-function attached to a lattice in the space of binary

[1] The author was informed of this proposition by Professor T. Ibukiyama. For a different proof, we refer to F. Sato: On zeta functions of ternary zero forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1982), 585-604.



quadratic forms. In our situation, it is expressed as

(5.9)
$$\zeta(s,\mathfrak{S})=\sum_{S_m\in L^+\bmod G(L)}\frac{1}{\sharp\Gamma(\mathfrak{S},m)({}^tm\mathfrak{S}m)^s}\qquad(\mathrm{Re}\,s>3/2).$$

By an easy argument using Lemma 5-1, we see that this function can be written as
$$\frac{2}{[G(L):(P_M\cap\Gamma)_0]}\sum_{S\in L^+\bmod(P_M\cap\Gamma)_0}\frac{1}{\sharp O_\Gamma(S)(\det S)^s}.$$
Let $\phi$ be the natural map from the set $\{x\in\boldsymbol{R}^3\,;{}^tx\mathfrak{S}x>0\}$ to $P^2(\boldsymbol{R})$, the real projective plane: $\phi(x)=(x_1:x_2:x_3)$. In [31] Satz 3, it is proved that the residue of $\zeta(s,\mathfrak{S})$ at $s=3/2$ is expressed by the following formula

(5.10)
$$\lim_{s\to3/2}(s-3/2)\zeta(s,\mathfrak{S})=\frac{1}{2}\int_{F(\mathfrak{S})}\omega,$$

where $\omega$ is a 2-form on $P^2(\boldsymbol{R})$ such that $\omega=({}^tx\mathfrak{S}x)^{-3/2}x_3dx_1dx_2$, and $F(\mathfrak{S})$ is a fundamental domain of $\Gamma(\mathfrak{S})=\rho_A(G(L))$ in the image of $\phi$. Now put $x=A^{-1}y$. Then we have the following commutative diagram
$$\begin{array}{ccccc}\{x\in\boldsymbol{R}^3\,;{}^tx\mathfrak{S}x>0\} & \xrightarrow{\ \phi\ } & \mathrm{Im}\,\phi & \xrightarrow{\ /\Gamma(\mathfrak{S})\ } & F(\mathfrak{S})\\ \downarrow A & & \downarrow A & & \downarrow A\\ \{y\in\boldsymbol{R}^3\,;{}^ty\mathfrak{S}_0y>0\} & \xrightarrow{\ \phi\ } & \mathrm{Im}\,\phi & \xrightarrow{\ /\Gamma(\mathfrak{S}_0)\ } & F(\mathfrak{S}_0)\end{array}$$
where $F(\mathfrak{S}_0)$ is a fundamental domain of $\Gamma(\mathfrak{S}_0)=\rho_1(G(L))$, $\rho_1$ being given as in (5.8) with $A=1_3$. Note that $A$ can be taken as an upper triangular matrix. It follows easily that we have $\omega=(\det A)^{-1}({}^ty\mathfrak{S}_0y)^{-3/2}y_3dy_1dy_2$. Now put $y_1/y_3=x^2+y^2$, $y_2/y_3=x$, and consider the map
$$f:\mathrm{Im}\,\phi\cap\{y_3\neq0\}\longrightarrow H_1$$
$$f((y_1:y_2:y_3))=x+iy\qquad(y>0).$$
Then it is easy to see that $f$ is a diffeomorphism and commute with the action of $SL_2(\boldsymbol{R})$ (cf. Lemma 5-2). Moreover, $\rho_1\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$ acts on $H_1$ as $x+iy\mapsto -x+iy$. Therefore the right hand side of (5.10) is equal to
$$\frac{1}{2[G(L):G(L)^+]}(\det A)^{-1}\int_{G(L)^+\backslash H_1}\frac{dxdy}{y^2}$$
$$=\frac{1}{2[G(L):G(L)^+]}\cdot\frac{\mathrm{vol}(G(L)^+\backslash H_1)}{\mathrm{vol}(L\backslash SM_2(\boldsymbol{R}))}.$$
This completes the proof. q.e.d.

**5-3.** Let us next consider $\gamma=\gamma(S)$, where $S$ defines an indefinite quadratic form over $Q$. Put

$$C_0(\gamma\,;G_R)=\begin{cases} C(\gamma\,;G_R) & \text{if } -\det S\notin(Q^\times)^2 \\ \begin{pmatrix}1 & SM_2(\boldsymbol{R})\\ 0 & 1\end{pmatrix} & \text{if } -\det S\in(Q^\times)^2.\end{cases}$$

Note that the second case occurs if and only if $G_Q$ has $Q$-rank two (cf. Theorem 1-5). In [18], Theorem 9, it is showed that the integral $I_0(\gamma\,;s)$ vanishes in the first case; and in the second case, it is equal to

$$\left\{-\frac{2^5\pi}{(2k-2)(2k-3)(2k-4)}+o(s)\right\}\cdot\frac{1}{|\det S|^{3/2}},$$

where $o(s)$ is independent of $S$. Therefore, in the following we assume that $-\det S\in(Q^\times)^2$. Let $\gamma$ be an element of $\Gamma$, which is conjugate in $G_Q$ to $\gamma(S)$. Then the family $[\gamma]_\Gamma$ is given by

(5.11) $$[\gamma]_\Gamma=g\begin{pmatrix}1 & L^s\\ 0 & 1\end{pmatrix}g^{-1},$$

where $L$ is as in (5.2), and $L^s=\{S\in L\,;-\det S\in(Q^\times)^2\}$.

LEMMA 5-2. *Let* $S=\begin{pmatrix}s_1 & s_{12}\\ s_{12} & s_2\end{pmatrix}$ *be any non-degenerate symmetric matrix with real coefficients and let* $z,\ z'\in C\cup\{\infty\}$ *be defined by the identity*

$$s_1X^2+2s_{12}XY+s_2Y^2=s_2(zX+Y)(z'X+Y)=s_1(X+z^{-1}Y)(X+z'^{-1}Y),$$

*with* $\operatorname{Im} z\geqq 0$, $\operatorname{Im} z'\leqq 0$. *Then, if we transform* $S$ *by* $A\in SL_2(\boldsymbol{R})$ *to* $AS^tA$, $(z,\ z')$ *is transformed to* $\left(\dfrac{az+b}{cz+d},\ \dfrac{az'+b}{cz'+d}\right)$, *where* $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

The proof is straightforward. By this lemma, we have a bijection

(5.12) $$f:L^s \bmod (P_M\cap\Gamma)_0^+\xrightarrow{\sim}(Q\cup\{\infty\})^2 \bmod (P_M\cap\Gamma)_0^+.$$

Let $\beta_j$ $(1\leqq j\leqq t)$ be a complete set of representatives of cusps of the Fuchsian group $(P_M\cap\Gamma)_0^+$. For each $\beta_j$, let $B_j$ be the stabilizor of $\beta_j$ in $(P_M\cap\Gamma)_0^+$. Put, for each positive rational number $d$,

(5.13) $$a_j(d)=\#(\{S\in L^s\,;\ -\det S=d^2,\ f(S)=(*,\ \beta_j)\}/B_j).$$

Then we have the following

PROPOSITION 5-2. *Notations being as above, there exist positive rational numbers* $d_j,\ c_j$ $(1\leqq j\leqq t)$, *such that*

(5.14) $$\begin{cases} a_j(d)=0 & \text{if } d\not\equiv 0 \pmod{d_j}\\ a_j(nd_j)=nc_j & \text{for all } n\in\boldsymbol{N}.\end{cases}$$

PROOF. Taking $V\in SL_2(Q)$ such that $V\langle\beta_j\rangle=\infty$ and considering $VB_jV^{-1}$ instead of $B_j$, we may assume that $\beta_j=\infty$. Then



(5.15) $$L_j^s=:f^{-1}((Q,\ \infty))=\left\{\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\right\}\cap L$$

is a lattice of $\left\{\begin{pmatrix}* & *\\ * & 0\end{pmatrix}\right\}\cap SM_2(\boldsymbol{R})$. It has a unique basis of the form

$$\begin{pmatrix}t_1 & 0\\ 0 & 0\end{pmatrix},\ \begin{pmatrix}t_2 & d_j\\ d_j & 0\end{pmatrix},\quad d_j>0,\quad t_1>|t_2|\geqq 0.$$

It follows that $-\det S\in d_j^2N$ for any $S\in L_j^s$, and that

(5.16) $$B_j=\left\{\pm\begin{pmatrix}1 & \frac{t_1c_j}{2d_j}\boldsymbol{Z}\\ 0 & 1\end{pmatrix}\right\}\quad (\text{for some } c_j\in\boldsymbol{N}).$$

The assertion follows easily from these facts. q. e. d.

By Lemmas 5-1, 5-2 and Proposition 5-2, the contribution of the family $[\gamma]_\Gamma$ is

$$\sum_{\delta\in[\gamma]_\Gamma/\widetilde{\Gamma}}\frac{I_0(\gamma'\,;s)}{[\bar C(\gamma'\,;\Gamma):\bar C_0(\gamma'\,;\Gamma)]}$$
$$=\frac{1}{2}\sum_{S\in L^s \bmod (P_M\cap\Gamma)_0^+} I_0(\gamma'\,;s)$$
$$=\left\{-\frac{1}{a(k)}\cdot\frac{1}{2^4\pi^2}+o(s)\right\}\sum_{j=1}^t\ \sum_{S\in L_j^s \bmod B_j}\frac{1}{|\det S|^{3/2}}$$
$$=\left\{-\frac{1}{a(k)}\cdot\frac{1}{2^4\pi^2}+o(s)\right\}\sum_{j=1}^t\frac{c_j}{d_j^3}\sum_{n=1}^\infty\frac{1}{n^2}.$$

Summing up, we have proved the following

THEOREM I-10. (i) *Suppose* $\gamma$ *is conjugate in* $G_Q$ *to* $\gamma(S)$, *and* $S$ *indefinite. Then*

(5.17) $$I_0(\gamma\,;s)=\begin{cases}\left\{-\dfrac{1}{a(k)}\cdot\dfrac{1}{2^3\pi^2}+o(s)\right\}\dfrac{1}{|\det S|^{3/2}} & \cdots \text{if } -\det S\in(Q^\times)^2\\ 0 \cdots\cdots\cdots\cdots & \text{if } -\det S\notin(Q^\times)^2.\end{cases}$$

(ii) *Assume that* $\gamma\in\Gamma$, *and let* $\beta_j$ $(1\leqq j\leqq t)$ *be a complete set of cusps of* $(P_M\cap\Gamma)_0^+$, *where* $P$ *is as in* § 5-1. *Then the totality of* $I_0(\gamma'\,;s)/[\bar C(\gamma'\,;\Gamma):\bar C_0(\gamma'\,;\Gamma)]$ *for* $\gamma'\in[\gamma]_\Gamma/\widetilde{\Gamma}$ *is expressed as above, and we have*

(5.18) $$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\widetilde{\Gamma}}\frac{I_0(\gamma'\,;s)}{[\bar C(\gamma'\,;\Gamma):\bar C_0(\gamma'\,;\Gamma)]}=-\frac{1}{a(k)}\cdot\frac{1}{2^53}\sum_{j=1}^t\frac{c_j}{d_j^3},$$

*where* $c_j,\ d_j$ *are as in Proposition* 5-2.

**5-4.** Finally consider $\gamma=\gamma(S)$, with $S=\begin{pmatrix}\lambda & 0\\ 0 & 0\end{pmatrix}$, $\lambda\neq 0$. We can put

$$C_0(\gamma\,;G_R)=C(\gamma\,;G_R)$$

(5.19)
$$=\left\{\pm\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix}\begin{pmatrix}1&s&u&t\\0&1&t&0\\0&0&1&-s\\0&0&0&1\end{pmatrix};\ \begin{matrix}ad-bc=1\\ s,\,t,\,u\in\boldsymbol{R}\end{matrix}\right\}.$$

From Lemma 3-1, it is easy to see that a fundamental domain of $C_0(\gamma\,;G_R)$ in $H_2$ can be taken as

$$F(\varepsilon_4)=\left\{\begin{pmatrix}iy_1&0\\0&i\end{pmatrix};\ y_1>0\right\}.$$

Expressing $Z\in H_2$ as

$$Z=\begin{pmatrix}1&0&0&0\\0&\sqrt{y_2}&0&x_2\sqrt{y_2}^{-1}\\0&0&1&0\\0&0&0&\sqrt{y_2}^{-1}\end{pmatrix}\begin{pmatrix}1&s&u&t\\0&1&t&0\\0&0&1&-s\\0&0&0&1\end{pmatrix}\left\langle\begin{pmatrix}iy_1&0\\0&i\end{pmatrix}\right\rangle$$
$$=\begin{pmatrix}(y_1+s^2)i+(u+st)&(si+t)\sqrt{y_2}\\(si+t)\sqrt{y_2}&x_2+iy_2\end{pmatrix},$$

it follows that

$$dZ=(y_1^{-3}dy_1)(y_2^{-2}dx_2dy_2)(dsdtdu).$$

Moreover, the stabilizer of $Z=\begin{pmatrix}iy_1&0\\0&i\end{pmatrix}$ in $C_0(\gamma\,;G_R)$ is easily seen to be $\left\{\begin{pmatrix}1&0&0&0\\0&\cos\theta&0&\sin\theta\\0&0&1&0\\0&-\sin\theta&0&\cos\theta\end{pmatrix}\right\}$, which is independent of $Z$. Therefore we get

PROPOSITION 5-3. *For any measurable function $f(Z)$ on $H_2$, the following integral formula holds*

(5.20)
$$\int_{H_2}f(Z)dZ=\frac{1}{2\pi}\int_{C_0(\gamma;G_R)}\int_{F(\varepsilon_4)}f\left(h\left\langle\begin{pmatrix}iy_1&0\\0&i\end{pmatrix}\right\rangle\right)dh\,d\hat{Z}$$

*where* $dh=d\alpha dsdtdu$ $\left(\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{R})\right)$ *is the standard Haar measure of* $C_0(\gamma\,;G_R)$, *and* $d\hat{Z}=y_1^{-3}dy_1$.

From this follows that

$$I_0(\gamma(S)\,;s)=\frac{1}{2\pi}\int_{F(\varepsilon_4)}H_\gamma(\hat{Z})d\hat{Z}$$
$$=\frac{1}{2\pi}\int_0^\infty\frac{y^{k-3}}{(y+i\lambda/2)^k}dy$$
$$=-\frac{2}{\pi}\frac{1}{(k-1)(k-2)}\cdot\frac{1}{|\lambda|^2}.$$



If $\gamma$ is an element of $\Gamma$ which is conjugate in $G_Q$ to $\gamma(S)$, then choosing $g$ in (5.2) suitably, the family $[\gamma]_\Gamma$ is given by

(5.21)
$$[\gamma]_\Gamma=g\left\{\begin{pmatrix}1&0&dn&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix};\ \begin{matrix}n\in\boldsymbol{Z}\\n\neq0\end{matrix}\right\}g^{-1}$$
$$\subseteq g\begin{pmatrix}1&L^d\\0&1\end{pmatrix}g^{-1},$$

where $L^d=\{S\in L\,;\ \det S=0\}$, and $d$ is a positive rational number. Then we easily have the following

THEOREM I-11. (i) *For $\gamma=\gamma(S)$ with $S=\begin{pmatrix}\lambda&0\\0&0\end{pmatrix}$, $\lambda\neq0$, we have*

(5.22)
$$I_0(\gamma\,;s)=-\frac{1}{a(k)}\cdot\frac{2k-3}{2^5\pi^4}\cdot\frac{1}{|\lambda|^2}.$$

(ii) *If $\gamma\in\Gamma$ is conjugate in $G_Q$ to $\gamma(S)$ above, then the family $[\gamma]_\Gamma$ can be expressed as in* (5.21), *and we have*

(5.23)
$$\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\widetilde{\Gamma}}\frac{I_0(\gamma'\,;s)}{[\bar{C}(\gamma'\,;\Gamma):\bar{C}_0(\gamma'\,;\Gamma)]}=-\frac{1}{a(k)}\cdot\frac{2k-3}{2^53\pi^2}\cdot\frac{1}{d^2}.$$

REMARK 5-1. In Theorem I-10, and I-11, we have used the fact $\zeta(2)=\pi^2/6$. It should be noted, in connection with our Problem 2 in the introduction, that this special value of Riemann's zeta-function may also be used to evaluate the volume of $(P_M\cap\Gamma)_0^+\backslash H_1$ which appeared in Theorem I-9.

**5-5.** We shall now resume the results obtained so far.

THEOREM 5-1. (Main Theorem) *Under the assumption* 5-1 *on $\Gamma$, the dimension of the space $S_k(\Gamma)$ of cusp forms of weight $k\geqq5$ for a lattice $\Gamma$ of* Sp(2, $\boldsymbol{R}$) *is given in the following formula*:

(5.24)
$$\dim S_k(\Gamma)=\frac{a(k)}{\sharp Z(\Gamma)}\sum_{[\gamma]_\Gamma}\frac{\mathrm{vol}(C_0(\gamma\,;\Gamma)\backslash C_0(\gamma\,;G_R))}{[\bar{C}(\gamma\,;\Gamma):\bar{C}_0(\gamma\,;\Gamma)]}I_0(\gamma)$$
$$+\frac{a(k)}{\sharp Z(\Gamma)}\sum_{[\gamma]_\Gamma}\frac{\mathrm{vol}(C_0(\gamma\,;\Gamma)\backslash C_0(\gamma\,;G_R))}{[\bar{C}(\gamma\,;\Gamma):\bar{C}_0(\gamma\,;\Gamma)]}\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma}I_0(\gamma'\,;s)$$
$$+\frac{a(k)}{\sharp Z(\Gamma)}\sum_{[\gamma]_\Gamma}\mathrm{vol}(C_0(\gamma\,;\Gamma)\backslash C_0(\gamma\,;G_R))\lim_{s\to+0}\sum_{\gamma'\in[\gamma]_\Gamma/\widetilde{\Gamma}}\frac{I_0(\gamma'\,;s)}{[\bar{C}(\gamma'\,;\Gamma):\bar{C}_0(\gamma'\,;\Gamma)]},$$

*where, in the first term the sum is extended over the set of conjugacy classes of elliptic and central elements in $\Gamma$, and in the second* (*resp. third*) *term, $[\gamma]_\Gamma$ runs over a complete set of representatives of conjugacy classes of elliptic/parabolic, paraelliptic, and $\delta$-parabolic* (*resp. parabolic*) *elements. $I_0(\gamma)$ and the limit of the* *sum of* $I_0(\gamma'; s)$ *have been evaluated in Theorems* I-1, $\cdots$, I-11.

REMARK 5-2. It should be noted that no two elements of $[\gamma]_\Gamma$ are conjugate in $\Gamma$, unless $\gamma$ is parabolic. If $\Gamma\backslash H_2$ is compact, the second and third terms do not appear. If $\Gamma$ satisfies the condition (1.5), the first term is expressed more simply as in Theorem 2-4.

## § 6. Application (1): the case of $\Gamma=\mathrm{Sp}(2, \boldsymbol{Z})$

In this section, as an illustration of the validity of our general formula for $\dim S_k(\Gamma)$ in Theorem 5-1, we shall specialize it to the case $\Gamma=\mathrm{Sp}(2, \boldsymbol{Z})$; thus putting all known data on conjugacy classes of $\mathrm{Sp}(2, \boldsymbol{Z})$ into our formula (5.24), we shall derive a completely explicit formula for $\dim S_k(\mathrm{Sp}(2, \boldsymbol{Z}))$, and see that it coincides with the one that has been given in Igusa [15] (see (0.3)). All data that we quote here will be used also in the next section.

**6-1.** We quote the following results on conjugacy classes, from Münchhausen [19], [20]. For elliptic classes, the same results are obtained also by Sakamoto [22].

THEOREM 6-1. *The complete list of representatives of* $\mathrm{Sp}(2, \boldsymbol{Z})$ *which are of type as in Theorem* 1-7 *are as follows*:

(a) *central*: $\pm\alpha_0=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\qquad (x-1)^4$

(b) *elliptic*:

(b-2) $\pm\alpha_1=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\sim\alpha(\pi/2, \pi/2)$ $\qquad (x^2+1)^2$

$\pm\alpha_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}\sim\alpha(2\pi/3, 2\pi/3)$ $\qquad (x^2+x+1)^2$

$\pm\alpha_3=\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim\alpha(-2\pi/3, -2\pi/3)$ $\qquad (x^2+x+1)^2$



(b-1) $\alpha_4=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sim\alpha(-\pi/4, 3\pi/4)$ $\qquad (x^4+1)$

$\alpha_5=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\sim\alpha(-3\pi/4, \pi/4)$ $\qquad (x^4+1)$

$\pm\alpha_6=\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim\alpha(\pi/4, 3\pi/4)$ $\qquad (x^4+1)$

$\pm\alpha_7=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim\alpha(-\pi/3, -2\pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\pm\alpha_8=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\sim\alpha(-2\pi/3, -\pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\alpha_9=\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sim\alpha(2\pi/3, -\pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\alpha_{10}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}\sim\alpha(-2\pi/3, \pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\alpha_{11}=\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\sim\alpha(-2\pi/3, \pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\alpha_{12}=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\sim\alpha(2\pi/3, -\pi/3)$ $\qquad (x^2+x+1)(x^2-x+1)$

$\alpha_{13}=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sim\alpha(-\pi/6, 5\pi/6)$ $\qquad (x^4-x^2+1)$

$$\alpha_{14}=\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&0&0&1\\-1&0&0&0\end{pmatrix}\sim\alpha(\pi/6,\ -5\pi/6)\qquad (x^4-x^2+1)$$

$$\pm\alpha_{15}=\begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\\-1&0&0&-1\end{pmatrix}\sim\alpha(2\pi/5,\ -4\pi/5)\qquad (x^4+x^3+x^2+x+1)$$

$$\pm\alpha_{16}=\begin{pmatrix}-1&0&1&0\\0&0&0&1\\-1&0&0&-1\\1&-1&-1&0\end{pmatrix}\sim\alpha(4\pi/5,\ 2\pi/5)\qquad (x^4+x^3+x^2+x+1)$$

$$\pm\alpha_{17}=\begin{pmatrix}0&-1&-1&0\\-1&0&0&-1\\1&-1&-1&0\\0&1&0&0\end{pmatrix}\sim\alpha(-4\pi/5,\ -2\pi/5)\qquad (x^4+x^3+x^2+x+1)$$

$$\pm\alpha_{18}=\begin{pmatrix}0&0&-1&-1\\1&-1&-1&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\sim\alpha(-2\pi/5,\ 4\pi/5)\qquad (x^4+x^3+x^2+x+1)$$

$$\pm\alpha_{19}=\begin{pmatrix}-1&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\sim\alpha(-2\pi/3,\ -\pi/2)\qquad (x^2+1)(x^2+x+1)$$

$$\pm\alpha_{20}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&-1&0\\0&-1&0&0\end{pmatrix}\sim\alpha(2\pi/3,\ \pi/2)\qquad (x^2+1)(x^2+x+1)$$

$$\pm\alpha_{21}=\begin{pmatrix}-1&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\sim\alpha(-2\pi/3,\ \pi/2)\qquad (x^2+1)(x^2+x+1)$$

$$\pm\alpha_{22}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&-1&0\\0&1&0&0\end{pmatrix}\sim\alpha(2\pi/3,\ -\pi/2)\qquad (x^2+1)(x^2+x+1)$$

(b-3)

$$\pm\beta_1=\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&-1&0\\0&0&0&1\end{pmatrix}\sim\alpha(-2\pi/3,\ 0)\qquad (x-1)^2(x^2+x+1)$$



$$\pm\beta_2=\begin{pmatrix}-1&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim\alpha(2\pi/3,\ 0)\qquad (x-1)^2(x^2+x+1)$$

$$\pm\beta_3=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{pmatrix}\sim\alpha(-\pi/3,\ 0)\qquad (x-1)^2(x^2-x+1)$$

$$\pm\beta_4=\begin{pmatrix}1&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\alpha(\pi/3,\ 0)\qquad (x-1)^2(x^2-x+1)$$

$$\pm\beta_5=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim\alpha(-\pi/2,\ 0)\qquad (x-1)^2(x^2+1)$$

$$\pm\beta_6=\begin{pmatrix}0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\alpha(\pi/2,\ 0)\qquad (x-1)^2(x^2+1)$$

(b-4)

$$\gamma_1=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\alpha(\pi/2,\ -\pi/2)\sim\gamma(\pi/2)\qquad (x^2+1)^2$$

$$\gamma_2=\begin{pmatrix}0&-1&1&0\\1&0&0&-1\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\alpha(\pi/2,\ -\pi/2)\sim\gamma(\pi/2)\qquad (x^2+1)^2$$

$$\pm\gamma_3=\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&0&-1&-1\\0&0&1&0\end{pmatrix}\sim\alpha(2\pi/3,\ -2\pi/3)\sim\gamma(2\pi/3)\qquad (x^2+x+1)^2$$

(b-5)

$$\delta_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\alpha(0,\ \pi)\qquad (x-1)^2(x+1)^2$$

$$\delta_2=\begin{pmatrix}1&0&0&-1\\0&-1&1&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\alpha(0,\ \pi)\qquad (x-1)^2(x+1)^2$$

(d) *elliptic/parabolic*

$$\pm\hat{\beta}_1(n)=\begin{pmatrix}0&0&1&0\\0&1&0&n\\-1&0&1&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(\pi/3,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2-x+1)$$

$$\pm\hat{\beta}_2(n)=\begin{pmatrix}1&0&-1&0\\0&1&0&n\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(-\pi/3,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2-x+1)$$

$$\pm\hat{\beta}_3(n)=\begin{pmatrix}0&0&-1&0\\0&1&0&n\\1&0&-1&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(4\pi/3,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2+x+1)$$

$$\pm\hat{\beta}_4(n)=\begin{pmatrix}-1&0&1&0\\0&1&0&n\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(2\pi/3,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2+x+1)$$

$$\pm\hat{\beta}_5(n)=\begin{pmatrix}-1&0&1&1\\-1&1&0&n\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(2\pi/3,\ n-1/3)\quad n\in\boldsymbol{Z}\qquad (x-1)^2(x^2+x+1)$$

$$\pm\hat{\beta}_6(n)=\begin{pmatrix}0&0&-1&0\\0&1&-1&n\\1&0&-1&-1\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(4\pi/3,\ n+1/3)\quad n\in\boldsymbol{Z}\qquad (x-1)^2(x^2+x+1)$$

$$\pm\hat{\beta}_7(n)=\begin{pmatrix}0&0&-1&0\\0&1&0&n\\1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(-\pi/2,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2+1)^2$$

$$\pm\hat{\beta}_8(n)=\begin{pmatrix}0&0&1&0\\0&1&0&n\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(\pi/2,\ n)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x^2+1)^2$$

$$\pm\hat{\beta}_9(n)=\begin{pmatrix}0&0&1&-1\\1&1&0&n\\-1&0&0&0\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(\pi/2,\ n-1/2)\quad n\in\boldsymbol{Z}\qquad (x-1)^2(x^2+1)^2$$



$$\pm\hat{\beta}_{10}(n)=\begin{pmatrix}0&0&-1&0\\0&1&1&n\\1&0&0&1\\0&0&0&1\end{pmatrix}\sim\hat{\beta}(-\pi/2,\ n+1/2)\quad n\in\boldsymbol{Z}\qquad (x-1)^2(x^2+1)^2$$

(j) *δ-parabolic*

(j-1) $$\hat{\hat{\hat{\delta}}}_1(m,\ n)=\begin{pmatrix}1&0&m&0\\0&-1&0&n\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(m,\ n)\quad \begin{matrix}(m,\ n)\in\boldsymbol{Z}^2\\ m,\ n\neq 0\end{matrix}\qquad (x-1)^2(x+1)^2$$

$$\hat{\hat{\hat{\delta}}}_2(m,\ n)=\begin{pmatrix}1&0&m&-1\\0&-1&1&n\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(m,\ n)\quad \begin{matrix}(m,\ n)\in\boldsymbol{Z}^2\\ m,\ n\neq 0\end{matrix}\qquad (x-1)^2(x+1)^2$$

$$\hat{\hat{\hat{\delta}}}_3(m,\ n)=\begin{pmatrix}1&0&2m&m+2\\1&-1&m-2&n\\0&0&1&1\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(m,\ 2n-m)\quad \begin{matrix}(m,\ n)\in\boldsymbol{Z}^2\\ m,\ 2n-m\neq 0\end{matrix}\qquad (x-1)^2(x+1)^2$$

$$\hat{\hat{\hat{\delta}}}_4(m,\ n)=\begin{pmatrix}1&0&2m-1&m\\1&-1&m-1&n\\0&0&1&1\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(m-1/2,\ 2n-m+1/2)\quad (m,\ n)\in\boldsymbol{Z}^2\qquad (x-1)^2(x+1)^2$$

(j-2) $$\pm\hat{\delta}_1(n)=\begin{pmatrix}1&0&n&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(n,\ 0)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x+1)^2$$

$$\pm\hat{\delta}_2(n)=\begin{pmatrix}1&0&n&-1\\0&-1&1&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\sim\hat{\delta}(n,\ 0)\quad \begin{matrix}n\in\boldsymbol{Z}\\ n\neq 0\end{matrix}\qquad (x-1)^2(x+1)^2$$

(k) *parabolic*

(k-1) $$\pm\varepsilon_1(S)=\begin{pmatrix}1&0&s_1&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad S=\begin{pmatrix}s_1&s_{12}\\s_{12}&s_2\end{pmatrix}\in SM_2(\boldsymbol{Z})\qquad (x-1)^4$$

*reduced, definite*

(k-2) $$\pm\varepsilon_2(S)=\begin{pmatrix}1&0&s_1&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad S=\begin{pmatrix}s_1&s_{12}\\s_{12}&s_2\end{pmatrix}\in SM_2(\boldsymbol{Z})\qquad (x-1)^4$$

*reduced, indefinite,*

$-\det S\notin(Q^\times)^2$

$$\pm\varepsilon_3(S)=\begin{pmatrix}1&0&0&s_{12}\\0&1&s_{12}&s_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad \begin{array}{l}S=\begin{pmatrix}0&s_{12}\\s_{12}&s_2\end{pmatrix}\in SM_2(\boldsymbol{Z})\\ 0\leqq s_2<2s_{12}\end{array}\qquad (x-1)^4$$

(k-3)
$$\pm\varepsilon_4(S)=\begin{pmatrix}1&0&0&0\\0&1&0&n\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad \begin{array}{l}S=\begin{pmatrix}0&0\\0&n\end{pmatrix}\\ n\in\boldsymbol{Z},\ n\neq0\end{array}\qquad (x-1)^4$$

(l) *paraelliptic*

$$\hat{\gamma}_1(n)=\begin{pmatrix}0&-1&0&-n\\1&0&n&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(-\pi/2,\ n)\quad \begin{array}{l}n\in\boldsymbol{Z}\\ n\neq0\end{array}\qquad (x^2+1)^2$$

$$\hat{\gamma}_2(n)=\begin{pmatrix}0&-1&0&-n\\1&0&n+1&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(-\pi/2,\ n+1/2)\quad n\in\boldsymbol{Z}\qquad (x^2+1)^2$$

$$\hat{\gamma}_3(n)=\begin{pmatrix}0&-1&1&-n\\1&0&n&-1\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(-\pi/2,\ n)\quad \begin{array}{l}n\in\boldsymbol{Z}\\ n\neq0\end{array}\qquad (x^2+1)^2$$

$$\hat{\gamma}_4(n)=\begin{pmatrix}0&-1&1&-n\\1&0&n+1&-1\\0&0&0&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(-\pi/2,\ n+1/2)\quad n\in\boldsymbol{Z}\qquad (x^2+1)^2$$

$$\pm\hat{\gamma}_5(n)=\begin{pmatrix}0&-1&-n&-2n\\1&-1&n&-n\\0&0&-1&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(2\pi/3,\ n)\quad \begin{array}{l}n\in\boldsymbol{Z}\\ n\neq0\end{array}\qquad (x^2+x+1)^2$$

$$\pm\hat{\gamma}_6(n)=\begin{pmatrix}0&-1&-n&-2n\\1&-1&n+1&-n\\0&0&-1&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(2\pi/3,\ n+1/3)\quad n\in\boldsymbol{Z}\qquad (x^2+x+1)^2$$

$$\pm\hat{\gamma}_7(n)=\begin{pmatrix}0&-1&-n&-2n\\1&-1&n+2&-n\\0&0&-1&-1\\0&0&1&0\end{pmatrix}\sim\hat{\gamma}(2\pi/3,\ n+2/3)\quad n\in\boldsymbol{Z}\qquad (x^2+x+1)^2.$$

Here we note that "$\pm$" means that $-\gamma$ should be added in the list, although we are writing $+\gamma$ alone.



REMARK 6-1. The parametrization of non-elliptic families are slightly different from that of Münchhausen [19]. Also we are taking different representatives from [19], in some cases, in order to make them to be mutually commutative in each family, or in other words, that they should be transformed into the canonical forms *simultaneously*.

**6-2.** Here we shall describe briefly the centralizers $C(\gamma;\Gamma)$, $\Gamma=\mathrm{Sp}(2,\boldsymbol{Z})$, for each representatives of conjugacy classes listed above. Again, the most part of the results are contained in Münchhausen [20], and in Sakamoto [22] for elliptic classes.

(b-1) *and* (b-2): For these classes, $C(\gamma;\Gamma)$ are finite groups. Their orders are as follows.

$$\begin{array}{l}\sharp C(\alpha_1;\Gamma)=32,\quad \sharp C(\alpha_2;\Gamma)=\sharp C(\alpha_3;\Gamma)=72,\\ \sharp C(\alpha_4;\Gamma)=\sharp C(\alpha_5;\Gamma)=8,\quad \sharp C(\alpha_6;\Gamma)=8,\quad \sharp C(\alpha_7;\Gamma)=12,\\ \sharp C(\alpha_8;\Gamma)=36,\quad \sharp C(\alpha_9;\Gamma)=\sharp C(\alpha_{10};\Gamma)=12,\\ \sharp C(\alpha_{11};\Gamma)=\sharp C(\alpha_{12};\Gamma)=36,\quad \sharp C(\alpha_{13};\Gamma)=\sharp C(\alpha_{14};\Gamma)=12,\\ \sharp C(\alpha_{15};\Gamma)=\sharp C(\alpha_{16};\Gamma)=\sharp C(\alpha_{17};\Gamma)=\sharp C(\alpha_{18};\Gamma)=10,\\ \sharp C(\alpha_{19};\Gamma)=\sharp C(\alpha_{20};\Gamma)=\sharp C(\alpha_{21};\Gamma)=\sharp C(\alpha_{22};\Gamma)=24.\end{array}$$

(b-3): Let $C_0(\beta;G_R)$ be as (3.2). Then $C_0(\beta;G_R)\cong SL_2(\boldsymbol{R})$ and $C_0(\beta;\Gamma)$ is a lattice of $C_0(\beta;G_R)$. We have

$$\begin{aligned}C(\beta_1;\Gamma)&=C(\beta_2;\Gamma)=C(\beta_3;\Gamma)=C(\beta_4;\Gamma)\\ &=\coprod_{j=0}^{5}\beta_3^j\,C_0(\beta_i;\Gamma)\qquad (1\leqq i\leqq 6),\end{aligned}$$

$$C_0(\beta_i;\Gamma)=\left\{\begin{pmatrix}1&0&0&0\\0&a&0&b\\0&0&1&0\\0&c&0&d\end{pmatrix};\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{Z})\right\}\qquad (1\leqq i\leqq 6).$$

$$[C(\beta_i;\Gamma):\{\pm1\}C_0(\beta_i;\Gamma)]=3\qquad (1\leqq i\leqq 4).$$

$$C(\beta_5;\Gamma)=C(\beta_6;\Gamma)=\coprod_{j=0}^{3}\beta_5^j\,C_0(\beta_i;\Gamma)\qquad (1\leqq i\leqq 6),$$

$$[C(\beta_i;\Gamma):\{\pm1\}C_0(\beta_i;\Gamma)]=2\qquad (i=5,\ 6).$$

(b-4): Let $\gamma=\gamma(\theta)$ be as in Theorem 1-1, and let $C_0(\gamma;G_R)$ be as (3.8). Then $C_0(\gamma;G_R)\cong SL_2(\boldsymbol{R})$, and for each $\gamma_i\in\Gamma$ such that $\gamma_i=g^{-1}\gamma(\theta)g$, $gC_0(\gamma_i;\Gamma)g^{-1}$ is a lattice of $C_0(\gamma;G_R)$. Put, for each rational number $p$, $q$,

(6.1)
$$\Gamma(\gamma;p,q)=\left\{\begin{pmatrix}a&0&b&0\\0&a&0&b\\c&0&d&0\\0&c&0&d\end{pmatrix};\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\begin{pmatrix}\boldsymbol{Z}&q\boldsymbol{Z}\\p\boldsymbol{Z}&\boldsymbol{Z}\end{pmatrix}\cap SL_2(Q)\right\}.$$

Then we have $C_0(\gamma_1;\Gamma)=\Gamma(\gamma;1,1)$, and

$$C(\gamma_1;\Gamma)=\coprod_{j=0,1}\gamma_1^j\,\Gamma(\gamma;1,1),$$

$$[C(\gamma_1;\Gamma):C_0(\gamma_1;\Gamma)]=2.$$

$$C(\gamma_2;\Gamma)=g\Big(\coprod_{j,k,l=0,1}h_1^jh_2^kh_3^l\,\Gamma(\gamma;4,1)\Big)g^{-1},$$

$$g=\begin{pmatrix}1&0&0&1/2\\0&1&1/2&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad h_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad h_3=\begin{pmatrix}4&0&3/2&0\\0&4&0&3/2\\10&0&4&0\\0&10&0&4\end{pmatrix},$$

$$h_2=\begin{pmatrix}3/2&0&1/4&0\\0&3/2&0&1/4\\1&0&1/2&0\\0&1&0&1/2\end{pmatrix}\begin{pmatrix}1&1&0&0\\-1&1&0&0\\0&0&1&1\\0&0&-1&1\end{pmatrix},$$

$$[C_0(\gamma_2;\Gamma):g\Gamma(\gamma;4,1)g^{-1}]=2,$$

$$[C(\gamma_2;\Gamma):C_0(\gamma_2;\Gamma)]=4.$$

$$C(\gamma_3;\Gamma)=g\Big(\coprod_{j=1}^{6}h_j\,\Gamma(\gamma;6,1/2)\Big)g^{-1},$$

$$g=\begin{pmatrix}2&0&0&0\\1&-\sqrt{3}&0&0\\0&0&1/2&\sqrt{3}/6\\0&0&0&-\sqrt{3}/3\end{pmatrix},\quad h_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$h_2=\begin{pmatrix}1/2&\sqrt{3}/2&0&0\\-\sqrt{3}/2&1/2&0&0\\0&0&1/2&\sqrt{3}/2\\0&0&-\sqrt{3}/2&1/2\end{pmatrix}$$

$$h_3=\begin{pmatrix}1/2&-\sqrt{3}/2&0&0\\\sqrt{3}/2&1/2&0&0\\0&0&1/2&-\sqrt{3}/2\\0&0&\sqrt{3}/2&1/2\end{pmatrix},$$

$$h_4=\begin{pmatrix}0&\sqrt{3}/3&0&0\\-\sqrt{3}/3&0&0&0\\0&0&0&\sqrt{3}/3\\0&0&-\sqrt{3}/3&0\end{pmatrix}\begin{pmatrix}0&0&1/2&0\\0&0&0&1/2\\-6&0&0&0\\0&-6&0&0\end{pmatrix}$$

$$h_5=\begin{pmatrix}1/2&\sqrt{3}/6&0&0\\-\sqrt{3}/6&1/2&0&0\\0&0&1/2&\sqrt{3}/6\\0&0&-\sqrt{3}/6&1/2\end{pmatrix}\begin{pmatrix}0&0&1/2&0\\0&0&0&1/2\\-6&0&0&0\\0&-6&0&0\end{pmatrix}$$



$$h_6=\begin{pmatrix}1/2&-\sqrt{3}/6&0&0\\\sqrt{3}/6&1/2&0&0\\0&0&1/2&-\sqrt{3}/6\\0&0&\sqrt{3}/6&1/2\end{pmatrix}\begin{pmatrix}0&0&1/2&0\\0&0&0&1/2\\-6&0&0&0\\0&-6&0&0\end{pmatrix},$$

$$C_0(\gamma_3;\Gamma)=g\Gamma(\gamma;6,1/2)g^{-1},$$

$$[C(\gamma_3;\Gamma):C_0(\gamma_3;\Gamma)]=6.$$

(b-5): Let $C_0(\delta_1;G_R)=C(\delta_1;G_R)$. Then $C_0(\delta_1;G_R)\cong SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$, and $C_0(\delta_1;\Gamma)$ is a lattice of $C_0(\delta_1;G_R)$, isomorphic to $SL_2(\boldsymbol{Z})\times SL_2(\boldsymbol{Z})$.

$$C_0(\delta_2;\Gamma)=C(\delta_2;\Gamma)=g\left\{\begin{pmatrix}a_1&0&b_1/2&0\\0&a_2&0&b_2/2\\2c_1&0&d_1&0\\0&2c_2&0&d_2\end{pmatrix};\ \begin{matrix}a_i,\,b_i,\,c_i,\,d_i\in\boldsymbol{Z}\\a_id_i-b_ic_i=1\\a_1\equiv d_2,\ d_1\equiv a_2\\b_1\equiv c_2,\ c_1\equiv b_2\end{matrix}\ (\mathrm{mod}\ 2)\right\}g^{-1},$$

$$g=\begin{pmatrix}1&0&0&1/2\\0&1&1/2&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad(\delta_2=g\delta_1g^{-1}).$$

*Non-elliptic conjugacy classes*

In the following we shall give, for each family $[\gamma]_\Gamma$, a matrix $T\in G_R$ which transforms the elements of $[\gamma]_\Gamma$ simultaneously into the canonical forms; namely $T^{-1}\gamma' T$ is a canonical form as given in Theorem 6-1, for all $\gamma'\in[\gamma]_\Gamma$. By using this $T$, we can write down explicitly the subgroup $C_0(\gamma;\Gamma)$ of $C(\gamma;\Gamma)$ so that we can compute the index

$$\mathrm{ind}(\gamma)=[\bar{C}(\gamma;\Gamma):\bar{C}_0(\gamma;\Gamma)],\tag{6.2}$$

which appears in our general formula (5.24). Note that $\mathrm{ind}(\gamma)$ depends only on the family $[\gamma]_\Gamma$, except for the parabolic classes.

(d) *elliptic/parabolic*

$$\hat\beta_1,\hat\beta_2,\hat\beta_3,\hat\beta_4:\ T=\begin{pmatrix}2c&0&0&0\\0&1&0&0\\c&0&\sqrt{3}c&0\\0&0&0&1\end{pmatrix}\quad c^2=\sqrt{3}/6.\quad \mathrm{ind}(\hat\beta_i)=6\quad(1\leqq i\leqq 4).$$

$$\hat\beta_5,\hat\beta_6:\quad T=\begin{pmatrix}c&0&0&1/3\\c/2&1&\sqrt{3}c/6&0\\c/2&0&\sqrt{3}c/2&-1/3\\0&0&0&1\end{pmatrix}\quad c^2=2\sqrt{3}/3.\ \ \mathrm{ind}(\hat\beta_i)=3\quad(i=5,6).$$

$$\hat\beta_7,\hat\beta_8:\quad T=1_4\quad \mathrm{ind}(\hat\beta_i)=4\quad(i=7,8).$$

$$\hat{\beta}_9,\ \hat{\beta}_{10}:\quad T=\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ -1/2 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\beta}_i)=4 \quad (i=9,\ 10).$$

(j) *δ-parabolic*

(j-1) $\hat{\hat{\delta}}_1$: $T=1_4$ $\qquad \mathrm{ind}(\hat{\hat{\delta}}_1)=2.$

$$\hat{\hat{\delta}}_2:\quad T=\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\hat{\delta}}_2)=2.$$

$$\hat{\hat{\delta}}_3:\quad T=\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1/2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \qquad \mathrm{ind}(\hat{\hat{\delta}}_3)=2.$$

$$\hat{\hat{\delta}}_4:\quad T=\begin{pmatrix} 1 & 0 & 0 & -1/4 \\ 1/2 & 1 & -1/4 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \qquad \mathrm{ind}(\hat{\hat{\delta}}_4)=2.$$

(j-2) $\hat{\delta}_1$: $T=1_4$ $\qquad \mathrm{ind}(\hat{\delta}_1)=1.$

$$\hat{\delta}_2:\quad T=\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\delta}_2)=1.$$

(l) *paraelliptic*

$\hat{\gamma}_1$: $T=1_4$ $\qquad \mathrm{ind}(\hat{\gamma}_1)=2.$

$$\hat{\gamma}_2:\quad T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_2)=2.$$

$$\hat{\gamma}_3:\quad T=\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_3)=2.$$

$$\hat{\gamma}_4:\quad T=\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_4)=2.$$



$$\hat{\gamma}_5:\quad T=\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2}/2 & -\sqrt{6}/2 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{6}/6 \\ 0 & 0 & 0 & -\sqrt{6}/3 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_5)=3.$$

$$\hat{\gamma}_6:\quad T=\begin{pmatrix} \sqrt{2} & 0 & 0 & -\sqrt{6}/18 \\ \sqrt{2}/2 & -\sqrt{6}/2 & \sqrt{2}/12 & -7\sqrt{6}/36 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{6}/6 \\ 0 & 0 & 0 & -\sqrt{6}/3 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_6)=3.$$

$$\hat{\gamma}_7:\quad T=\begin{pmatrix} \sqrt{2} & 0 & 0 & -\sqrt{6}/9 \\ \sqrt{2}/2 & -\sqrt{6}/2 & \sqrt{2}/6 & -7\sqrt{6}/18 \\ 0 & 0 & \sqrt{2}/2 & \sqrt{6}/6 \\ 0 & 0 & 0 & -\sqrt{6}/3 \end{pmatrix} \qquad \mathrm{ind}(\hat{\gamma}_7)=3.$$

**6-3.** Putting all these data into (5.24), we obtain an explicit formula for the dimension of $S_k(\mathrm{Sp}(2, \boldsymbol{Z}))$:

THEOREM 6-2. *We have, for $k \geqq 5$,*

$$\dim S_k(\mathrm{Sp}(2, \boldsymbol{Z}))=\sum_i t(\gamma_i\,;\,k)\,, \tag{6.3}$$

*where the sum is extended over the complete set of representatives of conjugacy classes modulo* $\pm 1$, *of families in* Sp(2, $\boldsymbol{Z}$) *listed in Theorem* 6-1. *The contribution* $t(\gamma\,;\,k)$ *of* $[\pm\gamma]_\Gamma$ *is given in the following.*

$t(\alpha_0\,;\,k)=2^{-9}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4)$

$t(\alpha_1\,;\,k)=2^{-7}(-1)^k$

$t(\alpha_2\,;\,k)+t(\alpha_3\,;\,k)=-2^{-2}3^{-3}[0,\ 1,\ -1\,;\,3]$

$t(\alpha_4\,;\,k)+t(\alpha_5\,;\,k)+t(\alpha_6\,;\,k)=2^{-3}[1,\ 0,\ 0,\ -1\,;\,4]$

$t(\alpha_7\,;\,k)+t(\alpha_8\,;\,k)=3^{-3}(-1)^k$

$t(\alpha_9\,;\,k)+t(\alpha_{10}\,;\,k)=2^{-2}3^{-2}[2,\ 1,\ -1,\ -2,\ -1,\ 1\,;\,6]$

$t(\alpha_{11}\,;\,k)+t(\alpha_{12}\,;\,k)=2^{-2}3^{-3}[2,\ 1,\ -1,\ -2,\ -1,\ 1\,;\,6]$

$t(\alpha_{13}\,;\,k)+t(\alpha_{14}\,;\,k)=2^{-2}3^{-1}[0,\ 1,\ -1\,;\,3]$

$t(\alpha_{15}\,;\,k)+t(\alpha_{16}\,;\,k)+\cdots+t(\alpha_{18}\,;\,k)=5^{-1}[1,\ 0,\ 0,\ -1,\ 0\,;\,5]$

$t(\alpha_{19}\,;\,k)+t(\alpha_{20}\,;\,k)+\cdots+t(\alpha_{22}\,;\,k)$

$\qquad\qquad =2^{-2}3^{-1}[1,\ 0,\ 0,\ -1,\ -1,\ -1,\ -1,\ 0,\ 0,\ 1,\ 1,\ 1\,;\,12]$

$t(\beta_1\,;\,k)+t(\beta_2\,;\,k)=2^{-3}3^{-3}[2k-3,\ -k+1,\ -k+2\,;\,3]$

$t(\beta_3\,;\,k)+t(\beta_4\,;\,k)=2^{-3}3^{-2}[-1,\ -k+1,\ -k+2,\ 1,\ k-1,\ k-2\,;\,6]$

$t(\beta_5\,;\,k)+t(\beta_6\,;\,k)=2^{-5}3^{-1}[k-2,\ -k+1,\ -k+2,\ k-1\,;\,4]$

$t(\gamma_1; k)=2^{-6}3^{-1}(2k-3)$

$t(\gamma_2; k)=2^{-7}(2k-3)$

$t(\gamma_3; k)=2^{-1}3^{-3}(2k-3)$

$t(\delta_1; k)=2^{-9}3^{-2}(-1)^k(2k-2)(2k-4)$

$t(\delta_2; k)=2^{-8}3^{-1}(-1)^k(2k-2)(2k-4)$

$t(\hat\beta_1; k)+t(\hat\beta_2; k)=2^{-2}3^{-1}[0, 1, 1, 0, -1, -1; 6]$

$t(\hat\beta_3; k)+t(\hat\beta_4; k)=-2^{-2}3^{-2}[2, -1, -1; 3]$

$t(\hat\beta_5; k)+t(\hat\beta_6; k)=-3^{-2}[1, -1, 0; 3]$

$t(\hat\beta_7; k)+t(\hat\beta_8; k)=-2^{-4}[1, -1, -1, 1; 4]$

$t(\hat\beta_9; k)+t(\hat\beta_{10}; k)=-2^{-4}[1, -1, -1, 1; 4]$

$t(\hat{\hat{\delta}}_1; k)=t(\hat{\hat{\delta}}_2; k)=2^{-5}(-1)^k$

$t(\hat{\hat{\hat{\delta}}}_3; k)+t(\hat{\hat{\delta}}_4; k)=2^{-5}(-1)^k$

$t(\hat\delta_1; k)=-2^{-6}3^{-1}(-1)^k(2k-3)$

$t(\hat\delta_2; k)=-2^{-6}(-1)^k(2k-3)$

$t(\varepsilon_1; k)=2^{-3}3^{-1}$

$t(\varepsilon_2; k)=0$

$t(\varepsilon_3; k)=-2^{-4}3^{-1}$

$t(\varepsilon_4; k)=-2^{-5}3^{-2}(2k-3)$

$t(\hat\gamma_1; k)+t(\hat\gamma_2; k)=t(\hat\gamma_3; k)+t(\hat\gamma_4; k)=-2^{-4}$

$t(\hat\gamma_5; k)+t(\hat\gamma_6; k)+t(\hat\gamma_7; k)=-2^{-1}3^{-1}$,

*where* $t=t(k)=[t_0, t_1, \cdots, t_{p-1}; p]$ *means that* $t=t_j$ *if* $k\equiv j \pmod p$ $(0\leqq j\leqq p-1)$.

An easy calculation shows that the generating function

$$\sum_{k=5}^{\infty}\dim S_k(\mathrm{Sp}(2, \boldsymbol{Z}))t^k$$

coincides with (0.3), which was given by Igusa [15].

REMARK 6-2. It would be interesting to observe that, if we put $k=4$ *formally* in (6.3), we get the correct value 0. On the other hand, if we put $k=3$, we get $-1$, while the correct value is 0.[2)]

COROLLARY 6-1. *For the principal congruence subgroup* $\Gamma(2)$ *of* $\mathrm{Sp}(2, \boldsymbol{Z})$, *the dimension of* $S_k(\Gamma(2))$ *is given by*

[2)] REMARK 6-3. In [9], we shall give a different proof of the above formulae for the elliptic contributions to $\dim S_k(\mathrm{Sp}(2, \boldsymbol{Z}))$ using (2.18), which is independent of Theorem 6-1.



$$\dim S_k(\Gamma(2))=t_2(\alpha_0; k)+t_2(\delta_1; k)+t_2(\hat{\hat{\delta}}_1; k)+t_2(\hat\delta_1; k)+t_2(\varepsilon_1; k)+t_2(\varepsilon_3; k)+t_2(\varepsilon_4; k), \tag{6.4}$$

*where* $t_2(\gamma; k)$ *denotes the contribution of the families of* $\Gamma(2)$ *contained in* $[\gamma]_{\mathrm{Sp}(2,Z)}\cap\Gamma(2)$, *which is given by*

$t_2(\alpha_0; k)=2^{-5}3^{-1}(2k-2)(2k-3)(2k-4)$

$t_2(\delta_1; k)=2^{-5}5(-1)^k(2k-2)(2k-4)$

$t_2(\hat{\hat{\delta}}_1; k)=2^{-3}3^25(-1)^k$

$t_2(\hat\delta_1; k)=-2^{-3}3.5(-1)^k(2k-3)$

$t_2(\varepsilon_1; k)=2^{-2}3.5$

$t_2(\varepsilon_3; k)=-2^{-3}3.5$

$t_2(\varepsilon_4; k)=-2^{-3}5(2k-3)$.

This is an immediate consequence of Theorems 6-1, 6-2, and the fact that $\Gamma(2)$ is a normal subgroup of $\mathrm{Sp}(2, \boldsymbol{Z})$ of index $[\mathrm{Sp}(2, \boldsymbol{Z}):\Gamma(2)]=720$ (see Remark 7-1 in §7-2).

## §7. Application (2): the case of $\Gamma=\Gamma_0(p)$

**7-1.** For each positive integer $N$, the group $\Gamma_0(N)$ is defined by

$$\Gamma_0(N)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(2, \boldsymbol{Z});\ C\equiv 0 \pmod N\right\}. \tag{7.1}$$

Throughout of this section, we assume that $N=p$ is an odd prime. We shall describe briefly how each conjugacy class $\{\gamma\}_{\mathrm{Sp}(2,Z)}$, listed in Theorem 6-1, decomposes into $\Gamma_0(p)$-conjugacy classes. Put, for each $\gamma\in\mathrm{Sp}(2, \boldsymbol{Z})$,

$$M(\gamma; \Gamma_0(p))=\{x\in\mathrm{Sp}(2, \boldsymbol{Z});\ x^{-1}\gamma x\in\Gamma_0(p)\}. \tag{7.2}$$

Then as in Lemma 1-1, we have a bijective map

$$\{\gamma\}_{\mathrm{Sp}(2,Z)}\cap\Gamma_0(p)/\underset{\Gamma_0(p)}{\sim}\xrightarrow{\ \sim\ } C(\gamma; \mathrm{Sp}(2, \boldsymbol{Z}))\backslash M(\gamma; \Gamma_0(p))/\Gamma_0(p), \tag{7.3}$$

which sends the class $\{x^{-1}\gamma x\}_{\Gamma_0(p)}$ to the double coset $[x]=C(\gamma; \mathrm{Sp}(2, \boldsymbol{Z}))x\Gamma_0(p)$.

The following lemma, which is proved easily, will play an essential role in the computations of this section.

LEMMA 7-1. *As a complete set of representatives of the coset space* $\mathrm{Sp}(2, \boldsymbol{Z})/\Gamma_0(p)$, *we can take the following* $[\mathrm{Sp}(2, \boldsymbol{Z}):\Gamma_0(p)]=(p+1)(p^2+1)$ *"generic" elements:*

$$(7.4)\qquad \begin{pmatrix} a & -b & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 \end{pmatrix} \left(\text{or, } \begin{pmatrix} -b & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}\right), \begin{pmatrix} a & b & -1 & 0 \\ b & c & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

*where $a$, $b$, $c$ runs over the rational integers modulo $p$.*

REMARK 7-1. We shall often *confuse* an integer modulo $p$, with an element of the finite field $\boldsymbol{Z}/p\boldsymbol{Z}$, and denote by $a^{-1}$ the integer $x \pmod p$ such that $ax \equiv 1 \pmod p$.

Combining this lemma and (7.3), we can decompose the set $\{\gamma\}_{\mathrm{Sp}(2,\boldsymbol{Z})} \cap \Gamma_0(p)$ into $\Gamma_0(p)$-conjugacy classes; Namely we first pick up, among the representatives $x$ in (7.4), those which satisfy $x^{-1}\gamma x \in \Gamma_0(p)$, and then check whether they belong to the same $C(\gamma; \mathrm{Sp}(2, \boldsymbol{Z}))$-$\Gamma_0(p)$ coset, or not. These are carried out by elementary, though somewhat lengthy, straightforward calculations. In the following, we shall list up the set of $x$'s which form a complete set of representatives of the double cosets in (7.3), for each representative of the conjugacy class of $\mathrm{Sp}(2, \boldsymbol{Z})$ in Theorem 6-1.

**7-2.** Here we treat the case (b-2), i.e., $\gamma = \alpha_1$, $\alpha_2$, and $\alpha_3$, separately, where the situations are most complicated. Suppose $\gamma = \alpha_1$ (resp. $\alpha_2$ or $\alpha_3$). Then a direct calculation shows that $x^{-1}\gamma x \in \Gamma_0(p)$ ($x$ is one of the elements in (7.4)) if and only if

$$x = \begin{pmatrix} -b & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix} \qquad \text{with } b^2+1 \equiv 0 \pmod p,$$

or,

$$x = \begin{pmatrix} a & 0 & -1 & 0 \\ 0 & c & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{with } a^2+1 \equiv c^2+1 \equiv 0 \pmod p$$
$$(\text{resp. } a^2+a+1 \equiv c^2+c+1 \equiv 0 \pmod p),$$

or,

$$x = \begin{pmatrix} a & b & -1 & 0 \\ b & c & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{with } b \not\equiv 0,\ a+c \equiv a^2+1+b^2 \equiv 0$$
$$(\text{resp. } b \not\equiv 0,\ a+c+1 \equiv a^2+a+1+b^2 \equiv 0) \pmod p.$$

It is easy to see that the first elements are equivalent (i.e., belong to the same double coset in (7.3)) to the third ones. Among the 4 elements in the second, two, which correspond to $(a, c) = (a, -a)$, and $(-a, a)$ (resp. $(a, -1-a)$, and $(-1-a, a)$), are equivalent. As for the elements in the third, we have



LEMMA 7-2. *The elements corresponding to the following values of $(a, b)$ are equivalent*:

$\gamma = \alpha_1$: $\quad (\pm a, \pm b)$, $(\pm 1/a, \pm b/a)$

$\gamma = \alpha_2, \alpha_3$: $\quad (a, \pm b)$, $(-1-a, \pm b)$, $(1/a, \pm b)$, $\left(\dfrac{-1-a}{a}, \pm\dfrac{b}{a}\right)$, $\left(\dfrac{-1}{1+a}, \pm\dfrac{-b}{1+a}\right)$, $\left(\dfrac{-a}{1+a}, \pm\dfrac{-b}{1+a}\right)$.

PROOF. Omitted.

By using these results, we get the following lists.

(7.5) $\alpha_1$:

| $x$ | $p$ mod 8 | number of double cosets in (7.3) | order of the centralizer |
|---|---|---|---|
| $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & c & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $a^2+1\equiv 0$, $c^2+1\equiv 0$ | 1 | 2+1 | 32, 16 |
| | 5 | 2+1 | 32, 16 |
| | 3 | 0 | |
| | 7 | 0 | |
| $\begin{pmatrix} a & b & -1 & 0 \\ b & -a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $b\not\equiv 0$, $a^2+1+b^2\equiv 0$ | 1 | $\frac{p-9}{8}+2$ | 4, 8 |
| | 5 | $\frac{p-5}{8}+1$ | 4, 8 |
| | 3 | $\frac{p-3}{8}+1$ | 4, 8 |
| | 7 | $\frac{p+1}{8}$ | 4 |

(7.6) $\alpha_2$ and $\alpha_3$:

| $x$ | $p$ mod 12 | number of double cosets in (7.3) | order of the centralizer |
|---|---|---|---|
| $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & c & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $a^2+a+1\equiv 0$, $c^2+c+1\equiv 0$ | 3 | 1 | 72 |
| | 1 | 2+1 | 72, 36 |
| | 7 | 2+1 | 72, 36 |
| | 5 | 0 | |
| | 11 | 0 | |

| | | | |
|---|---|---|---|
| | 3 | 0 | |
| $\begin{pmatrix} a & b & -1 & 0 \\ b & -1-a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $b\not\equiv 0$, $a^2+a+1+b^2\equiv 0$ | 1 | $\frac{p-13}{12}+2$ | 6, 12 |
| | 7 | $\frac{p-7}{12}+1$ | 6, 12 |
| | 5 | $\frac{p-5}{12}+1$ | 6, 12 |
| | 11 | $\frac{p+1}{12}$ | 6 |

DEFINITION 7-1. For each $\gamma\in \mathrm{Sp}(2, \boldsymbol{Z})$, let $\gamma_1, \cdots, \gamma_d$, $d=d(\gamma)$, be a complete set of representatives of $\Gamma_0(p)$-conjugacy classes contained in $\{\gamma\}_{\mathrm{Sp}(2,\boldsymbol{Z})}\cap\Gamma_0(p)$. We define the "*relative Maß of* $\gamma$", with respect to $\mathrm{Sp}(2, \boldsymbol{Z})/\Gamma_0(p)$, by

$$m(\gamma\,;\mathrm{Sp}(2, \boldsymbol{Z})/\Gamma_0(p))=\sum_{i=1}^{d}[C(\gamma_i\,;\mathrm{Sp}(2, \boldsymbol{Z})):C(\gamma_i\,;\Gamma_0(p))]. \tag{7.7}$$

We shall often denote it simply by $m(\gamma)$. Note that, it depends only on the conjugacy class $\{\gamma\}_{\mathrm{Sp}(2,\boldsymbol{Z})}$. Also note, from (7.3)

$$d(\gamma)=\#(C(\gamma\,;\mathrm{Sp}(2, \boldsymbol{Z})\backslash M(\gamma\,;\Gamma_0(p))/\Gamma_0(p)). \tag{7.8}$$

REMARK 7-2. Relative Maß can be defined similarly for any lattices $\Gamma_1\supset\Gamma_2$ such that $[\Gamma_1:\Gamma_2]<\infty$. If $\Gamma_2$ is a normal subgroup of $\Gamma_1$, we have $m(\gamma\,;\Gamma_1/\Gamma_2)=[\Gamma_1:\Gamma_2]$ unless $d(\gamma)=0$. The elliptic contributions for $\dim S_k(\Gamma_i)$ are related as

$$\dim S_k(\Gamma_2)\,|\,\{\gamma\}_{\Gamma_1}\cap\Gamma_2=m(\gamma\,;\Gamma_1/\Gamma_2)\cdot\dim S_k(\Gamma_1)\,|\,\{\gamma\}_{\Gamma_1}, \tag{7.9}$$

for each conjugacy class $\{\gamma\}_{\Gamma_1}$. We have used this fact for the computation of $\dim S_k(\Gamma(2))$ in Corollary 6-1.

As an immediate consequence of the above results, we get

PROPOSITION 7-1. *The relative Maß of* $\alpha_1$, $\alpha_2$, $\alpha_3$ *with respect to* $\mathrm{Sp}(2, \boldsymbol{Z})/\Gamma_0(p)$ *are given by*

$$\begin{aligned} &m(\alpha_1)=p+2+\left(\frac{-1}{p}\right),\\ &m(\alpha_2)=m(\alpha_3)=\begin{cases} p+2+\left(\frac{-3}{p}\right) & \cdots\ p>3\\ 1 & \cdots\ p=3. \end{cases}\end{aligned} \tag{7.10}$$

Compare this with (7.21). We can prove this also by using local computations as expained in § 1-3.



**7-3.** In the cases where $\gamma$ belongs to (b-1), we have always $C(\gamma\,;\mathrm{Sp}(2, \boldsymbol{Z}))=C(\gamma\,;\Gamma_0(p))$, so that $m(\gamma)=d(\gamma)$.

(7.11) $\gamma=\alpha_4, \alpha_5$:

| $p$ (mod 8) | $x$ | $d(\gamma)$ | |
|---|---|---|---|
| 1 | $\begin{pmatrix} -1 & b & -1 & 0 \\ b & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 4 | $b^2+2\equiv 0$, $a^2+1\equiv 0$ (mod $p$) |
| 3 | $\begin{pmatrix} -1 & b & -1 & 0 \\ b & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 2 | $b^2+2\equiv 0$ (mod $p$) |
| 5 | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 2 | $a^2+1\equiv 0$ (mod $p$) |
| 7 | | 0 | |

(7.12) $\gamma=\alpha_6$:

| $p$ (mod 8) | $x$ | $d(\gamma)$ | |
|---|---|---|---|
| 1 | $\begin{pmatrix} 2a+1 & a & -1 & 0 \\ a & 2a+1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} b & b & -1 & 0 \\ b & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 4 | $3a^2+2a+1\equiv 0$, $b^2+1\equiv 0$ (mod $p$) |
| 3 | $\begin{pmatrix} 2a+1 & a & -1 & 0 \\ a & 2a+1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 2 | $3a^2+2a+1\equiv 0$ (mod $p$) |
| 5 | $\begin{pmatrix} b & b & -1 & 0 \\ b & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 2 | $b^2+1\equiv 0$ (mod $p$) |
| 7 | | 0 | |

(7.13) $\gamma=\alpha_7$:

$$d(\gamma)=\left(1+\left(\frac{-3}{p}\right)\right)^2,\quad x=\begin{pmatrix}0&-b&-1&0\\0&1&0&0\\1&0&0&0\\b&0&0&1\end{pmatrix},\ \begin{pmatrix}2a&a&-1&0\\a&2a&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$b^2+b+1\equiv 3a^2+1\equiv 0 \pmod{p}.$$

(7.14) $\gamma=\alpha_8$:

$$d(\gamma)=\left(1+\left(\frac{-3}{p}\right)\right)^2,\quad x=\begin{pmatrix}a&0&-1&0\\0&c&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \begin{matrix}a^2+a+1\equiv 0\\ c^2+c+1\equiv 0\\ \pmod{p}.\end{matrix}$$

(7.15) $\gamma=\alpha_9,\ \alpha_{10}$:

$$d(\gamma)=\left(1+\left(\frac{-3}{p}\right)\right)^2,\quad x=\begin{pmatrix}-2&b&-1&0\\b&1&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},\ \begin{pmatrix}a&0&-1&0\\0&a&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$b^2+3\equiv a^2+a+1\equiv 0 \pmod{p}.$$

(7.16) $\gamma=\alpha_{11},\ \alpha_{12}$:

$$d(\gamma)=\left(1+\left(\frac{-3}{p}\right)\right)^2,\quad x=\begin{pmatrix}a&0&-1&0\\0&c&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \begin{matrix}a^2+a+1\equiv 0\\ c^2+c+1\equiv 0\\ \pmod{p}.\end{matrix}$$

(7.17) $\gamma=\alpha_{13},\ \alpha_{14}$:

| $p$ (mod 12) | $x$ | $d(\gamma)$ | |
|---|---|---|---|
| 1 | $\begin{pmatrix}-1&b&-1&0\\b&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},\ \begin{pmatrix}a&0&-1&0\\0&a&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ | 4 | $b^2+1\equiv 0$, $a^2+a+1\equiv 0$ (mod $p$) |
| 5 | $\begin{pmatrix}-1&b&-1&0\\b&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ | 2 | $b^2+1\equiv 0$ (mod $p$) |
| 7 (3) | $\begin{pmatrix}a&0&-1&0\\0&a&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ | 2 (1) | $a^2+a+1\equiv 0$ (mod $p$) |
| 11 | | 0 | |



(7.18) $\gamma=\alpha_{15},\ \cdots,\ \alpha_{18}$:

| $p$ (mod 5) | $x$ | $d(\gamma)$ | |
|---|---|---|---|
| 1 (5) | $\begin{pmatrix}a&b&-1&0\\b&c&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ | 4 (1) | $a^4+a^3+a^2+a+1\equiv 0$, $b\equiv -1/(a+1)$, $c\equiv -1/a$ (mod $p$) |
| 2, 3, 4 | | 0 | |

(7.19) $\gamma=\alpha_{19},\ \cdots,\ \alpha_{22}$:

$$d(\gamma)=\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right),\quad x=\begin{pmatrix}a&0&-1&0\\0&c&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad a^2+a+1\equiv c^2+1\equiv 0. \pmod{p}$$

**7-4.** For elliptic elements of types (b-3), (b-4), (b-5), we have the following results:

(7.20) $\gamma=\beta_1,\ \cdots,\ \beta_6$:

| $\gamma$ | $x$ | $d(\gamma)$ | $m(\gamma)$ | |
|---|---|---|---|---|
| $\beta_i$ $(1\leqq i\leqq 4)$ | $\begin{pmatrix}a&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $(p+1)\left(1+\left(\frac{-3}{p}\right)\right)$ | $a^2+a+1\equiv 0$ (mod $p$) |
| $\beta_5,\ \beta_6$ | $\begin{pmatrix}a&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $(p+1)\left(1+\left(\frac{-1}{p}\right)\right)$ | $a^2+1\equiv 0$ (mod $p$) |

(7.21) $\gamma=\gamma_1,\ \gamma_2,\ \gamma_3$:

| $\gamma$ | $x$ | $d(\gamma)$ | $m(\gamma)$ | |
|---|---|---|---|---|
| $\gamma_1$ | $1_4,\ \begin{pmatrix}-b&0&0&-1\\1&0&0&0\\0&1&0&0\\0&b&1&0\end{pmatrix}$ | $2+\left(\frac{-1}{p}\right)$ | $p+2+\left(\frac{-1}{p}\right)$ | $b^2+1\equiv 0$ (mod $p$) |
| $\gamma_2$ | $1_4,\ \begin{pmatrix}-b&b&0&-1\\1&0&0&0\\0&1&0&0\\0&b&1&0\end{pmatrix}$ | $2+\left(\frac{-1}{p}\right)$ | $p+2+\left(\frac{-1}{p}\right)$ | $b^2+1\equiv 0$ (mod $p$) |

| | | | | |
|---|---|---|---|---|
| $\gamma_3$ | $1_4, \begin{pmatrix} -b & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $2+\left(\frac{-3}{p}\right)$ | $p+2+\left(\frac{-3}{p}\right)$ $(p>3)$<br>$7$ $(p=3)$ | $b^2+b+1\equiv 0$ (mod $p$) |

(7.22) $\gamma=\delta_1,\ \delta_2:\ d(\gamma)=1,\ x=1_4,\ m(\gamma)=(p+1)^2.$

**7-5.** Now consider the non-elliptic elements. Let $P_0$, $P_1$ be the $Q$-parabolic subgroups of $\mathrm{Sp}(2, Q)$ as in (2.9). $\Gamma_0(p)$ has three (resp. two) point (resp. 1-dimensional) cusps, which correspond to the double cosets decomposition

(7.23) $$\mathrm{Sp}(2, \boldsymbol{Z})=\coprod_{i=1}^{3} P_0(\boldsymbol{Z})x_i\Gamma_0(p)$$
$$\left(\text{resp. } \mathrm{Sp}(2, \boldsymbol{Z})=\coprod_{i=1}^{2} P_1(\boldsymbol{Z})x_i\Gamma_0(p)\right),$$

where $x_1=1_4$, $x_2=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $x_3=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and $P_i(\boldsymbol{Z})=P_i\cap \mathrm{Sp}(2, \boldsymbol{Z})$.

Put

(7.24) $$P_j^i=x_i^{-1}P_jx_i \quad (i=1, 2, 3;\ j=0, 1).$$

Then it is easy to see that, every paraelliptic, $\delta$-parabolic, and unipotent (resp. elliptic/parabolic) elements of $\Gamma_0(p)$ are conjugate to an element of $P_0^i(\boldsymbol{Z})\cap\Gamma_0(p)$, $i=1, 2, 3$ (resp. $P_1^i(\boldsymbol{Z})\cap\Gamma_0(p)$, $i=1, 2$). The following lemmas are easily proved:

LEMMA 7-3. *Suppose $g\in \mathrm{Sp}(2, Q)$ is conjugate to $\begin{pmatrix} 1 & S \\ 0 & 1\end{pmatrix}$,* $\det S\neq 0$. *Then there exists a unique $Q$-parabolic subgroup $P_0^*$ of type $(P_0)$ such that $g\in(P_0^*)_U$. If $x^{-1}\begin{pmatrix} 1 & S \\ 0 & 1\end{pmatrix}x\in P_0$ for $x\in\mathrm{Sp}(2, Q)$, then $x\in P_0$.*

LEMMA 7-4. *Suppose $g\in\mathrm{Sp}(2, Q)$ is conjugate to* $\begin{pmatrix} x & 0 & y & 0 \\ 0 & 1 & 0 & u \\ z & 0 & w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}=g_0$, $x+w\neq 2$, $u\neq 0$. *Then there exists a unique $Q$-parabolic subgroup $P_1^*$ of type $(P_1)$, such that $g\in P_1^*$. If $h^{-1}g_0h\in P_1$ for $h\in \mathrm{Sp}(2, Q)$, then $h\in P_1$.*

By using these lemmas and Lemma 7-1, we can show that each paraelliptic, $\delta$-parabolic, and parabolic (resp. elliptic/parabolic) family of $\mathrm{Sp}(2, \boldsymbol{Z})$ decomposes into three (resp. two) families of $\Gamma_0(p)$, corresponding with $P_0^i$, $i=1, 2, 3$ (resp. $P_1^i$, $i=1, 2$), if it is non-degenerate, i.e., not of type (j-2), (k-3).

For each element $\gamma\in\Gamma_0(p)$ of above type, we put



(7.25) $$i_0(\gamma)=[C_0(\gamma\,;\mathrm{Sp}(2, \boldsymbol{Z}))\,:\,C_0(\gamma\,;\Gamma_0(p))].$$

**7-6.** For elliptic/parabolic elements, we have the following list of representatives of conjugacy classes of families:

(7.26) $\gamma=\beta_1(n), \cdots, \beta_{10}(n)$:

| $\gamma$ | $x$ | $d(\gamma)$ | $i_0(\gamma)$ | |
|---|---|---|---|---|
| $\beta_1(n)$ | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $p$ | $a^2-a+1\equiv 0$<br>$n\equiv 0$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ | $c^2-c+1\equiv 0$<br>(mod $p$) |
| $\beta_2(n)$ | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $p$ | $a^2-a+1\equiv 0$<br>$n\equiv 0$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ | $c^2-c+1\equiv 0$<br>(mod $p$) |
| $\beta_3(n)$ | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $p$<br>$(p>3)$ | $a^2-a+1\equiv 0$<br>$n\equiv 0$ |
| | $\begin{pmatrix} -1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $3$ | $p$<br>$(p=3)$ | $a=0, 1, 2$<br>$n\equiv a^2$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ | $c^2-c+1\equiv 0$<br>(mod $p$) |

| | | | | |
|---|---|---|---|---|
| $\hat{\beta}_4(n)$ | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $p$ $(p>3)$ | $a^2-a+1\equiv 0$ $n\equiv 0$ |
| | $\begin{pmatrix} -1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | 3 | $p$ $(p=3)$ | $a=0, 1, 2$ $n\equiv -a^2$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | 1 | $c^2-c+1\equiv 0$ (mod $p$) |
| $\hat{\beta}_5(n)$ | $\begin{pmatrix} 3a-1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ $(p>3)$; 0 $(p=3)$ | $p$ | $3a^2-3a+1\equiv 0$ $n\equiv 1/3$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | 1 | $c^2-c+1\equiv 0$ (mod $p$) |
| $\hat{\beta}_6(n)$ | $\begin{pmatrix} 3a-1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ $(p>3)$; 0 $(p=3)$ | $p$ | $3a^2-3a+1\equiv 0$ $n\equiv -1/3$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | 1 | $c^2-c+1\equiv 0$ (mod $p$) |
| $\hat{\beta}_7(n)$ $\hat{\beta}_8(n)$ | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $p$ | $a^2+1\equiv 0$ $n\equiv 0$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | 1 | $c^2+1\equiv 0$ (mod $p$) |



| | | | | |
|---|---|---|---|---|
| $\hat{\beta}_9(n)$ | $\begin{pmatrix} -2a-1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $p$ | $2a^2+2a+1\equiv 0$ $n\equiv 1/2$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | 1 | $c^2+1\equiv 0$ (mod $p$) |
| $\hat{\beta}_{10}(n)$ | $\begin{pmatrix} -2a-1 & a & -1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $p$ | $2a^2+2a+1\equiv 0$ $n\equiv -1/2$ |
| | $\begin{pmatrix} c & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | 1 | $c^2+1\equiv 0$ (mod $p$) |

**7-7.** For $\delta$-parabolic elements of non-degenerate type, we see that two families correspond to $P_0^3$, hence there are four in all.

(7.27) $\gamma=\hat{\hat{\delta}}_1(m, n), \cdots, \hat{\hat{\delta}}_4(m, n)$:

| $\gamma$ | $x$ | $i_0(\gamma)$ | |
|---|---|---|---|
| $\hat{\hat{\delta}}_1(m, n)$ resp. $\hat{\hat{\delta}}_2(m, n)$ | $1_4$ | 1 | |
| | resp. $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & r & -1 & 0 \\ r & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $2r\equiv 1$ (mod $p$) | $p^2$ | $m\equiv n\equiv 0$ |
| | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $p$ | $n\equiv 0$ |
| | $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | $p$ | $m\equiv 0$ (mod $p$) |

| | | | |
|---|---|---|---|
| $\hat{\hat{\delta}}_3(m,n)$ resp. $\hat{\hat{\delta}}_4(m,n)$ | $1_4$ | $1$ | |
| | $\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, resp. $\begin{pmatrix} r & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $p^2$ | $m\equiv n\equiv 0$ resp. $2m-1\equiv n\equiv 0$ |
| | $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | $p$ | $m\equiv 0$ resp. $2m-1\equiv 0$ |
| | $\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$ | $p$ | $m\equiv 2n$ resp. $2m-4n\equiv 1$ (mod $p$) |

(7.28) $\gamma=\hat{\delta}_1(n),\ \hat{\delta}_2(n)$:

| $\gamma$ | $x$ | $i_0(\gamma)$ | |
|---|---|---|---|
| $\hat{\delta}_1(n)$ resp. $\hat{\delta}_2(n)$ | $1_4$ | $p+1$ | |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, resp. $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | $p(p+1)$ | $n\equiv 0$ (mod $p$) |

**7-8.** For parabolic elements, we get in the same way the following list.

(7.29) $\gamma=\varepsilon_1(S),\ \varepsilon_2(S),\ \varepsilon_3(S),\ \varepsilon_4(S)$:

| $\gamma$ | $x$ | $i_0(\gamma)$ | |
|---|---|---|---|
| $\varepsilon_i(S)$ $1\leqq i\leqq 3$ | $1_4$ | $1$ | $S$: $GL_2(\boldsymbol{Z})$-reduced<br>$\varepsilon_3$: $t=1,\ c_1=2,\ d_1=1$ |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $p^3$ | $S$: $GL_2(\boldsymbol{Z})$-reduced<br>$S\equiv 0$<br>$\varepsilon_3$: $t=1,\ c_1=2,\ d_1=p$ |
| | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $p$ | $S$: $G\Gamma_0(p)$-reduced<br>$s_2\equiv 0$<br>$\varepsilon_3$: $t=2,\ c_j=2,\ d_j=1$ |



| | | | |
|---|---|---|---|
| $\varepsilon_4(S)$ | $1_4$ | $p+1$ | |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $p^2(p+1)$ | $S=\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}$<br>$n\equiv 0$ (mod $p$) |

$$G\Gamma_0(p)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\boldsymbol{Z});\ c\equiv 0 \pmod p\right\}.$$

**7-9.** Paraelliptic elements.

(7.30) $\gamma=\hat{\gamma}_1(n),\ \cdots,\ \hat{\gamma}_7(n)$:

| $\gamma$ | $x$ | $d(\gamma)$ | $i_0(\gamma)$ | |
|---|---|---|---|---|
| $\hat{\gamma}_1(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ | $n\equiv 0$ |
| | $\begin{pmatrix} -b & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $1$ | $b^2+1\equiv 0$ (mod $p$) |
| $\hat{\gamma}_2(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & r & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ | $n\equiv -r$<br>$2r\equiv 1$ |
| | $\begin{pmatrix} -b & -r & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $1$ | $b^2+1\equiv 0$ (mod $p$) |
| $\hat{\gamma}_3(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} 0 & r & -1 & 0 \\ r & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ | $n\equiv 0$ |

| | | | | |
|---|---|---|---|---|
| $\gamma_3(n)$ | $\begin{pmatrix} -b & b & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $1$ | $b^2+1\equiv 0$ (mod $p$) |
| $\gamma_4(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} 0 & r & -1 & 0 \\ r & r & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ | $n\equiv -r$ |
| | $\begin{pmatrix} -b & b-r & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-1}{p}\right)$ | $1$ | $b^2+1\equiv 0$ (mod $p$) |
| $\gamma_5(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ $(p>3)$ | $n\equiv 0$ |
| | $\begin{pmatrix} a & 0 & -1 & 0 \\ 0 & -a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $3$ | $p$ $(p=3)$ | $n\equiv a$ $(a=0, 1, 2)$ |
| | $\begin{pmatrix} -b & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ $(p>3)$ | $b^2+b+1\equiv 0$ |
| | $\begin{pmatrix} -1 & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ | $3$ | $1$ $(p=3)$ | $(a\equiv 0, 1, 2)$ (mod $p$) |
| $\gamma_6(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} -s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ $(p>3)$ | $n\equiv -s$ $3s\equiv 1$ |



| | | | | |
|---|---|---|---|---|
| $\gamma_6(n)$ | $\begin{pmatrix} -b & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ $(p>3)$ | $b^2+b+1\equiv 0$ $a\equiv -b/(2b+1)$ (mod $p$) |
| $\gamma_7(n)$ | $1_4$ | $1$ | $1$ | |
| | $\begin{pmatrix} -2s & 0 & -1 & 0 \\ 0 & 2s & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $1$ | $p$ $(p>3)$ | $n\equiv -2s$ $3s\equiv 1$ |
| | $\begin{pmatrix} -b & a & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \end{pmatrix}$ | $1+\left(\frac{-3}{p}\right)$ | $1$ $(p>3)$ | $b^2+b+1\equiv 0$ $a\equiv -2b/(2b+1)$ (mod $p$) |

**7-10.** Finally we put all these data into (5.24), and get the following explicit formula for the dimension of $S_k(\Gamma_0(p))$:

THEOREM 7-1. *We have, for $k\geqq 5$, and an odd prime $p$,*

$$\dim S_k(\Gamma_0(p))=\sum_i t_p(\gamma_i\,;\,k)\,, \tag{7.31}$$

*where the sum is as in Theorem 6-2, and $t_p(\gamma_i\,;\,k)$ is the contribution of conjugacy classes of families contained in $[\pm\gamma_i]_{Sp(2,Z)}\cap\Gamma_0(p)$, which is given as follows.*

$$t_p(\alpha_0\,;\,k)=2^{-9}3^{-3}5^{-1}(p+1)(p^2+1)(2k-2)(2k-3)(2k-4)$$

$$t_p(\alpha_1\,;\,k)=2^{-7}\left(p+2+\left(\frac{-1}{p}\right)\right)(-1)^k$$

$$t_p(\alpha_2\,;\,k)+t_p(\alpha_3\,;\,k)=-2^{-2}3^{-3}[0,\,1,\,-1\,;\,3]\times\begin{cases}\left(p+2+\left(\frac{-3}{p}\right)\right) & \cdots\ p>3\\ 1 & \cdots\ p=3\end{cases}$$

$$t_p(\alpha_4\,;\,k)+\cdots+t_p(\alpha_6\,;\,k)=2^{-3}[1,\,0,\,0,\,-1\,;\,4]\times\begin{cases}4\cdots\ p\equiv 1\\ 2\cdots\ p\equiv 3,\,5 \pmod 8\\ 0\cdots\ p\equiv 7\end{cases}$$

$$t_p(\alpha_7\,;\,k)+t_p(\alpha_8\,;\,k)=3^{-3}\left(1+\left(\frac{-3}{p}\right)\right)^2(-1)^k$$

$$t_p(\alpha_9\,;\,k)+t_p(\alpha_{10}\,;\,k)=2^{-2}3^{-2}\left(1+\left(\frac{-3}{p}\right)\right)^2[2,\,1,\,-1,\,-2,\,-1,\,1\,;\,6]$$

$$t_p(\alpha_{11}\,;\,k)+t_p(\alpha_{12}\,;\,k)=2^{-2}3^{-3}\left(1+\left(\frac{-3}{p}\right)\right)^2[2,\,1,\,-1,\,-2,\,-1,\,1\,;\,6]$$

$$t_p(\alpha_{13}\,;\,k)+t_p(\alpha_{14}\,;\,k)=2^{-2}3^{-1}[0,\ 1,\ -1\,;\ 3]\times\begin{cases}1\cdots p=3\\ 4\cdots p\equiv 1\\ 2\cdots p\equiv 5,\ 7\\ 0\cdots p\equiv 11\end{cases}\pmod{12}$$

$$t_p(\alpha_{15}\,;\,k)+\cdots+t_p(\alpha_{18}\,;\,k)=5^{-1}[1,\ 0,\ 0,\ -1,\ 0\,;\ 5]\times\begin{cases}4\cdots p\equiv 1\\ 0\cdots p\equiv 2,\ 3,\ 4\\ 1\cdots p=5\end{cases}\pmod{5}$$

$$t_p(\alpha_{19}\,;\,k)+\cdots+t_p(\alpha_{22}\,;\,k)$$
$$=2^{-2}3^{-1}[1,\ 0,\ 0,\ -1,\ -1,\ -1,\ -1,\ 0,\ 0,\ 1,\ 1,\ 1\,;\ 12]\Big(1+\Big(\frac{-1}{p}\Big)\Big)\Big(1+\Big(\frac{-3}{p}\Big)\Big)$$

$$t_p(\beta_1\,;\,k)+t_p(\beta_2\,;\,k)=2^{-3}3^{-3}[2k-3,\ -k+1,\ -k+2\,;\ 3](p+1)\Big(1+\Big(\frac{-3}{p}\Big)\Big)$$

$$t_p(\beta_3\,;\,k)+t_p(\beta_4\,;\,k)$$
$$=2^{-3}3^{-2}[-1,\ -k+1,\ -k+2,\ 1,\ k-1,\ k-2\,;\ 6](p+1)\Big(1+\Big(\frac{-3}{p}\Big)\Big)$$

$$t_p(\beta_5\,;\,k)+t_p(\beta_6\,;\,k)=2^{-5}3^{-1}[k-2,\ -k+1,\ -k+2,\ k-1\,;\ 4](p+1)\Big(1+\Big(\frac{-1}{p}\Big)\Big)$$

$$t_p(\gamma_1\,;\,k)=2^{-6}3^{-1}(2k-3)\Big(p+2+\Big(\frac{-1}{p}\Big)\Big)$$

$$t_p(\gamma_2\,;\,k)=2^{-7}(2k-3)\Big(p+2+\Big(\frac{-1}{p}\Big)\Big)$$

$$t_p(\gamma_3\,;\,k)=2^{-1}3^{-3}(2k-3)\times\begin{cases}p+2+\Big(\frac{-3}{p}\Big)\cdots p>3\\ 7\qquad\qquad\cdots p=3\end{cases}$$

$$t_p(\delta_1\,;\,k)=2^{-9}3^{-2}(-1)^k(2k-2)(2k-4)(p+1)^2$$

$$t_p(\delta_2\,;\,k)=2^{-8}3^{-1}(-1)^k(2k-2)(2k-4)(p+1)^2$$

$$t_p(\hat{\beta}_1\,;\,k)+t_p(\hat{\beta}_2\,;\,k)=2^{-1}3^{-1}[0,\ 1,\ 1,\ 0,\ -1,\ -1\,;\ 6]\Big(1+\Big(\frac{-3}{p}\Big)\Big)$$

$$t_p(\hat{\beta}_3\,;\,k)+t_p(\hat{\beta}_4\,;\,k)=-2^{-1}3^{-2}[2,\ -1,\ -1\,;\ 3]\Big(1+\Big(\frac{-3}{p}\Big)\Big)$$
$$+\begin{cases}-3^{-2}[1,\ -1,\ 0\,;\ 3]\Big(1+\Big(\frac{-3}{p}\Big)\Big)\cdots p=3\\ 0\qquad\qquad\cdots p>3\end{cases}$$

$$t_p(\hat{\beta}_5\,;\,k)+t_p(\hat{\beta}_6\,;\,k)=-3^{-2}[1,\ -1,\ 0\,;\ 3]\Big(1+\Big(\frac{-3}{p}\Big)\Big)\times\begin{cases}1\cdots p=3\\ 2\cdots p>3\end{cases}$$

$$t_p(\hat{\beta}_7\,;\,k)+t_p(\hat{\beta}_8\,;\,k)=-2^{-3}[1,\ -1,\ -1,\ 1\,;\ 4]\Big(1+\Big(\frac{-1}{p}\Big)\Big)$$



$$t_p(\hat{\beta}_9\,;\,k)+t_p(\hat{\beta}_{10}\,;\,k)=-2^{-3}[1,\ -1,\ -1,\ 1\,;\ 4]\Big(1+\Big(\frac{-1}{p}\Big)\Big)$$

$$t_p(\hat{\hat{\delta}}_1\,;\,k)=t_p(\hat{\hat{\delta}}_2\,;\,k)=2^{-3}(-1)^k$$

$$t_p(\hat{\hat{\delta}}_3\,;\,k)+t_p(\hat{\hat{\delta}}_4\,;\,k)=2^{-3}(-1)^k\times\begin{cases}1\cdots p\equiv 1\\ 2\cdots p\equiv 3\end{cases}\pmod{4}$$

$$t_p(\hat{\delta}_1\,;\,k)=-2^{-5}3^{-1}(-1)^k(2k-3)(p+1)$$

$$t_p(\hat{\delta}_2\,;\,k)=-2^{-5}(-1)^k(2k-3)(p+1)$$

$$t_p(\varepsilon_1\,;\,k)=2^{-3}3^{-1}(p+3)$$

$$t_p(\varepsilon_2\,;\,k)=0$$

$$t_p(\varepsilon_3\,;\,k)=-2^{-3}3^{-1}(p+1)$$

$$t_p(\varepsilon_4\,;\,k)=-2^{-4}3^{-2}(2k-3)(p+1)$$

$$t_p(\hat{\gamma}_1\,;\,k)+t_p(\hat{\gamma}_2\,;\,k)=t_p(\hat{\gamma}_3\,;\,k)+t_p(\hat{\gamma}_4\,;\,k)=-2^{-4}\Big(3+\Big(\frac{-1}{p}\Big)\Big)$$

$$t_p(\hat{\gamma}_5\,;\,k)+t_p(\hat{\gamma}_6\,;\,k)+t_p(\hat{\gamma}_7\,;\,k)=-2^{-1}3^{-1}\Big(3+\Big(\frac{-3}{p}\Big)\Big),$$

*where* $t=t(k)=[t_0,\ t_1,\ \cdots,\ t_{q-1}\,;\ q]$ *means that* $t=t_j$ *if* $k\equiv j$ *(mod* $q$*)* $(0\leqq j\leqq q-1)$.

**7-11.** *Numerical examples*

| $p$ \ $k$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2*) | 0 | 1 | 0 | 2 | 0 | 4 | 0 | 7 | 0 | 10 | 0 | 15 | 0 | 20 | 1 | 27 |
| 3 | 0 | 2 | 0 | 5 | 0 | 10 | 0 | 16 | 0 | 23 | 1 | 35 | 3 | 47 | 4 | 61 |
| 5 | 0 | 5 | 0 | 13 | 0 | 25 | 3 | 44 | 6 | 66 | 16 | 100 | 25 | 136 | 45 | 188 |
| 7 | 0 | 11 | 0 | 26 | 5 | 56 | 15 | 95 | 28 | 145 | 58 | 222 | 97 | 312 | 143 | 417 |
| 11 | 2 | 31 | 9 | 80 | 33 | 164 | 80 | 288 | 158 | 462 | 278 | 694 | 444 | 991 | 666 | 1365 |

REMARK 7-3. For $p=2$, the dimension of $S_k(\Gamma_0(2))$ has been computed by T. Ibukiyama [14], by using results of J. Igusa.

**7-12.** Finally, we make some remarks on our formula (7.31), concerning the *validity* for weight $k=4$, 3, 2. Since Selberg's trace formula (Theorem 2-1) works only for $k\geqq 5$, our result is valid only in these cases. It seems, nevertheless, not meaningless to consider the values we get in putting $k=4$, 3, 2, in view of Remark 6-2. Suppose first $k=4$. Then we make the following

*Conjecture* 7-1. *The dimension of* $S_4(\Gamma_0(p))$ *is given by* (7.31) *by putting*

$k=4$.

Note that the similar assertion is true for the principal congruence subgroup $\Gamma(N)$ of Sp(2, $\mathbf{Z}$) (compare [18] with [36]). Suppose next $k=3$. Then some correcting term is necessary; for Sp(2, $\mathbf{Z}$), it is $+1$, as we saw in Remark 6-2. This is also the case for elliptic modular cusp forms of weight *two*, in which the correcting term ($=1$) does not depend on the lattice $\Gamma$. So we can make the following

*Conjecture* 7-2. *The dimension of* $S_3(\Gamma_0(p))$ *is given by adding* $+1$ *to the value we obtain by putting* $k=3$ *in* (7.31).

These conjectures fit to the following numerical table, in which we give the values we get in putting to (7.31), $k=4$, 3, 2.

(7.32)

| $k$ \ $p$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 1 | 3 | 7 | 11 | 20 | 27 | 41 | 75 | 90 | 143 | 185 |
| 3 | −1 | −1 | −1 | −1 | −1 | 0 | 0 | 1 | 3 | 3 | 8 | 10 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Suppose finally $k=2$. In this case the situation is completely different; namely we have the following

*Observation* 7-1. *The fomula we get by putting* $k=2$ *in* (7.31) *is identically zero* !!

We do not know the real meaning of this vanishing, and we can not make any conjecture in this case.

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*Added in proof*:

After this paper was accepted, the author learned that the formula of Langlands [17] for the integral $I_0(\gamma)$ $(=\chi(\gamma)$ in [17], (2), p. 101) can be made explicit and useful also for singular elliptic elements $\gamma$, if we make a small modification of it. So the calculations of § 3 of this paper could have been avoided, as we did for regular element. However, to provide a completely elementary proof as presented here would not be meaningless and is, hopefully, of some interest.


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