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Dimension Formula for the Spaces of Siegel Cusp Forms of Half Integral Weight and Degree Two

by

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Let $\mathfrak{S}_g = \{Z \in M(g,\mathbf{C}) \mid {}^tZ = Z,\ \operatorname{Im} Z > 0\}$ be the Siegel upper half plane of degree g, $\Gamma_g = Sp\,(g,\mathbf{Z})$ the Siegel modular group of degree g and

$$\Gamma_g^* = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; \text{diagonal elements of } A\,{}^tB,\ C\,{}^tD \text{ are even} \right\} .$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we denote $(AZ+B)(CZ+D)^{-1}$ by $M\langle Z\rangle$. Let $\mathbf{e}(z) = \exp(2\pi i z)$ and for $Z \in \mathfrak{S}_g$ put

$$\theta(Z) = \sum_{\eta \in \mathbf{Z}^g} \mathbf{e}\left(\frac{1}{2}{}^t\eta Z \eta\right) .$$

If M belongs to Γ_g^*, $\theta(M\langle Z\rangle)/\theta(Z)$ is holomorphic on $\mathfrak{S}_g$. Let $\alpha = \begin{pmatrix} 2\cdot 1_g & O \\ O & 1_g \end{pmatrix}$ and let $\Theta(Z) = \theta(2Z) = \theta(\alpha\langle Z\rangle)$. Let

$$\Gamma_0^g(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; C \equiv O \pmod 4 \right\} .$$

Then $\Gamma_g^\alpha := \alpha^{-1}\Gamma_g^*\alpha \cap \Gamma_g$ contains $\Gamma_0^g(4)$. Hence if M belongs to $\Gamma_0^g(4)$ or more generally if M belongs to Γ_g^α, then

$$J(M,Z) := \Theta(M\langle Z\rangle)/\Theta(Z)$$

is holomorphic on $\mathfrak{S}_g$ and satisfies the equality:

$$J(M,Z)^2 = \det(CZ+D)\psi(\det D) ,$$

where $\psi : 1+2\mathbf{Z} \to \{\pm 1\}$ is the non-trivial Dirichlet character modulo 4 (cf. §1). $J(M,Z)$ is called *the automorphy factor of weight* $1/2$.

Let $\mu : GL(g,\mathbf{C}) \to GL(r,\mathbf{C})$ be an irreducible holomorphic representation. $\mu(CZ+D)$ is also an automorphy factor (with respect to Γ_g) and so is $J(M,Z)^{2k+1}\mu(CZ+D)$ (with respect to $\Gamma_0^g(4)$). Let Γ be a subgroup of $\Gamma_0^g(4)$ of finite index. A holomorphic mapping $f : \mathfrak{S}_g \to \mathbf{C}^r$ is called *a Siegel modular form of half integral weight* with respect to Γ, if f satisfies the following equality for any $M \in \Gamma$ and $Z \in \mathfrak{S}_g$:

$$f(M\langle Z\rangle) = J(M, Z)^{2k+1}\,\mu(CZ + D)\,f(Z)\,.$$

(We have to assume "the holomorphy at cusps" if $g = 1$.) We denote by $M_{\mu,k+1/2}(\Gamma)$ the $\mathbf{C}$-vector space of all such mappings. An element $f \in M_{\mu,k+1/2}(\Gamma)$ is called *a cusp form* if f belongs to the kernels of the Φ-operators. We denote the space of cusp forms by $S_{\mu,k+1/2}(\Gamma)$. Namely, f belongs to $S_{\mu,k+1/2}(\Gamma)$ if and only if

$$\Phi f(Z_1) := \lim_{\operatorname{Im} Z_2 \to \infty} f \mid [\xi]_{\mu,k+1/2}\,(Z) = 0$$

for any $\xi \in \widetilde{G}_g$ such that $p(\xi) \in \Gamma_g$, where $Z = \begin{pmatrix} Z_1 & \boldsymbol{o} \\ {}^t\boldsymbol{o} & Z_2 \end{pmatrix}$, $Z_1 \in \mathfrak{S}_{g-1}$ and $Z_2 \in \mathfrak{S}_1$ (cf. Definition 1.5 and Definition 1.7). If μ is the trivial representation, we denote $M_{\mu,k+1/2}(\Gamma)$ and $S_{\mu,k+1/2}(\Gamma)$ by $M_{k+1/2}(\Gamma)$ and $S_{k+1/2}(\Gamma)$, respectively. It is known that $M_{\mu,k+1/2}(\Gamma)$ is finite-dimensional.

Let χ be a character of Γ whose kernel is a subgroup of Γ of finite index. We denote by $M_{\mu,k+1/2}(\Gamma, \chi)$ the $\mathbf{C}$-vector space of the holomorphic mappings of $\mathfrak{S}_g$ to $\mathbf{C}^r$ which satisfy

$$f(M\langle Z\rangle) = J(M, Z)^{2k+1}\,\chi(M)\,\mu(CZ + D)\,f(Z)$$

for any $M \in \Gamma$ and $Z \in \mathfrak{S}_g$. We also denote by $S_{\mu,k+1/2}(\Gamma, \chi)$ its subspace of cusp forms.

Now we assume that $g = 2$ and μ is the symmetric tensor representation of degree j which we denote by Sym^j. We denote $M_{\mu,k+1/2}(\Gamma)$ and $S_{\mu,k+1/2}(\Gamma)$ by $M_{j,k+1/2}(\Gamma)$ and $S_{j,k+1/2}(\Gamma)$, respectively. Let ψ be as before. We define a character of $M \in \Gamma_0^2(4)$ by $\psi(\det D)$ where D is the lower right 2×2 matrix of M. If j is odd, then $M_{j,k+1/2}(\Gamma_0^2(4))$ and $M_{j,k+1/2}(\Gamma_0^2(4), \psi)$ are $\{0\}$ since $-1_4 \in \Gamma_0^2(4)$ and $\mathrm{Sym}^j(-1_2) = -1_{j+1}$. Therefore we assume that j is even. The purpose of this paper is to compute the dimension of $S_{2j,k+1/2}(\Gamma_0^2(4))$ and $S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ (Theorem 4.4 and Theorem 4.5). From these results we can prove that $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4))$ and $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4), \psi)$ are free modules of rank one over the graded ring of the automorphic forms of integral weights (Proposition 5.2 and Proposition 5.3). Their structures were explicitly determined by T. Ibukiyama ([Ib]). By using a similar method in [Sto], we can also determine the structure of the module $\bigoplus_{k=0}^{\infty} M_{2,k+1/2}(\Gamma_0^2(4))$ ([T6]).

More generally we can express the dimension of $S_{j,k+1/2}(\Gamma, \chi)$ by a finite sum for general Γ and χ (Theorem 3.2). Especially we will be able to compute the dimension of $S_{2j,k+1/2}(\Gamma_0^2(4p), \chi)$, where p is an odd prime and χ is a Dirichlet character modulo $4p$ (cf. [T5] for the case of integral weight). But this will be an exhausting job.

1. Transformation formula of $\Theta(Z)$ and the line bundle $\overline{H}_g$

In this section we recall the transformation formula of $\Theta(Z)$ (Theorem 1.4, cf. [Si] or [Smi]). Next we prove that the line bundle of the modular forms of half integral weight is extendable onto the Satake compactification of the Siegel space.

DEFINITION 1.1. Let $A \in M(g, \mathbf{C})$ be a symmetric matrix with $\mathrm{Re}\,(A) > 0$. Then there exists $T \in GL(g, \mathbf{R})$ such that

$$ {}^tTAT = \begin{pmatrix} 1+id_1 & & \\ & \ddots & \\ & & 1+id_g \end{pmatrix} . $$

We define $(\det A)^{1/2} = |\det T|^{-1} \prod_{j=1}^{g} (1+id_j)^{1/2}$, where we choose $z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \le \pi/2$ for $z \in \mathbf{C}$.

REMARK 1.2. If $g = 2$, $(\det A)^{1/2}$ is uniquely determined by the condition $-\pi/2 < \arg(\det A)^{1/2} < \pi/2$, because $-\pi/4 < \arg(1+id_j)^{1/2} < \pi/4$ $(j = 1, 2)$.

LEMMA 1.3. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ *and let* $m = \mathrm{rank}\, C$. *Then there exist* M', M_1, $M_2 \in \Gamma_g$ *such that*

$$ M = M_1 M' M_2 , \quad M_1 = \begin{pmatrix} A_1 & B_1 \\ O & D_1 \end{pmatrix} , \quad M_2 = \begin{pmatrix} A_2 & O \\ O & D_2 \end{pmatrix} , $$

$$ M' = \begin{pmatrix} A_0 & O & B_0 & O \\ O & 1_{g-m} & O & O \\ C_0 & O & D_0 & O \\ O & O & O & 1_{g-m} \end{pmatrix} , \quad \text{where } \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Gamma_m \text{ and } \det C_0 \neq 0 . $$

(*If* $m = 0$, *we suppose* $M' = 1_{2g}$.) *Moreover we can choose* C_0 *so that*

$$ C_0 = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_m \end{pmatrix} , \quad c_i \mid c_{i+1} \;\; (1 \le i \le m-1) . $$

Proof. The assertion is easily proved ([Smi], Theorem 8.1). But we give a proof here because we use the process of the proof later. There exist U, $V \in GL(g, \mathbf{Z})$ such that $UCV = \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix}$, where C_0 has the above form. Let $UD\,{}^tV^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ $(D_{11} \in M(m, \mathbf{Z}))$. Then since $C\,{}^tD = D\,{}^tC$, we have

$$ \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix} {}^t\!\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} {}^t\!\begin{pmatrix} C_0 & O \\ O & O \end{pmatrix} $$

and $D_{21}{}^tC_0 = O$. Hence $D_{21} = O$, since $\det C_0 \neq 0$. On the other hand $\begin{pmatrix} C_0 & O & D_{11} & D_{12} \\ O & O & O & D_{22} \end{pmatrix}$ is primitive. This means that $D_{22} \in GL(g-m, \mathbf{Z})$. Let

$$U_1 = \begin{pmatrix} 1_m & -D_{12}D_{22}^{-1} \\ O & D_{22}^{-1} \end{pmatrix}$$

and $D_0 = D_{11}$. Replacing U with U_1U we can assume that $UCV = \begin{pmatrix} C_0 & O \\ O & O \end{pmatrix}$ and $UD\,{}^tV^{-1} = \begin{pmatrix} D_0 & O \\ O & 1_{g-m} \end{pmatrix}$. Since $C_0{}^tD_0 = D_0{}^tC_0$, there exists $M_0 \in \Gamma_m$ such that $M_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$. We define M' by using M_0 as above. Let

$$M'' = \begin{pmatrix} {}^tU^{-1} & O \\ O & U \end{pmatrix} M \begin{pmatrix} V & O \\ O & {}^tV^{-1} \end{pmatrix}.$$

Then $M'M''^{-1}$ has the form $\begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix}$ (${}^tS = S$). So $M_1 = \begin{pmatrix} {}^tU & -{}^tUS \\ O & U^{-1} \end{pmatrix}$ and $M_2 = \begin{pmatrix} V^{-1} & O \\ O & {}^tV \end{pmatrix}$ satisfy the condition. □

Now for $Z \in \mathfrak{S}_g$, we put

$$M'M_2(Z) = \begin{pmatrix} Z_1 & Z_2 \\ {}^tZ_2 & Z_3 \end{pmatrix}, \quad \text{where } Z_1 \in \mathfrak{S}_m \text{ and } Z_3 \in \mathfrak{S}_{g-m}, \text{ if } m > 0,$$

and

$$j(M, Z) = \begin{cases} |\det C_0|^{1/2} \det(-i(Z_1 - A_0C_0^{-1}))^{1/2}, & \text{if } m > 0, \\ 1, & \text{if } m = 0. \end{cases}$$

Next we put

$$\lambda(M) = \begin{cases} |\det(C_0/2)|^{-1/2} \displaystyle\sum_{\eta \in \mathbf{Z}^g/({}^tC_0/2)\mathbf{Z}^g} \mathbf{e}(-{}^t\eta(C_0^{-1}D_0)\eta), & \text{if } m > 0, \\ 1, & \text{if } m = 0. \end{cases}$$

Then we have

THEOREM 1.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^g(4)$ *and let* $j(M, Z)$ *and* $\lambda(M)$ *be as above. Let* $J(M, Z) = j(M, Z)^{-1}\lambda(M)^{-1}$. *Then it holds that*

$$\Theta(M\langle Z\rangle) = J(M, Z)\Theta(Z)$$

and

$$J(M,Z)^2 = \det(CZ+D)\psi(\det D)\,.$$

DEFINITION 1.5. Let 1_g be the unit matrix of degree g and $J_g = \begin{pmatrix} O & 1_g \\ -1_g & O \end{pmatrix}$. Let

$$G_g = \{M \in GL(2g, \mathbf{R}) \mid {}^tMJ_gM = \nu(M)J_g,\ \text{ with some } \nu(M) > 0\}$$

be the symplectic group of degree g with similitudes. Let $\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$. We define a group $\widetilde{G}_g$ which consists of the pairs $\xi = (M, \phi(Z))$, where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_g$ and $\phi(Z)$ is a non-zero holomorphic function on $\mathfrak{S}_g$ such that

$$\phi(Z)^2 = t(\xi)\nu(M)^{-1/2}\det(CZ+D)$$

for any $Z \in \mathfrak{S}_g$ with some $t(\xi) \in \mathbf{T}$. The multiplicative law is defined as follows:

$$(M_1, \phi_1(Z))(M_2, \phi_2(Z)) = (M_1M_2, \phi_1(M_2\langle Z\rangle)\phi_2(Z))\,.$$

We denote the natural projection of $\widetilde{G}_g$ to G_g by p. By definition, if $p(\xi) = 1_{2g}$, then $\xi = (1_{2g}, t)$ where t is a constant.

COROLLARY 1.6. *We have an injective homomorphism* ι *of* $\Gamma_0^g(4)$ *to* $\widetilde{G}_g$:

$$\iota(M) = (M, J(M,Z))\,.$$

DEFINITION 1.7. For any holomorphic mapping $f : \mathfrak{S}_g \to \mathbf{C}^r$ and $\xi = (M, \phi(Z)) \in \widetilde{G}_g$, we put

$$f \mid [\xi]_{\mu,k+1/2}(Z) = \phi(Z)^{-(2k+1)}\mu(CZ+D)^{-1}f(M\langle Z\rangle)\,.$$

Then we have

$$f \mid [\xi\,\eta]_{\mu,k+1/2}(Z) = (f \mid [\xi]_{\mu,k+1/2}) \mid [\eta]_{\mu,k+1/2}(Z)$$

for any ξ and $\eta \in \widetilde{G}_g$. Such a mapping f belongs to $M_{\mu,k+1/2}(\Gamma_0^g(4))$ if and only if $f \mid [\iota(M)]_{\mu,k+1/2}(Z) = f(Z)$ for any $M \in \Gamma_0^g(4)$.

Let $\Gamma_g(N)$ be the principal congruence subgroup of level N of Γ_g. Namely,

$$\Gamma_g(N) = \{M \in \Gamma_g \mid M \equiv 1_{2g} \pmod N\}\,.$$

$\Gamma_g(N)$ is a normal subgroup of Γ_g. If $N \geq 3$, $\Gamma_g(N)$ acts on $\mathfrak{S}_g$ without fixed points and the quotient space $X_g(N) := \Gamma_g(N)\backslash\mathfrak{S}_g$ is a (non-compact) manifold. $X_g(N)$ is a open subspace of a projective variety $\overline{X}_g(N)$ which was constructed by I. Satake ([Sta], Satake compactification). If $g \geq 2$, $\overline{X}_g(N)$ has singularities along its cusps: $\overline{X}_g(N) - X_g(N)$. Cusps of $\overline{X}_g(N)$ is (as a set) a disjoint union of copies of $X_{g'}(N)$'s ($0 \leq g' < g$). A desingularization $\widetilde{X}_g(N)$ of $\overline{X}_g(N)$ was constructed by J.-I. Igusa ([Ig2]) and Y. Namikawa ([N]) ($g = 2, 3, 4$) and more generally by D. Mumford and others ([AMRT], Toroidal compactification).

Let $\mu : GL(g, \mathbf{C}) \to GL(r, \mathbf{C})$ be a holomorphic representation and let $\mathcal{V}_\mu$ be $\mathfrak{S}_g \times \mathbf{C}^r$, on which $\Gamma_g(N)$ acts as follows:

$$M(Z, v) = (M\langle Z\rangle, \mu(CZ + D)v)\,.$$

If $N \geq 3$, $V_\mu := \Gamma_g(N)\backslash\mathcal{V}_\mu$ is non-singular and is a holomorphic vector bundle over $X_g(N)$. V_μ is extended to a holomorphic vector bundle $\widetilde{V}_\mu$ on $\widetilde{X}_g(N)$ ([Mu]). In the case when $g = 2$ and $\mu = \mathrm{Sym}^j$, we denote V_μ and $\widetilde{V}_\mu$ by $\mathrm{Sym}^j(V)$ and $\mathrm{Sym}^j(\widetilde{V})$, respectively.

Let $\mathcal{H}_g$ be $\mathfrak{S}_g \times \mathbf{C}$. The group $\Gamma_g(4N)$ acts on $\mathcal{H}_g$ as follows:

$$M(Z, v) = (M\langle Z\rangle, J(M, Z)v)\,.$$

Then, $H_g := \Gamma_g(4N)\backslash\mathcal{H}_g$ is a holomorphic line bundle over $X_g(4N)$. We have

THEOREM 1.8. *The line bundle H_g is extendable to an ample line bundle $\overline{H}_g$ over the Satake compactification $\overline{X}_g(4N)$.*

Proof. Let f be a (local) section of $H_g^{\otimes(2k+1)}$. Then f is identified with a (local) modular form of weight $k+1/2$ with respect to $\Gamma_g(4N)$. We denote $\phi(Z)^{-(2k+1)}f(P\langle Z\rangle)$ by $f \mid [\xi]_{k+1/2}(Z)$ for $\xi = (P, \phi(Z)) \in \widetilde{G}_g$. We prove that

$$f \mid [\xi]_{k+1/2}(Z + S) = f \mid [\xi]_{k+1/2}(Z)$$

for any $\xi \in p^{-1}(\Gamma_g)$ and any integral symmetric matrix S whose entries are divisible by $4N$. Then $f \mid [\xi]_{k+1/2}(Z)$ is expanded to a Fourier series:

$$f \mid [\xi]_{k+1/2}(Z) = \sum_{T\geq 0} a(T)\,\mathbf{e}(\mathrm{tr}(TZ)/4N)\,,$$

where T is over all half-integral semi-positive symmetric matrices and from this fact it is proved that H_g is extendable onto $\overline{X}_g(4N)$ similarly as in [Sta]. $\overline{H}_g^{\otimes 2}$ is isomorphic to the line bundle $\overline{L}_g$ which is defined by the automorphy factor $\det(CZ+D)$. Since $\overline{L}_g$ is ample ([B]), $\overline{H}_g$ is also ample.

Let $M = \begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix} \in \Gamma_g(4N)$ and $\xi = (P, \phi(Z)) \in p^{-1}(\Gamma_g)$. Then PMP^{-1} belongs to $\Gamma_g(4N)$ since $\Gamma_g(4N)$ is a normal subgroup of Γ_g. We prove that

$$\xi\,\iota(M)\,\xi^{-1} = \iota(PMP^{-1})\,.$$

Then we have

$$f \mid [\xi\,\iota(M)\,\xi^{-1}]_{k+1/2}(Z) = f \mid [\iota(PMP^{-1})]_{k+1/2}(Z) = f(Z)$$

from the assumption that f is a (local) modular form with respect to $\Gamma_g(4N)$. Hence it follows that

$$f \mid [\xi]_{k+1/2}(Z + S) = f \mid [\xi\,\iota(M)]_{k+1/2}(Z) = f \mid [\xi]_{k+1/2}(Z)\,.$$

Now we prove our assertion. Since $\xi^{-1} = (P^{-1}, \phi(P^{-1}\langle Z\rangle)^{-1})$, we have

$$\iota(PMP^{-1})(\xi\,\iota(M)\,\xi^{-1})^{-1} = \iota(PMP^{-1})\,\xi\,\iota(M^{-1})\,\xi^{-1} = (1_{2g}, t)\,,$$

where

$$t = J(PMP^{-1}, PM^{-1}P^{-1}\langle Z\rangle)\,\phi(M^{-1}P^{-1}\langle Z\rangle)\,J(M^{-1}, P^{-1}\langle Z\rangle)\,\phi(P^{-1}\langle Z\rangle)^{-1}$$

is a constant. We prove that $t = 1$. Let $Z = P\langle Z' + S\rangle$. Since $J(M^{-1}, P^{-1}\langle Z\rangle) = 1$, t is equal to

$$\frac{\Theta(Z)}{\Theta(PM^{-1}P^{-1}\langle Z\rangle)} \cdot \frac{\phi(M^{-1}P^{-1}\langle Z\rangle)}{\phi(P^{-1}\langle Z\rangle)} = \frac{\Theta(P\langle Z' + S\rangle)}{\Theta(P\langle Z'\rangle)} \cdot \frac{\phi(Z')}{\phi(Z' + S)}\,.$$

Let $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then by definition we have

$$\frac{\phi(Z')}{\phi(Z' + S)} = \frac{\sqrt{\det(CZ' + D)}}{\sqrt{\det(C(Z' + S) + D)}}\,.$$

Since $\sqrt{\det(CZ' + D)}$ is a non-zero function on the simply connected space $\mathfrak{S}_g$, the sign of $\sqrt{\det(C(Z' + S) + D)}$ is uniquely determined by the sign of $\sqrt{\det(CZ' + D)}$ and we have

$$\lim_{\operatorname{Im} Z' \to \infty} \frac{\phi(Z')}{\phi(Z' + S)} = 1\,.$$

Hence the assertion is equivalent to

$$\lim_{\operatorname{Im} Z' \to \infty} J(PMP^{-1}, P\langle Z'\rangle) = \lim_{\operatorname{Im} Z' \to \infty} \frac{\Theta(P\langle Z' + S\rangle)}{\Theta(P\langle Z'\rangle)} = 1\,.$$

We fix P and assume that

$$\lim_{\operatorname{Im} Z \to \infty} J(PMP^{-1}, P\langle Z\rangle) = 1$$

for any $M = \begin{pmatrix} 1_g & S \\ O & 1_g \end{pmatrix} \in \Gamma_g(4N)$. Let $Q \in \Gamma_0^g(4)$. Then we have

$$J(QPMP^{-1}Q^{-1}, QP\langle Z\rangle) = J(Q, PM\langle Z\rangle)J(PMP^{-1}, P\langle Z\rangle)J(Q^{-1}, QP\langle Z\rangle)\,.$$

Since

$$\lim_{\operatorname{Im} Z \to \infty} J(Q, PM\langle Z\rangle)J(Q^{-1}, QP\langle Z\rangle) = \lim_{\operatorname{Im} Z \to \infty} \frac{J(Q, P\langle Z + S\rangle)}{J(Q, P\langle Z\rangle)} = 1\,,$$

it follows that

$$\lim_{\operatorname{Im} Z \to \infty} J(QPMP^{-1}Q^{-1}, QP\langle Z\rangle) = 1\,,$$

from the assumption.

Next let $N(B_0, \Gamma_g)$ be the subgroup of Γ_g consisting of the elements of the form:

$$\begin{pmatrix} U & T\,{}^tU^{-1} \\ O & {}^tU^{-1} \end{pmatrix}, \quad U \in GL(g, \mathbf{Z})\,, \quad T \in M(g, \mathbf{Z})\,, \quad {}^tT = T\,.$$

Let $R \in N(B_0, \Gamma_g)$ be an element of the above form. Then

$$J(PRMR^{-1}P^{-1}, PR\langle Z\rangle) = J(PM_1P^{-1}, P\langle UZ\,{}^tU + T\rangle)\,,$$

where

$$M_1 = \begin{pmatrix} 1_g & US\,{}^tU \\ O & 1_g \end{pmatrix}.$$

Hence it follows that

$$\lim_{\operatorname{Im} Z\to\infty} J(PRMR^{-1}P^{-1}, PR\langle Z\rangle) = 1$$

from the assumption.

Therefore it suffices to prove the assertion for the representatives of the double cosets in $\Gamma_0^g(4)\backslash\Gamma_g/N(B_0, \Gamma_g)$. Let $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$. Let $\overline{C} = (c_{ij})$ be the matrix such that $\overline{C} \equiv C \pmod 4$ and $-1 \le c_{ij} \le 2$ $(1 \le i,\ j \le g)$. There exists $P' = \begin{pmatrix} A' & B' \\ \overline{C} & D' \end{pmatrix} \in \Gamma_g$ such that $P \equiv P' \pmod 4$ (cf. [Ig3], Chap. V, Lemma 25). Notice that we can apply the proof of this lemma without changing η' which is the first row of $\overline{C}$. Then $P'P^{-1} \in \Gamma_g(4) \subset \Gamma_0^g(4)$. Hence we can replace P with P'. Let $m = \operatorname{rank} \overline{C}$ and represent P' as $M_1M'M_2$ in Lemma 1.3. We can replace P' with M'. So we assume that

$$P = \begin{pmatrix} A_0 & O & B_0 & O \\ O & 1_{g-m} & O & O \\ C_0 & O & D_0 & O \\ O & O & O & 1_{g-m} \end{pmatrix}, \quad \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Gamma_m,\ \det C_0 \ne 0$$

and

$$C_0 = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_m \end{pmatrix}, \quad c_i = 1 \text{ or } 2\ \ (1 \le i \le m)\,.$$

It suffices to prove the case when $N = 1$. Let $E_{ij} = (a_{kl})$ be the matrix such that $a_{ij} = 1$ and $a_{kl} = 0$, otherwise. Let $M_1 = \begin{pmatrix} 1_g & S_1 \\ O & 1_g \end{pmatrix}$, $M_2 = \begin{pmatrix} 1_g & S_2 \\ O & 1_g \end{pmatrix} \in \Gamma_g(4)$. Then we have

$$\begin{aligned}&\lim_{\operatorname{Im} Z\to\infty} J(PM_1M_2P^{-1}, P\langle Z\rangle) \\ &\quad = \lim_{\operatorname{Im} Z\to\infty} J(PM_1P^{-1}, P\langle Z\rangle) \lim_{\operatorname{Im} Z\to\infty} J(PM_2P^{-1}, P\langle Z\rangle)\,.\end{aligned}$$

Hence it suffices to prove the assertion for the case when $S = 4E_{ii}$ or $S = 4E_{ij} + 4E_{ji}$ $(i \ne j)$. First we prove the case when $S = 4E_{ii}$. Let V_{ij} be the matrix corresponding to the transposition (ij). Namely, $V_{ij} = 1_g - E_{ii} - E_{jj} + E_{ij} + E_{ji}$. Let $\sigma = (1i)$ and $V^\sigma = \begin{pmatrix} V_{1i} & O \\ O & V_{1i} \end{pmatrix}$. As we showed before, we have

$$\lim_{\operatorname{Im} Z\to\infty} J(PMP^{-1}, P\langle Z\rangle) = \lim_{\operatorname{Im} Z\to\infty} J(P^\sigma M^\sigma P^{\sigma\,-1}, P^\sigma\langle V_{1i} Z V_{1i}\rangle)\,,$$

where $P^\sigma = V^\sigma P V^\sigma$ and $M^\sigma = V^\sigma M V^\sigma = \begin{pmatrix} 1_g & 4E_{11} \\ O & 1_g \end{pmatrix}$. Let $P^\sigma = \begin{pmatrix} A^\sigma & B^\sigma \\ C^\sigma & D^\sigma \end{pmatrix}$. Then

$$P^\sigma M^\sigma P^{\sigma\,-1} = \begin{pmatrix} 1_g - 4A^\sigma E_{11}{}^t C^\sigma & 4A^\sigma E_{11}{}^t A^\sigma \\ -4C^\sigma E_{11}{}^t C^\sigma & 1_g + 4C^\sigma E_{11}{}^t A^\sigma \end{pmatrix}.$$

If $i > m$, then the assertion is trivial because $-4C^\sigma E_{11}{}^t C^\sigma = O$. So we assume that $i \le m$. Then

$$-4C^\sigma E_{11}{}^t C^\sigma = \begin{pmatrix} -4c_i^2 & {}^t\boldsymbol{o} \\ \boldsymbol{o} & O \end{pmatrix}, \quad 1_g + 4C^\sigma E_{11}{}^t A^\sigma = \begin{pmatrix} 1+4a_{ii}c_i & * \\ \boldsymbol{o} & 1_{g-1} \end{pmatrix},$$

$$1_g - 4A^\sigma E_{11}{}^t C^\sigma = \begin{pmatrix} 1-4a_{ii}c_i & {}^t\boldsymbol{o} \\ * & 1_{g-1} \end{pmatrix}, \quad 4A^\sigma E_{11}{}^t A^\sigma = \begin{pmatrix} 4a_{ii}^2 & * \\ * & * \end{pmatrix}.$$

Hence $P^\sigma M^\sigma P^{\sigma\,-1}$ is represented as $M_1 M' M_2$ where $M_2 = 1_{2g}$ and

$$M' = \begin{pmatrix} 1-4a_{ii}c_i & {}^t\boldsymbol{o} & 4a_{ii}^2 & {}^t\boldsymbol{o} \\ \boldsymbol{o} & 1_{g-1} & \boldsymbol{o} & O \\ -4c_i^2 & {}^t\boldsymbol{o} & 1+4a_{ii}c_i & {}^t\boldsymbol{o} \\ \boldsymbol{o} & O & \boldsymbol{o} & 1_{g-1} \end{pmatrix}.$$

Let $P^\sigma\langle V_{1i} Z V_{1i}\rangle = \begin{pmatrix} W_1 & W_2 \\ {}^t W_2 & W_3 \end{pmatrix}$ ($W_1 \in \mathfrak{S}_1$). Then

$$\lim_{\operatorname{Im} Z\to\infty} W_1 = \frac{a_{ii}}{c_i}\,.$$

$M_0 = \begin{pmatrix} 1-4a_{ii}c_i & 4a_{ii}^2 \\ -4c_i^2 & 1+4a_{ii}c_i \end{pmatrix}$ fixes $\dfrac{a_{ii}}{c_i}$. Hence we have

$$\lim_{\operatorname{Im} Z\to\infty} j(P^\sigma M^\sigma P^{\sigma\,-1}, P^\sigma\langle V_{1i} Z V_{1i}\rangle) = \frac{1-i}{\sqrt{2}}\,.$$

On the other hand from Lemma 1.9 exhibited just after this proof we have

$$\lambda(P^\sigma M^\sigma P^{\sigma\,-1}) = \frac{1}{\sqrt{2}c_i}\sum_{x=0}^{2c_i^2-1} \mathbf{e}\left(\frac{(1+4a_{ii}c_i)x^2}{4c_i^2}\right) = \frac{1+i}{\sqrt{2}}\,.$$

Therefore it follows that

$$\lim_{\operatorname{Im} Z\to\infty} J(P^\sigma M^\sigma P^{\sigma\,-1}, P^\sigma\langle V_{1i} Z V_{1i}\rangle) = 1\,.$$

Next we prove the case when $S = 4E_{ij} + 4E_{ji}$ $(i \neq j)$. Let $\sigma = (1i)(2j)$ and $V^\sigma = \begin{pmatrix} V_{1i}V_{2j} & O \\ O & V_{1i}V_{2j} \end{pmatrix}$. As we showed before, we have

$$\lim_{\operatorname{Im} Z \to \infty} J(PMP^{-1}, P\langle Z\rangle) = \lim_{\operatorname{Im} Z \to \infty} J(P^\sigma M^\sigma P^{\sigma\,-1}, P^\sigma\langle V_{1i}V_{2j}Z\,V_{2j}V_{1i}\rangle)\,,$$

where $P^\sigma = V^\sigma P V^\sigma = \begin{pmatrix} A^\sigma & B^\sigma \\ C^\sigma & D^\sigma \end{pmatrix}$ and $M^\sigma = V^\sigma M V^\sigma = \begin{pmatrix} 1_g & 4E_{12} + 4E_{21} \\ O & 1_g \end{pmatrix}$. Then

$$P^\sigma M^\sigma P^{\sigma\,-1} = \begin{pmatrix} 1_g - 4A^\sigma(E_{12}+E_{21})^t C^\sigma & 4A^\sigma(E_{12}+E_{21})^t A^\sigma \\ -4C^\sigma(E_{12}+E_{21})^t C^\sigma & 1_g + 4C^\sigma(E_{12}+E_{21})^t A^\sigma \end{pmatrix}.$$

If $i > m$ or $j > m$, then the assertion is trivial. So we assume that $i,\ j \le m$. Then

$$-4C^\sigma(E_{12}+E_{21})^t C^\sigma = \begin{pmatrix} 0 & -4c_ic_j & {}^t\boldsymbol{o} \\ -4c_ic_j & 0 & {}^t\boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} & O \end{pmatrix},$$

$$1_g + 4C^\sigma(E_{12}+E_{21})^t A^\sigma = \begin{pmatrix} 1+4a_{ij}c_i & 4a_{jj}c_i & * \\ 4a_{ii}c_j & 1+4a_{ji}c_j & * \\ \boldsymbol{o} & \boldsymbol{o} & 1_{g-2} \end{pmatrix},$$

$$1_g - 4A^\sigma(E_{12}+E_{21})^t C^\sigma = \begin{pmatrix} 1-4a_{ij}c_i & -4a_{ii}c_j & {}^t\boldsymbol{o} \\ -4a_{jj}c_i & 1-4a_{ji}c_j & {}^t\boldsymbol{o} \\ * & * & 1_{g-2} \end{pmatrix},$$

$$4A^\sigma(E_{12}+E_{21})^t A^\sigma = \begin{pmatrix} 8a_{ii}a_{ij} & 4a_{ii}a_{jj}+4a_{ij}a_{ji} & * \\ 4a_{ii}a_{jj}+4a_{ij}a_{ji} & 8a_{jj}a_{ji} & * \\ * & * & * \end{pmatrix}.$$

Since $A^\sigma\,{}^tC^\sigma = C^\sigma\,{}^tA^\sigma$, we have $a_{ij}c_i = a_{ji}c_j$. Hence $P^\sigma M^\sigma P^{\sigma\,-1}$ is represented as $M_1 M' M_2$ where $M_2 = 1_{2g}$ and

$$M' = \begin{pmatrix} 1-4a_{ij}c_i & -4a_{ii}c_j & {}^t\boldsymbol{o} & 8a_{ii}a_{ij} & 4a_{ii}a_{jj}+4a_{ij}a_{ji} & {}^t\boldsymbol{o} \\ -4a_{jj}c_i & 1-4a_{ji}c_j & {}^t\boldsymbol{o} & 4a_{ii}a_{jj}+4a_{ij}a_{ji} & 8a_{jj}a_{ji} & {}^t\boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} & 1_{g-2} & \boldsymbol{o} & \boldsymbol{o} & O \\ 0 & -4c_ic_j & {}^t\boldsymbol{o} & 1+4a_{ij}c_i & 4a_{jj}c_i & {}^t\boldsymbol{o} \\ -4c_ic_j & 0 & {}^t\boldsymbol{o} & 4a_{ii}c_j & 1+4a_{ji}c_j & {}^t\boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} & O & \boldsymbol{o} & \boldsymbol{o} & 1_{g-2} \end{pmatrix}.$$

Let $P^\sigma\langle V_{1i}V_{2j}Z\,V_{2j}V_{1i}\rangle = \begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_3 \end{pmatrix}$ $(W_1 \in \mathfrak{S}_2)$. Then

$$\lim_{\operatorname{Im} Z \to \infty} W_1 = \frac{1}{c_ic_j}\begin{pmatrix} a_{ii}c_j & a_{ij}c_i \\ a_{ji}c_j & a_{jj}c_i \end{pmatrix}$$

which is fixed by

$$M_0 = \begin{pmatrix} 1-4a_{ij}c_i & -4a_{ii}c_j & 8a_{ii}a_{ij} & 4a_{ii}a_{jj}+4a_{ij}a_{ji} \\ -4a_{jj}c_i & 1-4a_{ji}c_j & 4a_{ii}a_{jj}+4a_{ij}a_{ji} & 8a_{jj}a_{ji} \\ 0 & -4c_ic_j & 1+4a_{ij}c_i & 4a_{jj}c_i \\ -4c_ic_j & 0 & 4a_{ii}c_j & 1+4a_{ji}c_j \end{pmatrix}.$$

Hence we have

$$\lim_{\operatorname{Im} Z\to\infty} j(P^\sigma M^\sigma P^{\sigma-1}, P^\sigma\langle V_{1i}V_{2j}Z\,V_{2j}V_{1i}\rangle) = 1\,.$$

On the other hand from Lemma 1.9 we have

$$\lambda(P^\sigma M^\sigma P^{\sigma-1}) = \frac{1}{2c_ic_j}\sum_{x,\,y=0}^{2c_ic_j-1} \mathbf{e}\left(\frac{4a_{ii}c_jx^2+2(1+4a_{ij}c_i)xy+4a_{jj}c_iy^2}{4c_ic_j}\right) = 1\,.$$

Therefore it follows that

$$\lim_{\operatorname{Im} Z\to\infty} J(P^\sigma M^\sigma P^{\sigma-1}, P^\sigma\langle V_{1i}V_{2j}Z\,V_{2j}V_{1i}\rangle) = 1\,.$$

Now the proof of Theorem 1.8 was completed. □

LEMMA 1.9. (1) *If* $(c_i, a_{ii}) = (1,0),\ (2,0)$ *or* $(2,1)$, *then*

$$\sum_{x=0}^{2c_i^2-1} \mathbf{e}\left(\frac{(1+4a_{ii}c_i)x^2}{4c_i^2}\right) = (1+i)c_i\,.$$

(2) *If* $(c_i, c_j, a_{ii}, a_{ij}, a_{ji}, a_{jj}) = (1,1,0,0,0,0),\ (1,2,0,0,0,0),\ (1,2,0,0,0,1),\ (2,2,0,0,0,0),\ (2,2,1,0,0,0),\ (2,2,0,0,0,1)$ *or* $(2,2,1,0,0,1)$, *then*

$$\sum_{x,\,y=0}^{2c_ic_j-1} \mathbf{e}\left(\frac{4a_{ii}c_jx^2+2(1+4a_{ij}c_i)xy+4a_{jj}c_iy^2}{4c_ic_j}\right) = 2c_ic_j\,.$$

Proof. Directly proved by computation. □

REMARK 1.10. There are some cases such that S is not divisible by 4, $PMP^{-1} \in \Gamma_0^g(4)$ and $\lim_{\operatorname{Im} Z\to\infty} J(PMP^{-1}, P\langle Z\rangle) = i^a$ ($a \not\equiv 0 \pmod 4$) (cf. Theorem 3.9 (15) Φ_{15c}). Hence H_g is not extendable onto the Satake compactification $\overline{\Gamma\backslash\mathfrak{S}_g}$ for general Γ. Actually H_g is not extendable onto $\overline{\Gamma_0^g(4)\backslash\mathfrak{S}_g}$.

NOTATION 1.11. Let $\overline{H}_g$ and $\overline{L}_g$ be as above. Then we denote by $\widetilde{H}_g$ and $\widetilde{L}_g$ the pullbacks of $\overline{H}_g$ and $\overline{L}_g$ by the natural morphism of $\widetilde{X}_g(4N)$ to $\overline{X}_g(4N)$, respectively.

2. Classification of the fixed points (sets)

Let Γ be a subgroup of $\Gamma_0^g(4)$ of finite index. If $g \geq 2$, Γ contains $\Gamma_g(4N)$ for some N ([BLS], [Me]). In the following we assume that $g = 2$ and μ is Sym^j. The space of

Siegel modular forms $M_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)})) ,$$

which is the space of the global holomorphic sections of $\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)}$. The divisor at infinity $D := \widetilde{X}_2(4N) - X_2(4N)$ is a divisor with simple normal crossings. The space of cusp forms $S_{j,k+1/2}(\Gamma_2(4N))$ is canonically identified with the space

$$\Gamma(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} - D)) .$$

Here $\mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} - D)$ is the sheaf of germs of holomorphic sections which vanish along D and this is isomorphic to $\mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})$ where $[D]$ is the holomorphic line bundle which is associated with D.

Let χ be a character of Γ whose kernel is a subgroup of Γ of finite index. We may assume that the kernel of χ contains $\Gamma_2(4N)$. Let $f \in S_{j,k+1/2}(\Gamma_2(4N))$ and $M \in \Gamma$. We define an action of M on $S_{j,k+1/2}(\Gamma_2(4N))$ as follows:

$$Mf\,(M\langle Z\rangle) = J(M,Z)^{2k+1}\,\chi(M)\,\mathrm{Sym}^j(CZ+D)\,f(Z) .$$

Since $\Gamma_2(4N)$ acts trivially on $S_{j,k+1/2}(\Gamma_2(4N))$, this action induces an action of the factor group $\Gamma/\Gamma_2(4N)$ on $S_{j,k+1/2}(\Gamma_2(4N))$ and $S_{j,k+1/2}(\Gamma, \chi)$ is identified with the invariant subspace of $S_{j,k+1/2}(\Gamma_2(4N))$. Thus we have

$$S_{j,k+1/2}(\Gamma, \chi) = S_{j,k+1/2}(\Gamma_2(4N))^{\Gamma/\Gamma_2(4N)} .$$

Therefore the dimension of $S_{j,k+1/2}(\Gamma, \chi)$ is computed by applying the holomorphic Lefschetz fixed point formula ([AS]) and the vanishing theorem (Theorem 4.1) to the above situation.

To use the holomorphic Lefschetz fixed point formula we have to classify the fixed points (sets) of Γ_2 and $\Gamma_2/\Gamma_2(4N)$ acting on $\widetilde{X}_2(4N)$. We classify (the irreducible components of) the fixed points (sets) of Γ_2 in the following sense. Let Φ and Φ' be the fixed points (sets). Φ and Φ' are called *equivalent* if there is an element of Γ_2 which maps Φ biholomorphically to Φ'. The fixed points in the quotient space $X_2(4N)$ were classified in [G]. The fixed points in the divisor at infinity are classified easily. In total there are 25 kinds of fixed points (sets).

LEMMA 2.1. *Among the* 25 *kinds of fixed points* (*sets*) *the following* 10 *fixed points* (*sets*) *are not fixed by the elements of* Γ_2 *which are conjugate to elements of* $\Gamma_0^2(4)$, *where* $\rho = \mathbf{e}(1/3)$, $\omega = \mathbf{e}(1/5)$, $\eta = (1+2\sqrt{-2})/3$ *and* $Z \in \mathfrak{S}_1$. *To represent the fixed points* (*sets*) *we use the same notations* $\Phi_7, \Phi_8, \cdots, \Phi_{21}$ *as in* [T2].

$$\Phi_7 : \left\{\begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix}\right\}, \quad \Phi_8 : \left\{\begin{pmatrix} \rho & 0 \\ 0 & Z \end{pmatrix}\right\}, \quad \Phi_9 : \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \Phi_{11} : \begin{pmatrix} \rho & 0 \\ 0 & i \end{pmatrix},$$

$$\Phi_{13} : \begin{pmatrix} \eta & (\eta-1)/2 \\ (\eta-1)/2 & \eta \end{pmatrix}, \quad \Phi_{14} : \begin{pmatrix} \omega & \omega+\omega^3 \\ \omega+\omega^3 & -\omega^4 \end{pmatrix}, \quad \Phi_{18} : \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix},$$

$$\Phi_{19} : \begin{pmatrix} i & (i+1)/2 \\ (i+1)/2 & \infty \end{pmatrix}, \quad \Phi_{20} : \begin{pmatrix} \rho & 0 \\ 0 & \infty \end{pmatrix}, \quad \Phi_{21} : \begin{pmatrix} \rho & (\rho+2)/3 \\ (\rho+2)/3 & \infty \end{pmatrix}.$$

Proof. If M belongs to $\Gamma_0^2(4)$, we have

$$M \equiv \begin{pmatrix} U & V \\ O & {}^tU^{-1} \end{pmatrix} \pmod 4,$$

where $U \in GL(2, \mathbf{Z})$. Since $(\det U)^2 \equiv 1$ and $\det(x1_2 - {}^tU^{-1}) \equiv \det(x1_2 - U^{-1}) \cdot (\det U)^2 \equiv \det(xU - 1_2) \cdot \det U \pmod 4$, the characteristic polynomial $P_M(x)$ of M is equivalent to one of the following polynomials modulo 4:

$$(x^2+1)^2,\ (x^2+x+1)^2,\ (x^2+x-1)(x^2-x-1),\ (x^2+2x+1)^2,$$
$$(x^2-1)^2,\ (x^2-x+1)^2,\ (x^2-x-1)(x^2+x-1),\ (x^2+2x-1)(x^2-2x-1).$$

Therefore if $M \in \Gamma_2$ is conjugate to an element of $\Gamma_0^2(4)$, then the characteristic polynomial $P_M(x)$ of M is equivalent to one of the following three polynomials modulo 4:

$$x^4+2x^2+1, \quad x^4+2x^3+3x^2+2x+1, \quad x^4+x^2+1.$$

From this fact we can show that the above points (sets) except Φ_9 are not fixed by the elements of Γ_2 which are conjugate to elements of $\Gamma_0^2(4)$. Since the characteristic polynomial of P_2 (cf. Proposition 2.5) which fixes Φ_9 is $(x^2+1)^2$, the above argument is not valid in this case. In this case we have to check more carefully and the assertion is proved in Theorem 2.8 (9). □

REMARK 2.2. Although we represented Φ_7 by $\left\{\begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix}\right\} \subset \mathfrak{S}_2$ symbolically, Φ_7 means the image of $\left\{\begin{pmatrix} i & 0 \\ 0 & Z \end{pmatrix}\right\}$ to $\widetilde{X}_2(4N)$. The same applies to Φ_8 and also to the following cases.

The remaining 15 fixed points (sets) have the contributions to the dimension formula. But since the automorphic factor $J(M, Z)$ is defined with respect to $\Gamma_0^2(4)$, we have to classify the remaining 15 fixed points (sets) with respect to $\Gamma_0^2(4)$. Let Φ be one of the following 15 fixed points (sets):

$$\Phi_1 : \left\{\begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix}\right\}, \quad \Phi_2 : \left\{\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}\right\}, \quad \Phi_3 : \left\{\begin{pmatrix} Z_1 & 1/2 \\ 1/2 & Z_2 \end{pmatrix}\right\},$$

$$\Phi_4 : \left\{\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}\right\}, \quad \Phi_5 : \left\{\begin{pmatrix} Z & 1/2 \\ 1/2 & Z \end{pmatrix}\right\}, \quad \Phi_6 : \left\{\begin{pmatrix} Z & Z/2 \\ Z/2 & Z \end{pmatrix}\right\},$$

$$\Phi_{10} : \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \Phi_{12} : \frac{\sqrt{-3}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \Phi_{15} : \left\{\begin{pmatrix} Z & W \\ W & \infty \end{pmatrix}\right\},$$

$$\Phi_{16} : \left\{\begin{pmatrix} Z & 0 \\ 0 & \infty \end{pmatrix}\right\}, \quad \Phi_{17} : \left\{\begin{pmatrix} Z & 1/2 \\ 1/2 & \infty \end{pmatrix}\right\}, \quad \Phi_{22} : \left\{\begin{pmatrix} \infty & W \\ W & \infty \end{pmatrix}\right\},$$

$$\Phi_{23} : \begin{pmatrix} \infty & 0 \\ 0 & \infty \end{pmatrix}, \quad \Phi_{24} : \begin{pmatrix} \infty & 1/2 \\ 1/2 & \infty \end{pmatrix}, \quad \Phi_{25} : \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix},$$

where $\begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix} \in \mathfrak{S}_2,\ Z,\ Z_1,\ Z_2 \in \mathfrak{S}_1$ and $W \in \mathbf{C}$. Strictly speaking Φ_{17} should be represented as

$$\left\{\begin{pmatrix} Z & 1/2 \\ 1/2 & \infty \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} Z & 2N+1/2 \\ 2N+1/2 & \infty \end{pmatrix}\right\}$$
$$\bigcup \left\{\begin{pmatrix} Z & 2NZ+1/2 \\ 2NZ+1/2 & \infty \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} Z & 2N(Z+1)+1/2 \\ 2N(Z+1)+1/2 & \infty \end{pmatrix}\right\}.$$

This appears as a boundary of Φ_3 and is a four fold cover of a one-dimensional cusp.

DEFINITION 2.3. Let us denote by $\mathrm{Fix}(M)$ the fixed points in $\widetilde{X}_2(4N)$ of M and let

$$C(\Phi) = \{M \in \Gamma_2/\Gamma_2(4N) \mid M\langle Z\rangle = Z \text{ for any } Z \in \Phi\},$$
$$C^p(\Phi) = \{M \in C(\Phi) \mid \Phi \text{ is an irreducible component of } \mathrm{Fix}(M)\},$$
$$C(\Phi, \Gamma_2) = \{M \in \Gamma_2 \mid M\langle Z\rangle = Z \text{ for any } Z \in \Phi\},$$
$$C^p(\Phi, \Gamma_2) = \{M \in C(\Phi, \Gamma_2) \mid \Phi \text{ is an irreducible component of } \mathrm{Fix}(M)\},$$
$$N(\Phi, \Gamma_2) = \{M \in \Gamma_2 \mid M \text{ maps } \Phi \text{ into } \Phi\}.$$

We call $C^p(\Phi)$ and $C^p(\Phi, \Gamma_2)$ the sets of *proper* elements in $C(\Phi)$ and in $C(\Phi, \Gamma_2)$, respectively.

What we have to do is to classify the double cosets in $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi, \Gamma_2)$. Since Γ_2 is an infinite group, it is not an easy task to classify $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi, \Gamma_2)$. But since $\Gamma_0^2(4)$ contains $\Gamma_2(4)$ which is a normal subgroup of Γ_2, we can take the quotient by $\Gamma_2(4)$ and reduce the problem to a task in the finite group $\Gamma_2/\Gamma_2(4) \simeq Sp\,(2, \mathbf{Z}/4\mathbf{Z})$ and we can use a computer. So first we classify $\Gamma_0^2(4)\backslash\Gamma_2$ which consists of 120 cosets and next classify these cosets with respect to the action of $N(\Phi, \Gamma_2)$ from the right. We have to execute this computation many times in the following.

Let $P_1,\ P_2, \cdots, P_n$ be the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi, \Gamma_2)$. Next we have to check $P_i\, C^p(\Phi, \Gamma_2)P_i^{-1} \cap \Gamma_0^2(4)$ $(i = 1, 2, \cdots, n)$ is empty or not. Let

$$P_i\,\Phi = \{P_i\,\langle Z\rangle \mid Z \in \Phi\}.$$

The following assertion is trivial.

LEMMA 2.4. *If $P_i\,C^p(\Phi, \Gamma_2)P_i^{-1} \cap \Gamma_0^2(4)$ is empty, then $P_i\,\Phi$ is not fixed by the elements of $\Gamma_0^2(4)$.*

Before we classify the fixed points (sets), we classify the rational boundary components of $\mathfrak{S}_2$ with respect to $\Gamma_0^2(4)$ and determine the configuration of the cusps of the Satake compactification $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$ of $\Gamma_0^2(4)\backslash\mathfrak{S}_2$. Let B_1 be the one-dimensional boundary component of $\mathfrak{S}_2$ which is defined by $\operatorname{Im} Z_2 = \infty$. Let $N(B_1, \Gamma_2)$ be the stabilizer in Γ_2 of B_1. The elements of $N(B_1, \Gamma_2)$ have the following form:

$$\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

The one-dimensional cusps of the Satake compactification correspond bijectively to the double cosets in $\Gamma_0^2(4)\backslash\Gamma_2/N(B_1, \Gamma_2)$. Similarly as above we classify the double cosets by a computer. We have

PROPOSITION 2.5. *$\Gamma_0^2(4)\backslash\Gamma_2/N(B_1, \Gamma_2)$ consists of four double cosets. The representatives are P_1, P_2, P_3 and P_4, where $P_1 = 1_4$ and*

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in M acts on the one-dimensional rational boundary component at infinity and $P_i M P_i^{-1}$ $(i = 1, 2, 3, 4)$ belongs to $\Gamma_0^2(4)$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $\Gamma_0^1(4)$, respectively. Hence each one-dimensional cusps of the Satake compactification is biholomorphic to $\overline{\Gamma_0^1(4)\backslash\mathfrak{S}_1}$. $\Gamma_0^1(4)\backslash\mathfrak{S}_1$ is a rational curve with three holes.

Let B_0 be the zero-dimensional boundary component of $\mathfrak{S}_2$ which is defined by $\operatorname{Im} Z_1 = \operatorname{Im} Z_2 = \infty$. Let $N(B_0, \Gamma_2)$ be the stabilizer in Γ_2 of B_0. The elements of

$N(B_0, \Gamma_2)$ have the following form:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

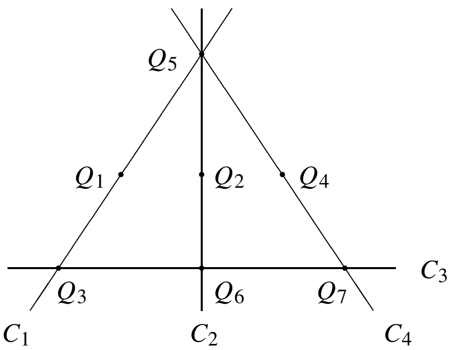
The zero-dimensional cusps of the Satake compactification correspond bijectively to the double cosets in $\Gamma_0^2(4)\backslash\Gamma_2/N(B_0, \Gamma_2)$. We have

PROPOSITION 2.6. *$\Gamma_0^2(4)\backslash\Gamma_2/N(B_0, \Gamma_2)$ consists of seven double cosets. The representatives are $P_1, P_2, \cdots, P_7$, where*

$$P_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad P_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

Let C_i be the one-dimensional cusp corresponding to the double coset $\Gamma_0^2(4)P_iN(B_1, \Gamma_2)$ $(i = 1, 2, 3, 4)$, respectively and let Q_i be the zero-dimensional cusp corresponding to the double coset $\Gamma_0^2(4)P_iN(B_0, \Gamma_2)$ $(i = 1, 2, \cdots, 7)$, respectively. Then the cusps of the Satake compactification look like as follows.

Cusps of $\overline{\Gamma_0^2(4)\backslash\mathfrak{S}_2}$:



This is proved as follows. The cusps Q_1, Q_2, Q_3 and Q_4 are on C_1, C_2, C_3 and C_4, respectively. Since

$$\Gamma_0^2(4)P_5N(B_1, \Gamma_2) = \Gamma_0^2(4)P_1N(B_1, \Gamma_2),$$
$$\Gamma_0^2(4)P_6N(B_1, \Gamma_2) = \Gamma_0^2(4)P_3N(B_1, \Gamma_2),$$
$$\Gamma_0^2(4)P_7N(B_1, \Gamma_2) = \Gamma_0^2(4)P_3N(B_1, \Gamma_2),$$

Q_5, Q_6 and Q_7 are on C_1, C_3 and C_3, respectively. Let P_{11} be as in Theorem 2.8. Since

$$\Gamma_0^2(4)P_5N(B_0, \Gamma_2) = \Gamma_0^2(4)P_{11}N(B_0, \Gamma_2),$$
$$\Gamma_0^2(4)P_{11}N(B_1, \Gamma_2) = \Gamma_0^2(4)P_4N(B_1, \Gamma_2),$$

Q_5 is also on C_4. Similarly we can prove that Q_5 and Q_6 are also on C_2, Q_3 is also on C_1 and Q_7 is also on C_4.

PROPOSITION 2.7. *We have* $[\Gamma_g^\alpha : \Gamma_0^g(4)] = 2^{g(g-1)/2}$. *Especially* $[\Gamma_2^\alpha : \Gamma_0^2(4)] = 2$ *and* $\Gamma_0^2(4)$ *is a normal subgroup of* Γ_2^α.

Proof. We have

$$\alpha\Gamma_g^\alpha\alpha^{-1} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; B \equiv O \pmod 2,\ \text{diagonal elements of } C\,{}^tD \text{ are even} \right\},$$

$$\alpha\Gamma_0^g(4)\alpha^{-1} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; B \equiv C \equiv O \pmod 2 \right\}.$$

We map them into $Sp(g, \mathbf{F}_2)$. Namely,

$$\alpha\Gamma_g^\alpha\alpha^{-1}/\Gamma_g(2) = \left\{ \begin{pmatrix} A & O \\ TA & {}^tA^{-1} \end{pmatrix} \;\middle|\; A \in GL(g, \mathbf{F}_2),\ {}^tT = T, \right.$$
$$\left. \text{diagonal elements of } T \text{ are } 0 \right\},$$

$$\alpha\Gamma_0^g(4)\alpha^{-1}/\Gamma_g(2) = \left\{ \begin{pmatrix} A & O \\ O & {}^tA^{-1} \end{pmatrix} \;\middle|\; A \in GL(g, \mathbf{F}_2) \right\}.$$

Hence $[\alpha\Gamma_g^\alpha\alpha^{-1} : \Gamma_g(2)] = 2^{g(g-1)/2}|GL(g, \mathbf{F}_2)|$ and $[\alpha\Gamma_0^g(4)\alpha^{-1} : \Gamma_g(2)] = |GL(g, \mathbf{F}_2)|$. Therefore $[\Gamma_g^\alpha : \Gamma_0^g(4)] = [\alpha\Gamma_g^\alpha\alpha^{-1} : \alpha\Gamma_0^g(4)\alpha^{-1}] = 2^{g(g-1)/2}$. □

As a matter of fact we classify the fixed points (sets) with respect to Γ_2^α instead of $\Gamma_0^2(4)$ (cf. Remark 3.6). In the following theorem we represent the representatives with respect to $\Gamma_0^2(4)$ as Φ_a, $\Phi_{a'}$, Φ_b, Φ_c. These notations mean that Φ_a and $\Phi_{a'}$ are equivalent with respect to Γ_2^α and Φ_a, Φ_b and Φ_c are not equivalent with respect to Γ_2^α.

THEOREM 2.8. *Let*

$$P_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad P_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, \quad P_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix},$$

$$P_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 2 & 0 & 0 \end{pmatrix}, \quad P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix}, \quad P_{13} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

$$P_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}, \quad P_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad P_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix},$$

$$P_{17}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&2&1&0\\2&2&0&1\end{pmatrix},\quad P_{18}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&1&0\\0&-1&0&0\end{pmatrix},\quad P_{19}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&1&0\\0&-1&0&2\end{pmatrix},$$

$$P_{20}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&1&2\\0&-1&2&0\end{pmatrix},\quad P_{21}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&1&2\\0&-1&2&2\end{pmatrix},\quad P_{22}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&0&1&0\\0&-1&0&1\end{pmatrix},$$

$$P_{23}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&2&1&0\\2&-1&0&1\end{pmatrix},\quad P_{24}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&-1&1&0\\-1&2&0&1\end{pmatrix},\quad P_{25}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\-1&-1&0&1\end{pmatrix},$$

$$P_{26}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&1&1&0\\1&-1&0&1\end{pmatrix},\quad P_{27}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&-1&0\\0&-1&0&0\end{pmatrix},\quad P_{28}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&-1&2\\0&-1&2&0\end{pmatrix},$$

$$P_{29}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&1&1&0\\1&2&0&1\end{pmatrix},\quad P_{30}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&2&1\\-1&2&2&0\end{pmatrix},\quad P_{31}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&2&2&1\\-1&2&2&0\end{pmatrix},$$

$$P_{32}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&2&1&1\\-1&1&0&0\end{pmatrix},\quad P_{33}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&-1&1\\-1&-1&0&0\end{pmatrix},\quad P_{34}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&2\\0&-1&2&0\end{pmatrix}.$$

Then fixed points (sets) of $\Gamma_0^2(4)$ are classified as follows.

(1) *$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_1,\Gamma_2)$ consists of one double coset. The representative is P_1. $\Phi_{1a}:=P_1\,\Phi_1$ is the total space $\widetilde{X}_2(4N)$.*

(2) *$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_2,\Gamma_2)$ consists of three double cosets. The representatives are P_1, P_4 and P_8. Only $\Phi_{2a}:=P_1\,\Phi_2$ and $\Phi_{2a'}:=P_4\,\Phi_2$ are fixed by elements of $\Gamma_0^2(4)$.*

(3) *$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_3,\Gamma_2)$ consists of five double cosets. The representatives are P_1, P_2, P_5, P_6 and P_7. Only $\Phi_{3a}:=P_1\,\Phi_3$, $\Phi_{3b}:=P_5\,\Phi_3$ and $\Phi_{3c}:=P_7\,\Phi_3$ are fixed by elements of $\Gamma_0^2(4)$.*

(4) *$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_4,\Gamma_2)$ consists of eleven double cosets. The representatives are P_1, P_3, P_4, P_5, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13} and P_{14}. Only $\Phi_{4a}:=P_1\,\Phi_4$ and $\Phi_{4a'}:=P_4\,\Phi_4$ are fixed by elements of $\Gamma_0^2(4)$.*

(5) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_5,\Gamma_2)$ *consists of eight double cosets. The representatives are* P_1, P_2, P_6, P_7, P_{10}, P_{13}, P_{15} *and* P_{16}. *Only* $\Phi_{5a} := P_1\,\Phi_5$ *and* $\Phi_{5b} := P_7\,\Phi_5$ *are fixed by elements of* $\Gamma_0^2(4)$.

(6) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_6,\Gamma_2)$ *consists of six double cosets. The representatives are* P_1, P_3, P_5, P_{10}, P_{14} *and* P_{16}. *Only* $\Phi_{6a} := P_1\,\Phi_6$ *is fixed by elements of* $\Gamma_0^2(4)$.

(9) Φ_9 *is not fixed by the elements of* Γ_2 *which are conjugate to elements of* $\Gamma_0^2(4)$.

(10) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{10},\Gamma_2)$ *consists of ten double cosets. The representatives are* P_1, P_3, P_4, P_7, P_8, P_9, P_{12}, P_{13}, P_{14} *and* P_{17}. *Only* $\Phi_{10a} := P_{14}\,\Phi_{10}$ *is fixed by elements of* $\Gamma_0^2(4)$.

(12) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{12},\Gamma_2)$ *consists of twenty four double cosets. The representatives are* P_1, P_3, P_4, P_7, P_9, P_{10}, P_{13}, P_{14}, P_{15}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{27}, P_{28}, P_{29}, P_{30}, P_{31}, P_{32} *and* P_{33}. *Only* $\Phi_{12a} := P_{24}\,\Phi_{12}$ *and* $\Phi_{12a'} := P_{29}\,\Phi_{12}$ *are fixed by elements of* $\Gamma_0^2(4)$.

(15) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{15},\Gamma_2)$ *consists of four double cosets. The representatives are* P_1, P_2, P_3 *and* P_4. *Let* $\Phi_{15a} := P_1\,\Phi_{15}$, $\Phi_{15a'} := P_4\,\Phi_{15}$, $\Phi_{15b} := P_2\,\Phi_{15}$ *and* $\Phi_{15c} := P_3\,\Phi_{15}$. *All of them are fixed by elements of* $\Gamma_0^2(4)$.

(16) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{16},\Gamma_2)$ *consists of seven double cosets. The representatives are* P_1, P_2, P_3, P_4, P_8, P_{12} *and* P_{34}. *Only* $\Phi_{16a} := P_1\,\Phi_{16}$, $\Phi_{16a'} := P_4\,\Phi_{16}$, $\Phi_{16b} := P_2\,\Phi_{16}$, $\Phi_{16c} := P_3\,\Phi_{16}$, $\Phi_{16d} := P_{12}\,\Phi_{16}$ *and* $\Phi_{16e} := P_{34}\,\Phi_{16}$ *are fixed by elements of* $\Gamma_0^2(4)$.

(17) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{17},\Gamma_2)$ *consists of ten double cosets. The representatives are* P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_{11}, P_{13} *and* P_{14}. *Only* $\Phi_{17a} := P_1\,\Phi_{17}$, $\Phi_{17a'} := P_4\,\Phi_{17}$, $\Phi_{17b} := P_3\,\Phi_{17}$, $\Phi_{17c} := P_5\,\Phi_{17}$, $\Phi_{17c'} := P_{11}\,\Phi_{17}$, $\Phi_{17d} := P_7\,\Phi_{17}$ *and* $\Phi_{17e} := P_{14}\,\Phi_{17}$ *are fixed by elements of* $\Gamma_0^2(4)$.

(22) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{22},\Gamma_2)$ *consists of twelve double cosets. The representatives are* P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_{11}, P_{12}, P_{13}, P_{17} *and* P_{32}. *Let* $\Phi_{22a} := P_1\,\Phi_{22}$, $\Phi_{22a'} := P_4\,\Phi_{22}$, $\Phi_{22b} := P_2\,\Phi_{22}$, $\Phi_{22c} := P_7\,\Phi_{22}$, $\Phi_{22c'} := P_{17}\,\Phi_{22}$, $\Phi_{22d} := P_{13}\,\Phi_{22}$, $\Phi_{22e} := P_{32}\,\Phi_{22}$, $\Phi_{22f} := P_5\,\Phi_{22}$, $\Phi_{22f'} := P_{11}\,\Phi_{22}$, $\Phi_{22g} := P_6\,\Phi_{22}$, $\Phi_{22h} := P_3\,\Phi_{22}$ *and* $\Phi_{22h'} := P_{12}\,\Phi_{22}$. *All of them are fixed by elements of* $\Gamma_0^2(4)$.

(23) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{23},\Gamma_2)$ *consists of fifteen double cosets. The representatives are* P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_{11}, P_{12}, P_{13}, P_{17}, P_{31}, P_{32} *and* P_{34}. *Only* $\Phi_{23a} := P_1\,\Phi_{23}$, $\Phi_{23a'} := P_4\,\Phi_{23}$, $\Phi_{23b} := P_2\,\Phi_{23}$, $\Phi_{23b'} := P_{34}\,\Phi_{23}$, $\Phi_{23c} := P_3\,\Phi_{23}$, $\Phi_{23c'} := P_{12}\,\Phi_{23}$, $\Phi_{23d} := P_5\,\Phi_{23}$, $\Phi_{23d'} := P_{11}\,\Phi_{23}$, $\Phi_{23e} := P_6\,\Phi_{23}$, $\Phi_{23e'} := P_{31}\,\Phi_{23}$, $\Phi_{23f} := P_7\,\Phi_{23}$, $\Phi_{23f'} := P_{17}\,\Phi_{23}$, *and* $\Phi_{23g} := P_{32}\,\Phi_{23}$ *are fixed by elements of* $\Gamma_0^2(4)$.

(24) $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{24},\Gamma_2)$ *consists of thirteen double cosets. The representatives are* P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_{11}, P_{12}, P_{13}, P_{17}, P_{32} *and* P_{34}. *Only* $\Phi_{24a} := P_1\,\Phi_{24}$, $\Phi_{24a'} := P_4\,\Phi_{24}$, $\Phi_{24b} := P_3\,\Phi_{24}$, $\Phi_{24b'} := P_{12}\,\Phi_{24}$, $\Phi_{24c} := P_5\,\Phi_{24}$, $\Phi_{24c'} := P_{11}\,\Phi_{24}$, $\Phi_{24d} := P_7\,\Phi_{24}$, $\Phi_{24d'} := P_{17}\,\Phi_{24}$, *and* $\Phi_{24e} := P_{32}\,\Phi_{24}$ *are fixed by elements of* $\Gamma_0^2(4)$.

(25) *$\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_{25},\Gamma_2)$ consists of eight double cosets. The representatives are P_1, P_2, P_3, P_4, P_5, P_6, P_7 and P_{11}. Let $\Phi_{25a} := P_1\,\Phi_{25}$, $\Phi_{25a'} := P_4\,\Phi_{25}$, $\Phi_{25b} := P_2\,\Phi_{25}$, $\Phi_{25c} := P_3\,\Phi_{25}$, $\Phi_{25d} := P_5\,\Phi_{25}$, $\Phi_{25d'} := P_{11}\,\Phi_{25}$, $\Phi_{25e} := P_6\,\Phi_{25}$ and $\Phi_{25f} := P_7\,\Phi_{25}$. All of them are fixed by elements of $\Gamma_0^2(4)$.*

Proof. We prove only (9). Other cases are similarly proved. $C^p(\Phi_9,\Gamma_2)$ has ten elements. It consists of $\pm P_2$, $\pm P_5$, $\pm P_5^{-1}$ and other four elements. Other four elements are conjugate to $\pm P_5$ or $\pm P_5^{-1}$. Since the characteristic polynomials of $\pm P_5$ and $\pm P_5^{-1}$ are x^4+1, they are not conjugate to elements of $\Gamma_0^2(4)$ (cf. Proof of Lemma 2.1). $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_9,\Gamma_2)$ consists of eighteen double cosets. The representatives are P_1, P_3, P_4, P_7, P_8, P_9, P_{10}, P_{12}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}, P_{23}, P_{24}, P_{25} and P_{26}. $P_iP_2P_i^{-1}$ ($i = 1, 3, 4, 7, 8, 9, 10, 12, 17, \cdots, 26$) does not belong to $\Gamma_0^2(4)$. Hence Φ_9 is not fixed by the elements of Γ_2 which are conjugate to elements of $\Gamma_0^2(4)$ (cf. Lemma 2.4). □

3. Detailed data

In this section we list the data which we use to compute the dimension formula. First we recall the holomorphic Lefschetz fixed point formula. Let X be a compact complex manifold and V a holomorphic vector bundle on X, and let G be a finite group of automorphisms of the pair (X, V). For $g \in G$ let X^g be the set of fixed points of g. Then, X^g is a disjoint union of submanifolds of X. Let

$$X^g = \sum_\alpha X_\alpha^g$$

be the irreducible decomposition of X^g, and let

$$N_\alpha^g = \sum_\theta N_\alpha^g(\theta)$$

denote the normal bundle of X_α^g decomposed according to the eigenvalues $e^{i\theta}$ of g. We put

$$\mathcal{U}^\theta(N_\alpha^g(\theta)) = \prod_\beta \left(\frac{1-e^{-x_\beta - i\theta}}{1-e^{-i\theta}}\right)^{-1},$$

where the Chern class of $N_\alpha^g(\theta)$ is

$$c(N_\alpha^g(\theta)) = \prod_\beta (1+x_\beta)\,.$$

Let $\mathcal{T}(X_\alpha^g)$ be the Todd class of X_α^g. Let $V|X_\alpha^g$ be the restriction of V to X_α^g and $ch(V|X_\alpha^g)(g)$ the Chern character of $V|X_\alpha^g$ with g-action ([AS]). Put

$$\tau(g, X_\alpha^g) = \left\{\frac{ch(V|X_\alpha^g)(g)\cdot\prod_\theta \mathcal{U}^\theta(N_\alpha^g(\theta))\cdot\mathcal{T}(X_\alpha^g)}{\det(1-g|(N_\alpha^g)^*)}\right\}[X_\alpha^g]$$

and

$$\tau(g) = \sum_{\alpha} \tau(g, X_{\alpha}^{g}) .$$

We have

THEOREM 3.1 ([AS]).

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(g | H^i(X, \mathcal{O}(V))) = \tau(g) .$$

Let Γ be a subgroup of $\Gamma_0^2(4)$ of finite index and χ a character of Γ whose kernel is a subgroup of Γ of finite index. The kernel of χ contains $\Gamma_2(4N)$ for some N. In our case X, V and G are $\widetilde{X}_2(4N)$, $\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)}$ and $\Gamma/\Gamma_2(4N)$, respectively. But in the following we assume that V is $\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes k} \otimes [D]^{\otimes(-1)}$ for the sake of simplicity. When we apply the data, we replace k with $2k+1$.

Applying the holomorphic Lefschetz theorem we have the dimension formula. We state the general dimension formula (cf. [T5], Theorem 1.6). Let $g \in \Gamma/\Gamma_2(4N)$. We denote the centralizer of g in $\Gamma_0^2(4)/\Gamma_2(4N)$ and in $\Gamma/\Gamma_2(4N)$ by $C(g, \Gamma_0^2(4)/\Gamma_2(4N))$ and $C(g, \Gamma/\Gamma_2(4N))$, respectively. Let

$$N(\Phi, \Gamma_0^2(4)/\Gamma_2(4N)) = \{M \in \Gamma_0^2(4)/\Gamma_2(4N) \mid M \text{ maps } \Phi \text{ into } \Phi\} .$$

THEOREM 3.2. *Under the assumption that the higher cohomology groups vanish, the dimension of* $S_{j,k+1/2}(\Gamma, \chi)$ *is expressed as*

$$\sum_{\Phi} \sum_{P} \sum_{M} \frac{\tau(PMP^{-1}, P\Phi)}{|N(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|} \left(\sum_{g} \frac{|C(g, \Gamma_0^2(4)/\Gamma_2(4N))|}{|C(g, \Gamma/\Gamma_2(4N))|} \cdot \chi(g) \right) .$$

Here Φ *is over the* 15 *fixed points* (*sets*) *in* §2, P *is over the representatives of* $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi, \Gamma_2)$ *and* M *is over* $C^p(\Phi) \cap P^{-1}\Gamma_0^2(4)P$. *Let* $\mathrm{Conj}(\Gamma/\Gamma_2(4N))$ *be the set of the representatives of the conjugacy classes of* $\Gamma/\Gamma_2(4N)$. *Moreover* g *runs over* $\mathrm{Conj}(\Gamma/\Gamma_2(4N))$ *such that* g *is conjugate to* PMP^{-1} *in* $\Gamma_0^2(4)/\Gamma_2(4N)$.

Let Φ be an irreducible component of fixed points sets and let $M \in C^p(\Phi)$. The Chern character with M-action $ch : W \mapsto ch(W)(M)$ is also a ring homomorphism of the ring of the holomorphic vector bundles to the cohomology ring as in the case of the usual Chern character. Hence we have

$$\begin{aligned} ch(V|\Phi)(M) &= ch(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes k} \otimes [D]^{\otimes(-1)}|\Phi)(M) \\ &= ch(\mathrm{Sym}^j(\widetilde{V})|\Phi)(M) \otimes ch(\widetilde{H}_2^{\otimes k}|\Phi)(M) \otimes ch([D]^{\otimes(-1)}|\Phi)(M) . \end{aligned}$$

Let $Z \in \Phi$. Then by definition we have

$$ch(\widetilde{H}_2^{\otimes k}|\Phi)(M) = J(M, Z)^k ch(\widetilde{H}_2^{\otimes k}|\Phi) .$$

Let

$$ch_0(V|\Phi)(M) = ch(\mathrm{Sym}^j(\widetilde{V})|\Phi)(M) \otimes ch(\widetilde{H}_2^{\otimes k}|\Phi) \otimes ch([D]^{\otimes(-1)}|\Phi)(M)$$

and let

$$\tau_0(M,\Phi)=\left\{\frac{ch_0(V|\Phi)(M)\cdot\prod_\theta \mathcal{U}^\theta(N^M(\theta))\cdot\mathcal{T}(\Phi)}{\det(1-M|(N^M)^*)}\right\}[\Phi]\,.$$

Then we have

$$ch(V|\Phi)(M)=J(M,Z)^k ch_0(V|\Phi)(M)$$

and

$$\tau(M,\Phi)=J(M,Z)^k\tau_0(M,\Phi)\,.$$

Let $\overline{L}_2$ and $\widetilde{L}_2$ be as in Notation 1.11. We have $\widetilde{H}_2^{\otimes 2}\simeq\widetilde{L}_2$ and

$$\begin{aligned}ch(\widetilde{H}_2)&=1+c_1(\widetilde{H}_2)+\frac{1}{2}c_1(\widetilde{H}_2)^2+\frac{1}{6}c_1(\widetilde{H}_2)^3\\&=1+\frac{1}{2}c_1(\widetilde{L}_2)+\frac{1}{8}c_1(\widetilde{L}_2)^2+\frac{1}{48}c_1(\widetilde{L}_2)^3\,.\end{aligned}$$

Since $\mathrm{Sym}^j(\widetilde{V})$ and $\widetilde{L}_2$ correspond to the automorphy factors (which are defined with respect to Γ_2) $\mathrm{Sym}^j(CZ+D)$ and $\det(CZ+D)$, respectively and the divisor D is invariant with respect to Γ_2, the terms in $\tau_0(M,\Phi)$ are invariant with respect to Γ_2. Namely, we have

PROPOSITION 3.3. *Let $M\in C^p(\Phi,\Gamma_2)$ and $P\in\Gamma_2$. If M and PMP^{-1} belong to $\Gamma_0^2(4)$, then*

$$\tau_0(PMP^{-1},P\Phi)=\tau_0(M,\Phi)\,.$$

Hence the only term in $\tau(M,\Phi)$ which depends on $\Gamma_0^2(4)$ is $J(M,Z)$. What we have to do to get the dimension formula is to compute $|N(P\Phi,\Gamma_0^2(4)/\Gamma_2(4N))|$ and $\tau(PMP^{-1},P\Phi)$ for every $P\in\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi,\Gamma_2)$ and $M\in C^p(\Phi)\cap P^{-1}\Gamma_0^2(4)P$. From the above observation it suffices to compute $\tau_0(M,\Phi)$, $|N(P\Phi,\Gamma_0^2(4)/\Gamma_2(4N))|$ and $J(PMP^{-1},P\langle Z\rangle)$ $(Z\in\Phi)$. We list $\tau_0(M,\Phi)$ in Theorem 3.4, $|N(P\Phi,\Gamma_0^2(4)/\Gamma_2(4N))|$ in Theorem 3.8 and $J(PMP^{-1},P\langle Z\rangle)$ $(Z\in\Phi)$ in Theorem 3.9, respectively.

In the following theorem we assume that j is even. Hence we replace j with $2j$ and assume G is $\Gamma_0^2(4)/\pm\Gamma_2(4N)$. The notations $\varphi_1,\varphi_2,\cdots,\varphi_{25}(6,r,s,t)\in\Gamma_2/\pm\Gamma_2(4N)$ are same as in [T2]. We do not show them explicitly here. If one does not know them, he can obtain the dimension formula from the data in Theorem 3.4, Theorem 3.8 and Theorem 3.9. The elements in $C^p(\Phi_{10})$ except $\varphi_{10}(i)$ $(i=1,2,4,5)$ are not conjugate to the elements in $\Gamma_0^2(4)$.

THEOREM 3.4. *Let V be $\mathrm{Sym}^{2j}(\widetilde{V})\otimes\widetilde{H}_2^{\otimes k}\otimes[D]^{\otimes(-1)}$. Let $\zeta=\mathbf{e}(1/4N)$ and $\rho=\mathbf{e}(1/3)$. We have the following results. There p in $\prod$ is over the odd prime numbers which divide N, while Tr_ρ means the trace map $\mathbf{Q}(\rho)\to\mathbf{Q}$.*

(1) $\tau_0(\varphi_1,\Phi_1)=2^3 3^{-1}(2j+1)(2(k-4)(4j+k-2)(2j+k-3)N^{10}$
$-30(2j+k-3)N^8+45N^7)\prod(1-p^{-2})(1-p^{-4})$

(2) $\tau_0(\varphi_2, \Phi_2) = 2^{-1}((k-4)(4j+k-2)N^6 - 6(2j+k-3)N^5 + 36N^4)\prod(1-p^{-2})^2$

(3) $\tau_0(\varphi_3, \Phi_3) = 2^{2}((k-4)(4j+k-2)N^6 - 3(2j+k-3)N^5 + 3N^4)\prod(1-p^{-2})^2$

(4) $\tau_0(\varphi_4, \Phi_4) = 2^{-1}(-1)^j((2j+k-3)N^3 - 3N^2)\prod(1-p^{-2})$

(5) $\tau_0(\varphi_5, \Phi_5) = 2^{-1}3(-1)^j((2j+k-3)N^3 - 2N^2)\prod(1-p^{-2})$

(6) $$\tau_0(\varphi_6, \Phi_6) = \mathrm{Tr}_\rho(\rho^j(1-\rho))((2j+2k-3)N^3 - 9N^2) \times \begin{cases} 2^{-1}3^{-3}\prod(1-p^{-2}), & \textit{if } 3 \nmid N \\ 2^{-3}3^{-2}\prod(1-p^{-2}), & \textit{if } 3 \mid N \end{cases}$$

$\tau_0(\varphi_6^{-1}, \Phi_6) = \tau_0(\varphi_6, \Phi_6)$

(10) $\tau_0(\varphi_{10}(1), \Phi_{10}) = 3^{-2}(\rho)^j(2\rho+1)(2j+1)$

$\tau_0(\varphi_{10}(2), \Phi_{10}) = 3^{-2}(\rho^2)^j(2\rho^2+1)(2j+1)$

$\tau_0(\varphi_{10}(4), \Phi_{10}) = 3^{-1}(\rho)^j$

$\tau_0(\varphi_{10}(5), \Phi_{10}) = 3^{-1}(\rho^2)^j$

(12) $\tau_0(\varphi_{12}, \Phi_{12}) = 2^{-1}3^{-1}\mathrm{Tr}_\rho((\rho)^j(-\rho^2))$

(15) $$\tau_0(\varphi_{15}(r), \Phi_{15}) = 2^{-3}3^{-1}(2j+1)N^3\prod(1-p^{-2}) \times\left(\frac{9-(2j+2k-3)N}{(1-\zeta^r)} + \frac{(2j+2k-3)N-6}{(1-\zeta^r)^2} - \frac{4}{(1-\zeta^r)^3}\right)$$

(16) $$\tau_0(\varphi_{16}(r), \Phi_{16}) = 2^{-5}3^{-1}\left(\frac{12-(2j+2k-3)N}{(1-\zeta^r)}\right)N^3\prod(1-p^{-2})$$

(17) $$\tau_0(\varphi_{17}(r), \Phi_{17}) = \left(\frac{8-(2j+2k-3)N}{(1-\zeta^r)} + \frac{4}{(1-\zeta^r)^2}\right)N^3\prod(1-p^{-2})$$

(22) $$\tau_0(\varphi_{22}(1,r,t), \Phi_{22}) = \frac{(2j+1)}{(\zeta^r-1)(\zeta^t-1)}\left(\frac{2}{(\zeta^r-1)} + \frac{2}{(\zeta^t-1)} + 3\right)$$

$$\tau_0(\varphi_{22}(3,r,t), \Phi_{22}) = \frac{1}{(\zeta^{r+t}-1)}\left(\frac{4}{(\zeta^{r+t}-1)} + 3\right)$$

(23) $\tau_0(\varphi_{23}(2,r,t), \Phi_{23}) = 2^{-1}(-1)^j(\zeta^{r+t}-1)^{-1}$

$\tau_0(\varphi_{23}(4,r,t), \Phi_{23}) = 2^{-1}(\zeta^r-1)^{-1}(\zeta^t-1)^{-1}$

(24) $\tau_0(\varphi_{24}(2,r,t), \Phi_{24}) = 2^{-1}(-1)^j(\zeta^{r+t}-1)^{-1}$

$\tau_0(\varphi_{24}(4,r,t), \Phi_{24}) = 2^{-1}(\zeta^r-1)^{-1}(\zeta^t-1)^{-1}$

$$\begin{aligned}
(25)\quad &\tau_0(\varphi_{25}(1,r,s,t),\Phi_{25}) = (2j+1)(\zeta^{r+s}-1)^{-1}(\zeta^{s+t}-1)^{-1}(\zeta^{-s}-1)^{-1}\\
&\tau_0(\varphi_{25}(2,r,s,t),\Phi_{25}) = 3^{-1}\mathrm{Tr}_\rho(\rho^j(1-\rho))(\zeta^{s+r+t}-1)^{-1}\\
&\tau_0(\varphi_{25}(3,r,s,t),\Phi_{25}) = 3^{-1}\mathrm{Tr}_\rho(\rho^j(1-\rho))(\zeta^{s+r+t}-1)^{-1}\\
&\tau_0(\varphi_{25}(4,r,s,t),\Phi_{25}) = (\zeta^{r+2s+t}-1)^{-1}(\zeta^{-s}-1)^{-1}\\
&\tau_0(\varphi_{25}(5,r,s,t),\Phi_{25}) = (\zeta^{s+t}-1)^{-1}(\zeta^{r}-1)^{-1}\\
&\tau_0(\varphi_{25}(6,r,s,t),\Phi_{25}) = (\zeta^{r+s}-1)^{-1}(\zeta^{t}-1)^{-1}
\end{aligned}$$

Proof. Due to [T5], Theorem 3.2 which is the result in the case of weight k and level N. It suffices to remove $\det(CZ+D)^k$ of $\tau(\varphi,\Phi)$ in [T5], Theorem 3.2 and replace k and N with $k/2$ and $4N$, respectively. □

Let Γ_2^α be as above. We have the following

PROPOSITION 3.5. *If* $\Gamma_2^\alpha PN(\Phi,\Gamma_2) = \Gamma_2^\alpha P'N(\Phi,\Gamma_2)$, *then*

$$|N(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))| = |N(P'\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))|\,.$$

Proof. From the assumption we have elements $\gamma\in\Gamma_2^\alpha$ and $n\in N(\Phi,\Gamma_2)$ such that $P'=\gamma Pn$. $N(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))$ is isomorphic to $(N(P\,\Phi,\Gamma_2)\cap\Gamma_0^2(4))/(N(P\,\Phi,\Gamma_2)\cap\Gamma_2(4N))$. Since $\Gamma_0^2(4)$ is a normal subgroup of Γ_2^α and $\Gamma_2(4N)$ is a normal subgroup of Γ_2, we have

$$\gamma\,(N(P\,\Phi,\Gamma_2)\cap\Gamma_0^2(4))\,\gamma^{-1} = N(P'\,\Phi,\Gamma_2)\cap\Gamma_0^2(4),$$
$$\gamma\,(N(P\,\Phi,\Gamma_2)\cap\Gamma_2(4N))\,\gamma^{-1} = N(P'\,\Phi,\Gamma_2)\cap\Gamma_2(4N).$$

The assertion is proved from these relations. □

Let

$$C(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N)) = \{M\in\Gamma_0^2(4)/\Gamma_2(4N)\mid M\langle Z\rangle = Z \text{ for any } Z\in P\,\Phi\}$$

and let $C^p(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))$ be the set of proper elements.

REMARK 3.6. Let $P\,\Phi$, $P'\,\Phi$ and γ be as in the above proposition. It is obvious that $C^p(P'\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N)) = \gamma\,C^p(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))\,\gamma^{-1}$. Since the automorphy factor $J(M,Z)$ is defined with respect to Γ_2^α, we have

$$J(\gamma M\gamma^{-1},\gamma\langle Z\rangle) = J(M,Z)$$

for $M\in C^p(P\,\Phi,\Gamma_0^2(4)/\Gamma_2(4N))$ and $Z\in P\,\Phi$. From the above proposition and this observation it follows that the contributions of $P\,\Phi$ and $P'\,\Phi$ to the dimension of $S_{2j,k+1/2}(\Gamma_0^2(4))$ are same.

Next we prove that the contributions of $P\,\Phi$ and $P'\,\Phi$ are the same also in the case of $S_{2j,k+1/2}(\Gamma_0^2(4),\psi)$. It suffices to prove the following

LEMMA 3.7. *Let* $M = \begin{pmatrix} A & B \\ 4C & D \end{pmatrix} \in \Gamma_0^2(4)$ *and* $\gamma \in \Gamma_2^\alpha$. *Put* $\widetilde{\psi}(M) = \psi(\det D)$. *Then* $\widetilde{\psi}(M) = \widetilde{\psi}(\gamma M \gamma^{-1})$.

Proof. It suffices to prove that $\widetilde{\psi}$ is extendable to a character of Γ_2^α. Let P_4 be as in Proposition 2.5. Then $\Gamma_2^\alpha = \Gamma_0^2(4) \cup \Gamma_0^2(4)\, P_4$. Let $M = \begin{pmatrix} A & B \\ 4C & D \end{pmatrix}$, $M' = \begin{pmatrix} A' & B' \\ 4C' & D' \end{pmatrix} \in \Gamma_0^2(4)$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$M P_4 = \begin{pmatrix} A + 2BF & B \\ 4C + 2DF & D \end{pmatrix}.$$

Put $\widetilde{\psi}(MP_4) = \psi(\det D)$. We have to prove that $\widetilde{\psi}(MM'P_4) = \widetilde{\psi}(M)\widetilde{\psi}(M'P_4)$, $\widetilde{\psi}(MP_4M') = \widetilde{\psi}(MP_4)\widetilde{\psi}(M')$ and $\widetilde{\psi}(MP_4M'P_4) = \widetilde{\psi}(MP_4)\widetilde{\psi}(M'P_4)$ for any $M, M' \in \Gamma_0^2(4)$. The first case is trivial. We prove only the second case. The third case is similarly proved. The lower right 2×2 matrix of MP_4M' is $(4C + 2DF)B' + DD'$. Let $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$, $B' = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix}$ and $D' = \begin{pmatrix} d'_{11} & d'_{12} \\ d'_{21} & d'_{22} \end{pmatrix}$. Then

$$\begin{aligned} &\det((4C + 2DF)B' + DD') - (\det D)(\det D') \\ &\quad\equiv 2(d_{11}d_{22} + d_{12}d_{21})(b'_{11}d'_{12} + b'_{21}d'_{22} - b'_{12}d'_{11} - b'_{22}d'_{21}) \pmod 4 . \end{aligned}$$

On the other hand we have

$$b'_{11}d'_{12} + b'_{21}d'_{22} = b'_{12}d'_{11} + b'_{22}d'_{21} ,$$

because it holds that ${}^tB'D' = {}^tD'B'$. Hence it follows that

$$\det((4C + 2DF)B' + DD') \equiv (\det D)(\det D') \pmod 4 .$$

This proves the assertion. □

In the following theorem we list $|N(P\,\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$. If $P\,\Phi$ and $P'\,\Phi$ are not equivalent with respect to $\Gamma_0^2(4)$ but equivalent with respect to Γ_2^α, we list only one of them and we mark the notations of the fixed points (sets) by $*$. We also list the order of $C(P\,\Phi, \Gamma_0^2(4)/\Gamma_2(4N))$. We list $P\,\Phi$, $|C(P\,\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$ and $|N(P\,\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|$ in this order. Similarly as before p in $\prod$ is over the odd prime numbers which divide N.

THEOREM 3.8. *The orders of the isotropy groups and the stabilizer groups of the fixed points* (*sets*) *of* $\Gamma_0^2(4)$ *are as follows.*

(1)	Φ_{1a}	2	$2^{11}3N^{10}\prod(1 - p^{-2})(1 - p^{-4})$
(2)	$\Phi_{2a}{}^*$	4	$2^7N^6\prod(1 - p^{-2})^2$
(3)	Φ_{3a}	4	$2^{10}N^6\prod(1 - p^{-2})^2$

	Φ_{3b}	4	$2^8 N^6 \prod(1-p^{-2})^2$
	Φ_{3c}	4	$2^{10} 3 N^6 \prod(1-p^{-2})^2$
(4)	$\Phi_{4a}{}^*$	8	$2^5 N^3 \prod(1-p^{-2})$
(5)	Φ_{5a}	8	$2^5 3 N^3 \prod(1-p^{-2})$
	Φ_{5b}	8	$2^5 3 N^3 \prod(1-p^{-2})$
(6)	Φ_{6a}	12	$\begin{cases} 2^5 3 N^3 \prod(1-p^{-2}), & \textit{if} \ \ 3 \nmid N \\ 2^3 3^2 N^3 \prod(1-p^{-2}), & \textit{if} \ \ 3 \mid N \end{cases}$
(10)	Φ_{10a}	12	12
(12)	$\Phi_{12a}{}^*$	24	24
(15)	$\Phi_{15a}{}^*$	$8N$	$2^{10} N^6 \prod(1-p^{-2})$
	Φ_{15b}	$2N$	$2^6 N^6 \prod(1-p^{-2})$
	Φ_{15c}	$8N$	$2^9 N^6 \prod(1-p^{-2})$
(16)	$\Phi_{16a}{}^*$	$16N$	$2^6 N^4 \prod(1-p^{-2})$
	Φ_{16b}	$4N$	$2^4 N^4 \prod(1-p^{-2})$
	Φ_{16c}	$16N$	$2^6 N^4 \prod(1-p^{-2})$
	Φ_{16d}	$16N$	$2^6 N^4 \prod(1-p^{-2})$
	Φ_{16e}	$4N$	$2^4 N^4 \prod(1-p^{-2})$
(17)	$\Phi_{17a}{}^*$	$16N$	$2^8 N^4 \prod(1-p^{-2})$
	Φ_{17b}	$16N$	$2^8 N^4 \prod(1-p^{-2})$
	$\Phi_{17c}{}^*$	$16N$	$2^7 N^4 \prod(1-p^{-2})$
	Φ_{17d}	$16N$	$2^8 N^4 \prod(1-p^{-2})$
	Φ_{17e}	$4N$	$2^5 N^4 \prod(1-p^{-2})$
(22)	$\Phi_{22a}{}^*$	$2^6 N^2$	$2^9 N^3$
	Φ_{22b}	$2^2 N^2$	$2^3 N^3$
	$\Phi_{22c}{}^*$	$2^6 N^2$	$2^9 N^3$
	Φ_{22d}	$2^4 N^2$	$2^6 N^3$

	Φ_{22e}	2^3N^2	2^6N^3
	${\Phi_{22f}}^*$	2^3N^2	2^6N^3
	Φ_{22g}	2^3N^2	2^5N^3
	${\Phi_{22h}}^*$	2^5N^2	2^8N^3
(23)	${\Phi_{23a}}^*$	2^7N^2	2^7N^2
	${\Phi_{23b}}^*$	2^3N^2	2^3N^2
	${\Phi_{23c}}^*$	2^6N^2	2^6N^2
	${\Phi_{23d}}^*$	2^4N^2	2^4N^2
	${\Phi_{23e}}^*$	2^4N^2	2^4N^2
	${\Phi_{23f}}^*$	2^7N^2	2^7N^2
	Φ_{23g}	2^4N^2	2^4N^2
(24)	${\Phi_{24a}}^*$	2^7N^2	2^7N^2
	${\Phi_{24b}}^*$	2^6N^2	2^6N^2
	${\Phi_{24c}}^*$	2^4N^2	2^4N^2
	${\Phi_{24d}}^*$	2^7N^2	2^7N^2
	Φ_{24e}	2^4N^2	2^4N^2
(25)	${\Phi_{25a}}^*$	2^83N^3	2^83N^3
	Φ_{25b}	2^23N^3	2^23N^3
	Φ_{25c}	2^8N^3	2^8N^3
	${\Phi_{25d}}^*$	2^6N^3	2^6N^3
	Φ_{25e}	2^5N^3	2^5N^3
	Φ_{25f}	2^8N^3	2^8N^3

Proof. We prove only the cases of $\Phi_{3b} = P_5\Phi_3$ and $\Phi_{3c} = P_7\Phi_3$. Other cases are proved easily. $N(P\Phi_3, \Gamma_0^2(4)/\Gamma_2(4N))$ is isomorphic to $(N(P\Phi_3, \Gamma_2) \cap \Gamma_0^2(4))/(N(P\Phi_3, \Gamma_2) \cap \Gamma_2(4N))$. From [T2], Theorem 2.2 we have

$$\begin{aligned}[N(P\Phi_3,\Gamma_2) : N(P\Phi_3,\Gamma_2) \cap \Gamma_2(4N)] &= [PN(\Phi_3,\Gamma_2)P^{-1} : PN(\Phi_3,\Gamma_2)P^{-1} \cap \Gamma_2(4N)] \\ &= [N(\Phi_3, \Gamma_2) : N(\Phi_3, \Gamma_2) \cap \Gamma_2(4N)] \\ &= 2^{11}3N^6\prod(1 - p^{-2})^2 .\end{aligned}$$

So it suffices to determine $[N(P\Phi_3, \Gamma_2) : N(P\Phi_3, \Gamma_2) \cap \Gamma_0^2(4)]$. Let ε, δ and γ be

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix},$$

respectively. Let

$$N_1 = \left\{ \begin{pmatrix} a_1 & 0 & b_1/2 & 0 \\ 0 & a_2 & 0 & b_2/2 \\ 2c_1 & 0 & d_1 & 0 \\ 0 & 2c_2 & 0 & d_2 \end{pmatrix} \in Sp(2, \mathbf{Q}) \;\middle|\; \begin{matrix} a_i,\ b_i,\ c_i,\ d_i \in \mathbf{Z} \\ a_i \equiv d_{3-i} \pmod 2 \\ b_i \equiv c_{3-i} \pmod 2 \\ (i = 1,\ 2) \end{matrix} \right\}.$$

Then $N(\Phi_3, \Gamma_2) = \varepsilon N_1 \varepsilon^{-1} \cup \delta\varepsilon N_1 \varepsilon^{-1}$ ([T2], Theorem 2.6). Let l be a natural number and let $N_1(2l)$ be the subgroup of N_1 consisting of the elements such that $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2l}$ $(i = 1,\ 2)$. $N_1(2l)$ is a normal subgroup of N_1 which is isomorphic to $\Gamma_1(2l) \times \Gamma_1(2l)$ and we have $[N_1 : N_1(2)] = 6$.

Let N_2 be the subgroup of N_1 consisting of the elements such that $c_2 \equiv 0 \pmod 4$ and $b_1 \equiv c_2 \pmod 8$. Then

$$P_5 N(\Phi_3, \Gamma_2) P_5^{-1} \cap \Gamma_0^2(4) = P_5(\varepsilon N_2 \varepsilon^{-1}) P_5^{-1} \cup P_5(\delta\varepsilon\gamma N_2 \varepsilon^{-1}) P_5^{-1}$$

and

$$[P_5 N(\Phi_3, \Gamma_2) P_5^{-1} : P_5 N(\Phi_3, \Gamma_2) P_5^{-1} \cap \Gamma_0^2(4)] = [N_1 : N_2]\,.$$

On the other hand

$$[N_1 : N_1(8)] = 6 \cdot [N_1(2) : N_1(8)] = 6 \cdot [\Gamma_1(2) : \Gamma_1(8)]^2 = 3 \cdot 2^{13}$$

and $[N_2 : N_1(8)] = 2^{10}$ because $N_2/N_1(8)$ is isomorphic to a subgroup of $SL(2, \mathbf{Z}/8\mathbf{Z}) \times SL(2, \mathbf{Z}/8\mathbf{Z})$ of order 2^{10}. Hence we have $[N(P_5\Phi_3, \Gamma_2) : N(P_5\Phi_3, \Gamma_2) \cap \Gamma_0^2(4)] = 24$. This proves the case of Φ_{3b}.

Let N_3 be the subgroup of N_1 consisting of the elements such that $a_1 + c_1 - d_1 \equiv c_1 + c_2 \equiv 0 \pmod 2$. Then

$$P_7 N(\Phi_3, \Gamma_2) P_7^{-1} \cap \Gamma_0^2(4) = P_7(\varepsilon N_3 \varepsilon^{-1}) P_7^{-1} \cup P_7(\delta\varepsilon N_3 \varepsilon^{-1}) P_7^{-1}$$

and

$$[P_7 N(\Phi_3, \Gamma_2) P_7^{-1} : P_7 N(\Phi_3, \Gamma_2) P_7^{-1} \cap \Gamma_0^2(4)] = [N_1 : N_3].$$

Since $[N_3 : N_1(2)] = 3$, we have $[N(P_7\Phi_3, \Gamma_2) : N(P_7\Phi_3, \Gamma_2) \cap \Gamma_0^2(4)] = 2$. This proves the case of Φ_{3c}. □

In the following theorem we list $J(P\varphi P^{-1}, P\langle Z\rangle)$ and $\psi(\det D)$, where φ is an element of $C^p(\Phi, \Gamma_2/\Gamma_2(4N))$ such that $P\varphi P^{-1} \in C^p(P\,\Phi, \Gamma_0^2(4)/\Gamma_2(4N))$, $Z \in \Phi$ and D is the lower right 2×2 matrix of $P\varphi P^{-1}$. We list $P\,\Phi$, φ, $J(P\varphi P^{-1}, P\langle Z\rangle)$ and $\psi(\det D)$ in this order. In the case where Φ is in the divisor at infinity, $J(P\varphi P^{-1}, P\langle Z\rangle)$ means the limit of $J(P\varphi P^{-1}, P\langle Z\rangle)$ when Z tends to Φ.

THEOREM 3.9. *The proper elements of the isotropy groups of the fixed points* (*sets*) *of* $\Gamma_0^2(4)$ *and the values of the automorphy factor of weight* $1/2$ *and the character* ψ *are as follows. We assume that* $r+t \equiv 0 \pmod 4$ *for the elements whose notations are marked by* $*1$) *and assume that* $r-t \equiv 0 \pmod 2$ *for the elements whose notations are marked by* $*2$). *The meaning of the mark* $*$ *of* Φ *is the same as in the above theorem.*

(1)	Φ_{1a}	φ_1	1	1
(2)	$\Phi_{2a}{}^*$	φ_2	1	-1
(3)	Φ_{3a}	φ_3	1	-1
	Φ_{3b}	φ_3	-1	-1
	Φ_{3c}	φ_3	1	-1
(4)	$\Phi_{4a}{}^*$	φ_4	1	1
(5)	Φ_{5a}	φ_5	1	1
	Φ_{5b}	φ_5	-1	1
(6)	Φ_{6a}	φ_6	1	1
		φ_6^{-1}	1	1
(10)	Φ_{10a}	$\varphi_{10}(1)$	ρ^2	1
		$\varphi_{10}(2)$	ρ	1
		$\varphi_{10}(4)$	$-\rho^2$	-1
		$\varphi_{10}(5)$	$-\rho$	-1
(12)	$\Phi_{12a}{}^*$	φ_{12}	-1	-1
		φ_{12}^{-1}	-1	-1
(15)	$\Phi_{15a}{}^*$	$\varphi_{15}(r)$	1	1
	Φ_{15b}	$\varphi_{15}(4r)$	1	1
	Φ_{15c}	$\varphi_{15}(r)$	$(i)^r$	$(-1)^r$
(16)	$\Phi_{16a}{}^*$	$\varphi_{16}(r)$	1	-1

	Φ_{16b}	$\varphi_{16}(4r)$	1	-1
	Φ_{16c}	$\varphi_{16}(r)$	$(i)^r$	$(-1)^{r+1}$
	Φ_{16d}	$\varphi_{16}(r)$	$(i)^r$	$(-1)^{r+1}$
	Φ_{16e}	$\varphi_{16}(4r)$	1	-1
(17)	$\Phi_{17a}{}^*$	$\varphi_{17}(r)$	1	-1
	Φ_{17b}	$\varphi_{17}(r)$	$(i)^r$	$(-1)^{r+1}$
	$\Phi_{17c}{}^*$	$\varphi_{17}(r)$	1	-1
	Φ_{17d}	$\varphi_{17}(r)$	$-(i)^r$	$(-1)^{r+1}$
	Φ_{17e}	$\varphi_{17}(4r)$	1	-1
(22)	$\Phi_{22a}{}^*$	$\varphi_{22}(1, r, t)$	1	1
		$\varphi_{22}(3, r, t)$	1	-1
	Φ_{22b}	$\varphi_{22}(1, 4r, 4t)$	1	1
		$\varphi_{22}(3, 4r, 4t)$	1	-1
	$\Phi_{22c}{}^*$	$\varphi_{22}(1, r, t)$	$(i)^{r+t}$	$(-1)^{r+t}$
		$\varphi_{22}(3, r, t)$	$(i)^{r+t}$	$(-1)^{r+t+1}$
	Φ_{22d}	$\varphi_{22}(1, r, t)^{*1)}$	1	1
		$\varphi_{22}(3, r, t)^{*1)}$	1	-1
	Φ_{22e}	$\varphi_{22}(1, 2r, 2t)^{*2)}$	$(-1)^t$	1
		$\varphi_{22}(3, 2r, 2t)^{*2)}$	$(-1)^t$	-1
	$\Phi_{22f}{}^*$	$\varphi_{22}(1, 4r, t)$	1	1
	Φ_{22g}	$\varphi_{22}(1, 4r, t)$	$(i)^t$	$(-1)^t$
	$\Phi_{22h}{}^*$	$\varphi_{22}(1, r, t)$	$(i)^t$	$(-1)^t$
(23)	$\Phi_{23a}{}^*$	$\varphi_{23}(2, r, t)$	1	1
		$\varphi_{23}(4, r, t)$	1	-1
	$\Phi_{23b}{}^*$	$\varphi_{23}(2, 4r, 4t)$	1	1
		$\varphi_{23}(4, 4r, 4t)$	1	-1
	$\Phi_{23c}{}^*$	$\varphi_{23}(4, r, t)$	$(i)^t$	$(-1)^{t+1}$
	$\Phi_{23d}{}^*$	$\varphi_{23}(4, 4r, t)$	1	-1
	$\Phi_{23e}{}^*$	$\varphi_{23}(4, 4r, t)$	$(i)^t$	$(-1)^{t+1}$

	$\Phi_{23f}{}^{*}$	$\varphi_{23}(2, r, t)$	$(i)^{r+t}$	$(-1)^{r+t}$
		$\varphi_{23}(4, r, t)$	$(i)^{r+t}$	$(-1)^{r+t+1}$
	Φ_{23g}	$\varphi_{23}(2, 2r+1, 2t+1)^{*2)}$	$i(-1)^{t}$	-1
		$\varphi_{23}(4, 2r+1, 2t+1)^{*2)}$	$i(-1)^{t}$	1
(24)	$\Phi_{24a}{}^{*}$	$\varphi_{24}(2, r, t)$	1	1
		$\varphi_{24}(4, r, t)$	1	-1
	$\Phi_{24b}{}^{*}$	$\varphi_{24}(4, r, t)$	$(i)^{t}$	$(-1)^{t+1}$
	$\Phi_{24c}{}^{*}$	$\varphi_{24}(4, 4r, t)$	1	-1
	$\Phi_{24d}{}^{*}$	$\varphi_{24}(2, r, t)$	$-(i)^{r+t}$	$(-1)^{r+t}$
		$\varphi_{24}(4, r, t)$	$-(i)^{r+t}$	$(-1)^{r+t+1}$
	Φ_{24e}	$\varphi_{24}(2, 2r, 2t)^{*2)}$	$(-1)^{t}$	1
		$\varphi_{24}(4, 2r, 2t)^{*2)}$	$(-1)^{t}$	-1
(25)	$\Phi_{25a}{}^{*}$	$\varphi_{25}(1, r, s, t)$	1	1
		$\varphi_{25}(2, r, s, t)$	1	1
		$\varphi_{25}(3, r, s, t)$	1	1
		$\varphi_{25}(4, r, s, t)$	1	-1
		$\varphi_{25}(5, r, s, t)$	1	-1
		$\varphi_{25}(6, r, s, t)$	1	-1
	Φ_{25b}	$\varphi_{25}(1, 4r, 4s, 4t)$	1	1
		$\varphi_{25}(2, 4r, 4s, 4t)$	1	1
		$\varphi_{25}(3, 4r, 4s, 4t)$	1	1
		$\varphi_{25}(4, 4r, 4s, 4t)$	1	-1
		$\varphi_{25}(5, 4r, 4s, 4t)$	1	-1
		$\varphi_{25}(6, 4r, 4s, 4t)$	1	-1
	Φ_{25c}	$\varphi_{25}(1, r, s, t)$	$(i)^{t}$	$(-1)^{t}$
		$\varphi_{25}(6, r, s, t)$	$(i)^{t}$	$(-1)^{t+1}$
	$\Phi_{25d}{}^{*}$	$\varphi_{25}(1, 4r, s, t)$	1	1
		$\varphi_{25}(5, 4r, s, t)$	1	-1
	Φ_{25e}	$\varphi_{25}(1, 4r, 2s, t)$	$(i)^{t}$	$(-1)^{t}$
		$\varphi_{25}(5, 4r, 2s, t)$	$(i)^{t}$	$(-1)^{t+1}$

Φ_{25f}	$\varphi_{25}(1,r,s,t)$	$(i)^{r+2s+t}$	$(-1)^{r+t}$
	$\varphi_{25}(4,r,s,t)$	$(i)^{r+2s+t}$	$(-1)^{r+t+1}$

Proof. Due to the transformation formula of $\Theta(Z)$ (Theorem 1.4). When Φ is in the divisor at infinity, φ includes parameters (for example "r" of $\varphi_{15}(r)$). In such cases we have a problem to evaluate the Gaussian sum $\lambda(P\varphi P^{-1})$. But we skip this problem as follows. Since $\varphi_{15}(r) = \varphi_{15}(1)^r$, we have

$$\lim_{Z\to\Phi_{15}} J(P\varphi_{15}(r)P^{-1}, P\langle Z\rangle) = \lim_{Z\to\Phi_{15}} J(P\varphi_{15}(1)P^{-1}, P\langle Z\rangle)^r .$$

Hence it suffices to compute $J(P\varphi P^{-1}, P\langle Z\rangle)$ for the generators of $C(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))$. □

4. The dimension formula

In this section we present the explicit dimension formulas and also prove $M_{2j,k+1/2}(\Gamma_0^2(4),\psi) = S_{2j,k+1/2}(\Gamma_0^2(4),\psi)$. We can prove the following vanishing theorem similarly as in [T5], Theorem 6.1 by using the vanishing theorem of Kawamata-Viehweg ([Ka], [V]).

THEOREM 4.1. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then*

$$H^i(\widetilde{X}_2(4N), \mathcal{O}(\mathrm{Sym}^j(\widetilde{V}) \otimes \widetilde{H}_2^{\otimes(2k+1)} \otimes [D]^{\otimes(-1)})) \simeq \{0\}$$

for $i > 0$.

By using this theorem and the theorem of Riemann-Roch-Hirzebruch we have

THEOREM 4.2. *If $j = 0$ and $k \geq 3$ or if $j \geq 1$ and $k \geq 4$, then*

$$S_{j,k+1/2}(\Gamma_2(4N)) = 2^3 3^{-1}(j+1)(2(2k-3)(2j+2k-1)(j+2k-2)N^{10} - 30(j+2k-2)N^8 + 45N^7) \times \prod(1-p^{-2})(1-p^{-4}),$$

where p is over odd prime numbers which divide N.

Proof. It suffices to replace k and N in the formula of the dimension of $S_{j,k}(\Gamma_2(N))$ ([T3]) with $k + 1/2$ and $4N$, respectively. □

To evaluate the sums which appear in the computation of Theorem 4.4 and Theorem 4.5 we use the following

LEMMA 4.3. *Let $\zeta = \mathbf{e}(1/4N)$. Then we have*

$$(1) \qquad \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1-\zeta^r)} = \begin{cases} -\dfrac{1-4N}{2}, & \textit{if } k \equiv 0 \pmod 4, \\ -\dfrac{1+2N}{2}, & \textit{if } k \equiv 1 \pmod 4, \\ -\dfrac{1}{2}, & \textit{if } k \equiv 2 \pmod 4, \\ -\dfrac{1-2N}{2}, & \textit{if } k \equiv 3 \pmod 4. \end{cases}$$

$$(2) \qquad \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1-\zeta^r)^2} = \begin{cases} \dfrac{-16N^2+24N-5}{12}, & \textit{if } k \equiv 0 \pmod 4, \\ \dfrac{2N^2-12N-5}{12}, & \textit{if } k \equiv 1 \pmod 4, \\ \dfrac{8N^2-5}{12}, & \textit{if } k \equiv 2 \pmod 4, \\ \dfrac{2N^2+12N-5}{12}, & \textit{if } k \equiv 3 \pmod 4. \end{cases}$$

$$(3) \qquad \sum_{r=1}^{4N-1} \frac{(i)^{kr}}{(1-\zeta^r)^3} = \begin{cases} \dfrac{-16N^2+16N-3}{8}, & \textit{if } k \equiv 0 \pmod 4, \\ \dfrac{4N^3+2N^2-8N-3}{8}, & \textit{if } k \equiv 1 \pmod 4, \\ \dfrac{8N^2-3}{8}, & \textit{if } k \equiv 2 \pmod 4, \\ \dfrac{-4N^3+2N^2+8N-3}{8}, & \textit{if } k \equiv 3 \pmod 4. \end{cases}$$

The dimension of $S_{2j,k+1/2}(\Gamma_0^2(4))$ is calculated as

$$\sum_{\Phi}\sum_{P}\sum_{M} J(PMP^{-1}, P\langle Z\rangle)^{2k+1} \frac{\tau_0(M,\Phi)}{|N(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|},$$

where Φ is over the 15 fixed points (sets) in §2, P is over the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi,\Gamma_2)$, M is over $C^p(\Phi)\cap P^{-1}\Gamma_0^2(4)P$ and $Z\in\Phi$. We have

THEOREM 4.4. *If* $j=0$ *and* $k\geq 3$ *or if* $j\geq 1$ *and* $k\geq 4$, *the dimension of* $S_{2j,k+1/2}(\Gamma_0^2(4))$ *is given by the following Mathematica function*:

```
SiegelHalf[j_,k_]:=Block[{a,ljk},
  mod[x_,y_]:=Mod[x,y]+1;
  a=(2*j+1)*(4*j+2*k-1)*(j+k-1)*(2*k-3)/2^5/3^2;
  a=a+(2*j+1)*If[Mod[k,2]==0,19-22*k-22*j,25-22*k-22*j]/2^6/3;
  a=a+3*(2*j+1)*If[Mod[k,2]==0,-1,1]/2^6;
```

```
(* contribution of φ1 *)
(* contribution of φ15(r) *)
(* contribution of φ22(1,r,t) *)
(* contribution of φ25(1,r,s,t) *)
a=a+(4*j+2*k-1)*(2*k-3)/2^6;
a=a+If[Mod[k,2]==0,17-12*k-12*j,49-20*k-20*j]/2^6;
(* contribution of φ2 *)
(* contribution of φ16(r) *)
(* contribution of φ23(4,r,t) *)
a=a+7*(4*j+2*k-1)*(2*k-3)/2^6/3;
a=a+(35-48*k-48*j)/2^5/3;
a=a-13/2^4/3;
a=a+If[Mod[k,2]==0,7,15]/2^6;
a=a+If[Mod[k,2]==0,2,3]/2^2;
(* contribution of φ3 *)
(* contribution of φ17(r) *)
(* contribution of φ22(3,r,t) *)
(* contribution of φ24(4,r,t) *)
(* contribution of φ25(i,r,s,t) (i = 4,5,6) *)
ljk={1,-1};
a=a+(j+k-1)*ljk[[mod[j,2]]]/2^3;
a=a-If[Mod[k,2]==0,3,5]*ljk[[mod[j,2]]]/2^4;
(* contribution of φ4 *)
(* contribution of φ23(2,r,t) *)
a=a-If[Mod[k,2]==0,3,1]*ljk[[mod[j,2]]]/2^4;
(* contribution of φ5 *)
(* contribution of φ24(2,r,t) *)
ljk={1,0,-1};
a=a+2*ljk[[mod[j,3]]]*(j+k-1)/3^2;
a=a-ljk[[mod[j,3]]]/2;
(* contribution of φ6 *)
(* contribution of φ25(2,r,s,t) and φ25(3,r,s,t) *)
ljk=(2*j+1)*{{1,0,-1},{0,-1,1},{-1,1,0}};
a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;
(* contribution of φ10(1) and φ10(2) *)
ljk={{1,-2,1},{-2,1,1},{1,1,-2}};
a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;
(* contribution of φ10(4) and φ10(5) *)
ljk={1,-2,1};
a=a-ljk[[mod[j,3]]]/2/3^2;
```

```
  (* contribution of φ12 *)
  Return[a];
  ]
```

The dimension of $S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ is calculated as

$$\sum_{\Phi}\sum_{P}\sum_{M} J(PMP^{-1}, P\langle Z\rangle)^{2k+1}\psi(\det D)\frac{\tau_0(M,\Phi)}{|N(P\Phi, \Gamma_0^2(4)/\Gamma_2(4N))|},$$

where D is the lower right 2×2 matrix of PMP^{-1}. We have

THEOREM 4.5. *If* $j = 0$ *and* $k \geq 3$ *or if* $j \geq 1$ *and* $k \geq 4$, *then the dimension of* $S_{2j,k+1/2}(\Gamma_0^2(4), \psi)$ *is given by the following Mathematica function*:

```
SiegelHalfpsi[j_,k_]:=Block[{a,ljk},
  mod[x_,y_]:=Mod[x,y]+1;
  a=(2*j+1)*(4*j+2*k-1)*(j+k-1)*(2*k-3)/2^5/3^2;
  a=a+(2*j+1)*If[Mod[k,2]==0,25-22*k-22*j,19-22*k-22*j]/2^6/3;
  a=a-3*(2*j+1)*If[Mod[k,2]==0,-1,1]/2^6;
  (* contribution of φ1 *)
  (* contribution of φ15(r) *)
  (* contribution of φ22(1,r,t) *)
  (* contribution of φ25(1,r,s,t) *)
  a=a-(4*j+2*k-1)*(2*k-3)/2^6;
  a=a-If[Mod[k,2]==0,49-20*k-20*j,17-12*k-12*j]/2^6;
  (* contribution of φ2 *)
  (* contribution of φ16(r) *)
  (* contribution of φ23(4,r,t) *)
  a=a-7*(4*j+2*k-1)*(2*k-3)/2^6/3;
  a=a-(35-48*k-48*j)/2^5/3;
  a=a+13/2^4/3;
  a=a-If[Mod[k,2]==0,15,7]/2^6;
  a=a-If[Mod[k,2]==0,3,2]/2^2;
  (* contribution of φ3 *)
  (* contribution of φ17(r) *)
  (* contribution of φ22(3,r,t) *)
  (* contribution of φ24(4,r,t) *)
  (* contribution of φ25(i,r,s,t) (i = 4,5,6) *)
  ljk={1,-1};
  a=a+(j+k-1)*ljk[[mod[j,2]]]/2^3;
  a=a-If[Mod[k,2]==0,5,3]*ljk[[mod[j,2]]]/2^4;
  (* contribution of φ4 *)
```

```
(* contribution of φ23(2,r,t) *)
a=a-If[Mod[k,2]==0,1,3]*ljk[[mod[j,2]]]/2^4;
(* contribution of φ5 *)
(* contribution of φ24(2,r,t) *)
ljk={1,0,-1};
a=a+2*ljk[[mod[j,3]]]*(j+k-1)/3^2;
a=a-ljk[[mod[j,3]]]/2;
(* contribution of φ6 *)
(* contribution of φ25(2,r,s,t) and φ25(3,r,s,t) *)
ljk=(2*j+1)*{{1,0,-1},{0,-1,1},{-1,1,0}};
a=a+ljk[[mod[j,3],mod[k,3]]]/2/3^2;
(* contribution of φ10(1) and φ10(2) *)
ljk={{1,-2,1},{-2,1,1},{1,1,-2}};
a=a-ljk[[mod[j,3],mod[k,3]]]/2/3^2;
(* contribution of φ10(4) and φ10(5) *)
ljk={1,-2,1};
a=a+ljk[[mod[j,3]]]/2/3^2;
(* contribution of φ12 *)
Return[a];
]
```

Now we prove

THEOREM 4.6.

$$M_{2j,k+1/2}(\Gamma_0^2(4),\psi) = S_{2j,k+1/2}(\Gamma_0^2(4),\psi)\,.$$

Proof. Let $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$ and $f \in M_{2j,k+1/2}(\Gamma_0^2(4),\psi)$. We have to prove that

$$(*)\qquad \lim_{\operatorname{Im} Z_2 \to \infty} f \mid [\xi]_{2j,k+1/2}(Z) = 0$$

for any $\xi \in p^{-1}(\Gamma_2)$. Let P_i $(i = 1,2,3,4)$ be the matrices which correspond to the representatives of one-dimensional cusps as before and let us recall that

$$\varphi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

To prove the assertion, it suffices to prove $(*)$ for $\xi = (P, \phi(Z))$ such that P is P_1, P_2, P_3 or P_4. Let Z be as above. From $\varphi_2 \langle Z \rangle = Z$, we have

$$P\langle Z \rangle = P\varphi_2 \langle Z \rangle = (P\varphi_2 P^{-1})P\langle Z \rangle\,.$$

for any P. Let $i = 1, 2$ or 3. Then $P_i \varphi_2 P_i^{-1} = \varphi_2$. Hence we have

$$\begin{aligned} f(P_i\langle Z\rangle) &= f((P_i\varphi_2 P_i^{-1})P_i\langle Z\rangle) \\ &= J(\varphi_2, P_i\langle Z\rangle)^{2k+1}\,\psi(-1)\,f(P_i\langle Z\rangle) \\ &= -f(P_i\langle Z\rangle). \end{aligned}$$

Therefore $f(P_i\langle Z\rangle) = 0$. Next let $i = 4$. Then we have

$$P_4\varphi_2 P_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 4 & 0 & 0 & -1 \end{pmatrix}$$

and $J(P_4\varphi_2 P_4^{-1}, P_4\langle Z\rangle)$ is identically equal to 1. Therefore similarly as above we have $f(P_4\langle Z\rangle) = 0$. □

REMARK 4.7. Note that $f(P_i\langle Z\rangle)$ $(i = 1, 2, 3, 4)$ is identically zero for $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$. So it may be natural to ask whether for any $P \in \Gamma_2$, $f(P\langle Z\rangle)$ is identically zero or not. But this is not true in general. Let us recall that Φ_2 is $\left\{\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}\right\}$ and let P_8 be as before. Then P_1, P_4 and P_8 are the representatives of $\Gamma_0^2(4)\backslash\Gamma_2/N(\Phi_2, \Gamma_2)$.

$$P_8\varphi_2 P_8^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}$$

does not belong to $\Gamma_0^2(4)$ but belongs to Γ_2^α and $J(P_8\varphi_2 P_8^{-1}, P_8\langle Z\rangle)$ is identically equal to 1. Therefore if $f(Z)$ belongs to $M_{2j,k+1/2}(\Gamma_2^\alpha, \psi)$, it holds that $f(P\langle Z\rangle) = 0$ for any $P \in \Gamma_2$ and $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$. ($\psi$ is extendable to a character of Γ_2^α (cf. Lemma 3.7).)

5. The case $j = 0$

In this section we prove $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4))$ and $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4), \psi)$ are modules of rank one over the graded ring of the modular forms of integral weights.

PROPOSITION 5.1.

$$\begin{aligned} \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4))\, t^k &= \sum_{k=0}^{\infty} \texttt{SiegelHalf[0,k]}\, t^k + t^2 \\ &= \frac{2t^5 + 2t^6 - t^7 - 2t^8 - t^9 + t^{10}}{(1-t)(1-t^2)^2(1-t^3)}. \end{aligned}$$

Proof. If $f(Z) \in S_{k+1/2}(\Gamma_0^2(4))$, then $f(Z)\Theta(Z)^2 \in S_{k+3/2}(\Gamma_0^2(4))$. Since $\dim S_{7/2}(\Gamma_0^2(4))$ is equal to `SiegelHalf[0,3]` $= 0$, we have $S_{5/2}(\Gamma_0^2(4)) \simeq S_{3/2}(\Gamma_0^2(4)) \simeq S_{1/2}(\Gamma_0^2(4)) \simeq \{0\}$. But since `SiegelHalf[0,2]` $= -1$, `SiegelHalf[0,1]` $= 0$ and `SiegelHalf[0,0]` $= 0$, we have the equality of the first line. □

Now we have

THEOREM 5.2.

$$\begin{aligned}
&\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4))\, t^k \\
&= \sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^2(4))\, t^k + 3\sum_{k=0}^{\infty} \dim S_{k+1/2}(\Gamma_0^1(4))\, t^k + 4\sum_{k=0}^{\infty} t^k - (3+3t+t^2) \\
&= \frac{2t^5+2t^6-t^7-2t^8-t^9+t^{10}}{(1-t)(1-t^2)^2(1-t^3)} + \frac{3(t^4+t^5)}{(1-t^2)^2} + \frac{4}{(1-t)} - (3+3t+t^2) \\
&= \frac{1}{(1-t)(1-t^2)^2(1-t^3)}\,.
\end{aligned}$$

Proof. Recall that

$$\varphi_{15}(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and put $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$. From Theorem 3.9 (15) Φ_{15c} we have

$$\lim_{\operatorname{Im} Z_2 \to \infty} J(P_3\varphi_{15}(r)P_3^{-1}, P_3\langle Z\rangle) = (i)^r\,,$$

where $i = \sqrt{-1}$. Hence if $f \in M_{k+1/2}(\Gamma_0^2(4))$ and r is an odd integer, then we have

$$\begin{aligned}
\lim_{\operatorname{Im} Z_2 \to \infty} f(P_3\langle Z\rangle) &= \lim_{\operatorname{Im} Z_2 \to \infty} f(P_3\langle \varphi_{15}(r)\langle Z\rangle) \\
&= \lim_{\operatorname{Im} Z_2 \to \infty} f((P_3\varphi_{15}(r)P_3^{-1})P_3\langle Z\rangle) \\
&= \lim_{\operatorname{Im} Z_2 \to \infty} J(P_3\varphi_{15}(r)P_3^{-1}, P_3\langle Z\rangle)^{2k+1} f(P_3\langle Z\rangle) \\
&= (i)^{r(2k+1)} \lim_{\operatorname{Im} Z_2 \to \infty} f(P_3\langle Z\rangle)\,.
\end{aligned}$$

Therefore $\lim_{\operatorname{Im} Z_2 \to \infty} f(P_3\langle Z\rangle)$ and $\lim_{\operatorname{Im} Z_2 \to \infty} f \mid [\xi]_{k+1/2}(Z)$ are identically 0 where $\xi = (P_3, \phi(Z))$. Namely, the Φ-operators to the one-dimensional cusp C_3 and to the zero-dimensional cusps Q_3, Q_6 and Q_7 are 0-maps.

Next we prove the surjectivity of the Φ-operators to other cusps. In general the Eisenstein series of Klingen type of degree g attached to a cusp form of degree r and weight k

converges if $k > g + r + 1$ ([Kl]). We define Eisenstein series of half integral weight in the following. In case k is a half integer, their convergence is also proved similarly as in the case of integral weight.

Let $N(B_0, \Gamma_2)$ and $N(B_1, \Gamma_2)$ be as in §2 and let P_i $(i = 1, 2, 4, 5)$ be as in §2. Let $\xi_i = (P_i, \phi_i(Z)) \in \widetilde{G}_2$ $(i = 1, 2, 4, 5)$. We assume that $\xi_1 = (1_4, 1)$ and $\xi_4 = \iota(P_4) = (P_4, J(P_4, Z))$ since $P_4 \in \Gamma_2^\alpha$. First we prove the case of zero-dimensional cusps. Let 1 be the function on $\mathfrak{S}_2$ which is identically 1. Let

$$E_i(Z) = \sum_\gamma 1 \mid [\xi_i^{-1}\iota(\gamma)]_{k+1/2}(Z)\,,$$

where γ is over $(P_i N(B_0, \Gamma_2) P_i^{-1} \cap \Gamma_0^2(4))\backslash\Gamma_0^2(4)$. Let $M \in N(B_0, \Gamma_2)$ and assume that $P_i M P_i^{-1} \in \Gamma_0^2(4)$. We prove $\xi_i\, \iota(M)\, \xi_i^{-1} = \iota(P_i M P_i^{-1})$ $(i = 1, 2, 4, 5)$. Then

$$\begin{aligned} 1 \mid [\xi_i^{-1}\iota(P_i M P_i^{-1}\gamma)]_{k+1/2}(Z) &= (1 \mid [\iota(M)]_{k+1/2})\, [\xi_i^{-1}\iota(\gamma)]_{k+1/2}(Z) \\ &= 1 \mid [\xi_i^{-1}\iota(\gamma)]_{k+1/2}(Z)\,. \end{aligned}$$

Therefore $1 \mid [\xi_i^{-1}\iota(\gamma)]_{k+1/2}(Z)$ is independent of the choice of γ.

We prove our assertion. The case of $i = 1$ or $i = 4$ is trivial. Similarly as in the proof of Theorem 1.8, we have

$$\iota(P_i M P_i^{-1})(\xi_i\, \iota(M)\, \xi_i^{-1})^{-1} = \iota(P_i M P_i^{-1})\, \xi_i\, \iota(M^{-1})\, \xi_i^{-1} = (1_4, t)\,,$$

where

$$t = J(P_i M P_i^{-1}, P_i M^{-1} P_i^{-1}\langle Z\rangle)\, \phi_i(M^{-1} P_i^{-1}\langle Z\rangle)\, J(M^{-1}, P_i^{-1}\langle Z\rangle)\, \phi_i(P_i^{-1}\langle Z\rangle)^{-1}$$

is a constant. We prove that $t = 1$. Let $Z = P_i M\langle Z'\rangle$. Since $J(M^{-1}, P_i^{-1}\langle Z\rangle) = 1$, t is equal to

$$J(P_i M P_i^{-1}, P_i\langle Z'\rangle)\, \phi_i(Z')\, \phi_i(M\langle Z'\rangle)^{-1}\,.$$

Let

$$M_1 = \begin{pmatrix} 1_2 & S \\ O & 1_2 \end{pmatrix},\ S \in M(2, \mathbf{Z}),\ S = {}^tS \ \text{ and }\ M_2 = \begin{pmatrix} U & O \\ O & {}^tU^{-1} \end{pmatrix},\ U \in GL(2, \mathbf{Z})\,.$$

Let $S = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$ and $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Elements of $N(B_0, \Gamma_2)$ have the form of $M_1 M_2$.

Let $i = 2$. Then $P_2 M_1 M_2 P_2^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if r, s and t are divisible by 4. Since $\lim_{\operatorname{Im} Z' \to \infty} \phi_2(Z')\, \phi_2(M_1\langle Z'\rangle)^{-1} = 1$ (cf. Proof of Theorem 1.8), the assertion for M_1 follows from Theorem 3.9 (25) $\Phi_{25b}\ \varphi_{25}(1, 4r, 4s, 4t)$. Since $P_2 M_2 P_2^{-1} \in N(B_0, \Gamma_2)$, we have $J(P_2 M_2 P_2^{-1}, P_2\langle Z\rangle) = 1$. On the other hand we have

$$\frac{\phi_2(Z')}{\phi_2(M_2\langle Z'\rangle)} = \frac{\sqrt{\det(-Z')}}{\sqrt{\det(-UZ'\,{}^tU)}} = 1\,.$$

(It suffices to check in the case Z is diagonal and U is over the generators of $GL(2,\mathbf{Z})$.) So the assertion for M_2 was proved.

Let $i=5$. $P_5M_1M_2P_5^{-1}$ belongs to $\Gamma_0^2(4)$ if and only if r and b are divisible by 4. Since $\lim_{\mathrm{Im}\,Z'\to\infty}\phi_5(Z')\,\phi_5(M_1\langle Z'\rangle)^{-1}=1$, the assertion for M_1 is due to Theorem 3.9 (25) $\Phi_{25d}{}^{*}\,\varphi_{25}(1,4r,s,t)$. Let

$$\widetilde{\Gamma}^{1,0}(4)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL(2,\mathbf{Z})\;\middle|\; b\equiv 0 \pmod 4\right\}.$$

$U_1=\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}$, $U_2=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and $U_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ are the generators of $\widetilde{\Gamma}^{1,0}(4)/(\pm 1_2)$. It suffices to prove the assertion for them. If $U=U_2$ or $U=U_3$, the assertion is trivial since $\phi_5(Z')=\phi_5(M_2\langle Z'\rangle)$ and $P_5M_2P_5^{-1}\in N(B_0,\Gamma_2)$. Let $U=U_1$ and $Z=\begin{pmatrix}Z_1 & 0\\ 0 & Z_2\end{pmatrix}$. Then

$$\frac{\phi_5(Z)}{\phi_5(M_2\langle Z\rangle)}=\frac{\sqrt{Z_1}}{\sqrt{Z_1+16Z_2}}.$$

We assume $\arg\sqrt{Z_1}$ is in $(0,\pi/2)$. Since Z and $UZ\,{}^tU$ are connected by the path

$$\begin{pmatrix}Z_1+t^2Z_2 & tZ_2\\ tZ_2 & Z_2\end{pmatrix}\qquad (0\le t\le 4)$$

which is on $\mathfrak{S}_2$, $\arg\sqrt{Z_1+16Z_2}$ is also in $(0,\pi/2)$. On the other hand

$$P_5M_2P_5^{-1}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -4 & 1 & 0\\ -4 & 0 & 0 & 1\end{pmatrix}.$$

From the transformation formula of $\Theta(Z)$ we have

$$J(P_5M_2P_5^{-1},P_5\langle Z\rangle)=\sqrt{\frac{Z_1+16Z_2}{Z_1}}.$$

Its argument is in $(-\pi/2,\pi/2)$ (cf. Remark 1.2). Hence the assrtion was proved.

If $k\ge 3$, the series of $E_i(Z)$ $(i=1,2,4,5)$ converges and $E_i(Z)\in S_{k+1/2}(\Gamma_0^2(4))$. Similarly as in the case of integral weight we can prove that $\lim_{\mathrm{Im}\,Z\to\infty}E_i\mid[\xi_i]_{k+1/2}(Z)=1$ and $\lim_{\mathrm{Im}\,Z\to\infty}E_i\mid[\xi_j]_{k+1/2}(Z)=0$ $(i\ne j)$. Hence Φ-operators to the zero-dimensional cusps Q_1, Q_2, Q_4, Q_5 are surjective if $k\ge 3$.

Next we construct Eisenstein series of Klingen type and prove the case of one-dimensional cusps. $M\in N(B_1,\Gamma_2)$ has the following form.

$$\begin{pmatrix} a & 0 & b & an-bm \\ mu & u & nu & ru \\ c & 0 & d & cn-dm \\ 0 & 0 & 0 & u \end{pmatrix}$$
$$= \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & n \\ m & 1 & n & 0 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$, $m, n, r \in \mathbf{Z}$ and $u = \pm 1$. We denote the matrices of the right hand side by M_1, M_2, M_3, M_4, respectively. Let $Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix}$. It is easily seen that

$$J(M, Z) = J(M_0, Z_1).$$

Let ξ_i $(i = 1, 2, 4)$ be as before. Let $f \in S_{k+1/2}(\Gamma_0^1(4))$. Since $S_{k+1/2}(\Gamma_0^1(4)) \simeq \{0\}$ $(k \leq 3)$, we can assume that $k \geq 4$. Let $M\langle Z\rangle_1$ be the upper-left entry of $M\langle Z\rangle$. We have $M\langle Z\rangle_1 = M_0\langle Z_1\rangle$. We put $\widetilde{f}(Z) = f(Z_1)$. Then

$$\begin{aligned} \widetilde{f} \mid [\iota(M)]_{k+1/2}(Z) &= \widetilde{f}(M\langle Z\rangle) J(M, Z)^{-2k-1} = f(M\langle Z\rangle_1) J(M_0, Z_1)^{-2k-1} \\ &= f(M_0\langle Z_1\rangle) J(M_0, Z_1)^{-2k-1} = f(Z_1) = \widetilde{f}(Z). \end{aligned}$$

Let $i = 1$ or 4 and define

$$E_{i,f}(Z) = \sum_{\gamma} \widetilde{f} \mid [\iota(P_i^{-1}\gamma)]_{k+1/2}(Z)$$

where γ is over $(P_i N(B_1, \Gamma_2) P_i^{-1} \cap \Gamma_0^2(4)) \backslash \Gamma_0^2(4)$. $\widetilde{f} \mid [\iota(P_i^{-1}\gamma)]_{k+1/2}(Z)$ is independent of the choice of γ from the above observation.

We return to the general case of degree g. Let

$$\Gamma^{g,0}(4) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \;\middle|\; B \equiv O \pmod 4 \right\}.$$

Then $\alpha \Gamma_g^* \alpha^{-1} \cap \Gamma_g$ contains $\Gamma^{g,0}(4)$. Let $\Theta^0(Z) = \theta(Z/2)$. If M belongs to $\Gamma^{g,0}(4)$, then

$$J^0(M, Z) := \Theta^0(M\langle Z\rangle)/\Theta^0(Z)$$

is holomorphic on $\mathfrak{S}_g$. By using $J^0(M, Z)$ we define the space $S_{k+1/2}(\Gamma^{g,0}(4))$ similarly as before. Let $Q_g = \begin{pmatrix} 4 \cdot 1_g & O \\ O & 1_g \end{pmatrix}$ and $\lambda_g = (Q_g, 1) \in \widetilde{G}_g$. Let $M \in \Gamma^{g,0}(4)$ and $\iota^0(M) = (M, J^0(M, Z))$. By definition we have $Q_g^{-1} \Gamma^{g,0}(4) Q_g = \Gamma_0^g(4)$ and $J(Q_g^{-1} M Q_g, Q_g^{-1}\langle Z\rangle) = J^0(M, Z)$. Hence it follows $\lambda_g^{-1} \iota^0(M) \lambda_g = \iota(Q_g^{-1} M Q_g)$. If $f \in S_{k+1/2}(\Gamma_0^g(4))$, then

$$\begin{aligned}(f \mid [\lambda_g^{-1}]_{k+1/2}) \mid [\iota^0(M)]_{k+1/2}(Z) &= (f \mid [\lambda_g^{-1}\iota^0(M)\lambda_g]_{k+1/2}) \mid [\lambda_g^{-1}]_{k+1/2}(Z)\\ &= f \mid [\lambda_g^{-1}]_{k+1/2}(Z)\,.\end{aligned}$$

Therefore $f \mapsto f \mid [\lambda_g^{-1}]_{k+1/2}$ is an isomorohism of $S_{k+1/2}(\Gamma_0^g(4))$ to $S_{k+1/2}(\Gamma^{g,0}(4))$.

Let $f \in S_{k+1/2}(\Gamma_0^1(4))$ and $f^0 = f \mid [\lambda_1^{-1}]_{k+1/2} \in S_{k+1/2}(\Gamma^{1,0}(4))$. We put $\widetilde{f^0}(Z) = f^0(Z_1)$ for $Z \in \mathfrak{S}_2$. We have $P_2\Gamma^{2,0}(4)P_2^{-1} = \Gamma_0^2(4)$. Let

$$E_{2,f}(Z) = \sum_{\gamma} \widetilde{f^0} \mid [\xi_2^{-1}\iota(\gamma)]_{k+1/2}(Z)$$

where γ is over $(P_2N(B_1,\Gamma_2)P_2^{-1} \cap \Gamma_0^2(4))\backslash\Gamma_0^2(4)$. $M \in N(B_1,\Gamma_2)$ is decomposed to a product $M_1M_2M_3M_4$ as before. We assume M belongs to $\Gamma^{2,0}(4)$. Namely, b, n and r are divisible by 4. We prove $\xi_2\,\iota^0(M)\,\xi_2^{-1} = \iota(P_2MP_2^{-1})$. Then $\widetilde{f^0} \mid [\xi_2^{-1}\iota(\gamma)]_{k+1/2}(Z)$ is independent of the choice of γ since $\widetilde{f^0} \mid [\iota^0(M)]_{k+1/2}(Z) = \widetilde{f^0}(Z)$.

Let $Z = PM\langle Z'\rangle$. Then

$$\iota(P_2MP_2^{-1})(\xi_2\,\iota^0(M)\,\xi_2^{-1})^{-1} = \iota(P_2MP_2^{-1})\,\xi_2\,\iota^0(M^{-1})\,\xi_2^{-1} = (1_4, t)\,,$$

where

$$t = J(P_2MP_2^{-1}, P_2\langle Z'\rangle)\,\phi_2(Z')\,J^0(M,Z')^{-1}\,\phi_2(M\langle Z'\rangle)^{-1}$$

is a constant. We prove that $t = 1$. Let $Z' = \begin{pmatrix} Z_1 & 0\\ 0 & Z_2\end{pmatrix}$. Then the case of M_3 is trivial. Since $J^0(M_4,Z') = 1$ and $\lim_{\operatorname{Im} Z_2\to\infty}\phi_2(Z')\phi_2(M_4\langle Z'\rangle)^{-1} = 1$, the assertion for M_4 is due to Theorem 3.9 (15) Φ_{15b}. The case of M_2 is easily proved if $m = 1$ and $n = 0$. Let $m = 0$ and $n = 4$. Then $J^0(M_2,Z') = 1$. When W moves on the segment from $Z' = \begin{pmatrix} Z_1 & 0\\ 0 & Z_2\end{pmatrix}$ to $M_2\langle Z'\rangle = \begin{pmatrix} Z_1 & 4\\ 4 & Z_2\end{pmatrix}$, $\det W$ moves on the segment from Z_1Z_2 to $Z_1Z_2 - 16$. Hence the argument of

$$\frac{\phi_2(Z')}{\phi_2(M_2\langle Z'\rangle)} = \frac{\sqrt{Z_1Z_2}}{\sqrt{Z_1Z_2-16}}$$

is in $(-\pi/2,\pi/2)$. On the other hand

$$P_2M_2P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -4 & 1 & 0\\ -4 & 0 & 0 & 1\end{pmatrix}.$$

From the transformation formula of $\Theta(Z)$ we have

$$J(P_2M_2P_2^{-1}, P_2\langle Z'\rangle) = \sqrt{\frac{Z_1Z_2-16}{Z_1Z_2}}\,.$$

Its argument is in $(-\pi/2, \pi/2)$. Hence the assrtion was proved. Now we prove the case of M_1. Since $\Gamma^{1,0}(4)/(\pm 1_2)$ is generated by $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, it suffices to prove the assertion for them. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. Then $J^0(M_1, Z') = 1$,

$$\frac{\phi_2(Z')}{\phi_2(M_1\langle Z'\rangle)} = \frac{\sqrt{Z_1 Z_2}}{\sqrt{(Z_1+4)Z_2}} \quad \text{and} \quad P_2 M_1 P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From the transformation formula of $\Theta(Z)$ we have

$$J(P_2 M_1 P_2^{-1}, P_2\langle Z'\rangle) = \sqrt{\frac{Z_1+4}{Z_1}}.$$

Hence the assrtion is similarly proved as before. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then since $P_2 M_1 P_2^{-1} \in N(B_0, \Gamma_2)$, $J(P_2 M_1 P_2^{-1}, P_2\langle Z'\rangle) = 1$.

$$\frac{\phi_2(Z')}{\phi_2(M_1\langle Z'\rangle)} = \frac{\sqrt{Z_1 Z_2}}{\sqrt{Z_1 Z_2/(Z_1+1)}}$$

and

$$J^0(M, Z') = J^0\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, Z_1\right) = J\left(\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \frac{Z_1}{4}\right).$$

This is calculated by the transformation formula and equal to $\sqrt{Z_1+1}$. Hence the assrtion is similarly proved as before.

Since $k \geq 4$, the series of $E_{i,f}(Z)$ $(i = 1, 2, 4)$ converges and $E_{i,f}(Z) \in S_{k+1/2}(\Gamma_0^2(4))$. Similarly as in the case of integral weight we can prove that $\lim_{\mathrm{Im}\, Z_2\to\infty} E_{i,f} \mid [\xi_i]_{k+1/2}(Z) = f(Z_1)$ $(i = 1, 4)$, $\lim_{\mathrm{Im}\, Z_2\to\infty} E_{2,f} \mid [\xi_2]_{k+1/2}(Z) = f^0(Z_2)$ and $\lim_{\mathrm{Im}\, Z_2\to\infty} E_{i,f} \mid [\xi_j]_{k+1/2}(Z) = 0$ $(i \neq j)$. Hence Φ-operators to the one-dimensional cusps C_1, C_2 and C_4 are surjective. Now the theorem was proved for $k \geq 3$.

We show that $\dim M_{1/2}(\Gamma_0^2(4)) = 1$, $\dim M_{3/2}(\Gamma_0^2(4)) = 1$ and $\dim M_{5/2}(\Gamma_0^2(4)) = 3$. Then the first equality of the theorem is proved. Since $\Theta(Z) \in M_{1/2}(\Gamma_0^2(4))$, $\dim M_{1/2}(\Gamma_0^2(4)) \geq 1$. We have the product map:

$$M_{1/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) \to M_{11}(\Gamma_0^2(4)).$$

Since $\dim M_{21/2}(\Gamma_0^2(4), \psi) = \dim M_{11}(\Gamma_0^2(4)) = 1$ (cf. Proposition 5.3, and Proposition 5.4), $\dim M_{1/2}(\Gamma_0^2(4)) = 1$. Similarly we have $\Theta(Z)^3 \in M_{3/2}(\Gamma_0^2(4))$ and the product map:

$$M_{3/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) \to M_{12}(\Gamma_0^2(4), \psi).$$

Since $\dim M_{12}(\Gamma_0^2(4), \psi) = 1$, we have $\dim M_{3/2}(\Gamma_0^2(4)) = 1$. Similarly we have the product maps:

$$M_{5/2}(\Gamma_0^2(4)) \times M_{21/2}(\Gamma_0^2(4), \psi) \to M_{13}(\Gamma_0^2(4)),$$
$$M_{1/2}(\Gamma_0^2(4)) \times M_2(\Gamma_0^2(4)) \to M_{5/2}(\Gamma_0^2(4)).$$

Since $\dim M_{13}(\Gamma_0^2(4)) = 3$, we have $\dim M_{5/2}(\Gamma_0^2(4)) \leq 3$ and since $\dim M_2(\Gamma_0^2(4)) = 3$, we have $\dim M_{5/2}(\Gamma_0^2(4)) \geq 3$. Thus we have completed the proof of Theorem 5.2. □

PROPOSITION 5.3.

$$\begin{aligned}\sum_{k=0}^{\infty} \dim M_{k+1/2}(\Gamma_0^2(4), \psi)\, t^k &= \sum_{k=0}^{\infty} \texttt{SiegelHalfpsi[0,k]}\, t^k + (3 + t + t^2) \\ &= \frac{t^{10}}{(1-t)(1-t^2)^2(1-t^3)}.\end{aligned}$$

Proof. From Theorem 4.6, we have $\dim M_{k+1/2}(\Gamma_0^2(4), \psi) = \dim S_{k+1/2}(\Gamma_0^2(4), \psi)$. Since we have $\dim S_{7/2}(\Gamma_0^2(4), \psi) = \texttt{SiegelHalfpsi[0,3]} = 0$, it follows that $S_{5/2}(\Gamma_0^2(4), \psi) \simeq S_{3/2}(\Gamma_0^2(4), \psi) \simeq S_{1/2}(\Gamma_0^2(4), \psi) \simeq \{0\}$. But since we have `SiegelHalfpsi[0,2]` $= -1$, `SiegelHalfpsi[0,1]` $= -1$ and `SiegelHal fpsi[0,0]` $= -3$, we have the equality of the first line. □

Let $M(\Gamma_0^2(4))$, $M(\Gamma_0^2(4), \psi)$ and $A(\Gamma_0^2(4), \psi)$ be $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4))$, $\bigoplus_{k=0}^{\infty} M_{k+1/2}(\Gamma_0^2(4), \psi)$ and $\bigoplus_{k=0}^{\infty} M_k(\Gamma_0^2(4), \psi^k)$, respectively. Then $A(\Gamma_0^2(4), \psi)$ is a graded ring and since it holds $J(M, Z)^2 = \det(CZ + D)\psi(\det D)$, $M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4), \psi)$ are $A(\Gamma_0^2(4), \psi)$-modules. From the result of J.-I. Igusa ([Ig1]), we have the following proposition. (We can also prove them by dimension formula.)

PROPOSITION 5.4.

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4))\, t^k = \frac{1 + t^4 + t^{11} + t^{15}}{(1-t^2)^3(1-t^6)},$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4), \psi)\, t^k = \frac{t + t^3 + t^{12} + t^{14}}{(1-t^2)^3(1-t^6)},$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^2(4), \psi^k)\, t^k = \frac{1 + t + t^3 + t^4}{(1-t^2)^3(1-t^6)} = \frac{1}{(1-t)(1-t^2)^2(1-t^3)}.$$

From this, Theorem 5.2 and Proposition 5.3, we have

COROLLARY 5.5. *$M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4), \psi)$ are free $A(\Gamma_0^2(4), \psi)$-modules of rank one.*

A generator of $M(\Gamma_0^2(4))$ as $A(\Gamma_0^2(4),\psi)$-module is given by $\Theta(Z)$. Let $f_{21/2}(Z)$ be a generator of $M(\Gamma_0^2(4),\psi)$. Then $f_{21/2}(Z)\Theta(Z)$ is an automorphic form with respect to $J(M,Z)^{22}\psi(\det D)=\det(CZ+D)^{11}$. Hence this belongs to $M_{11}(\Gamma_0^2(4))$. Let $f_{11}(Z)$ be the base of one-dimensional space $M_{11}(\Gamma_0^2(4))$. Then $f_{11}(Z)/\Theta(Z)$ is holomorphic and we may take $f_{21/2}(Z)=f_{11}(Z)/\Theta(Z)$. Since $A(\Gamma_0^2(4),\psi)$ is contained in $\bigoplus_{k=0}^{\infty}M_k(\Gamma_2(4))$ and $\bigoplus_{k=0}^{\infty}M_k(\Gamma_2(4))$ is contained in the ring of theta constants ([Ig1]), every elements of $M(\Gamma_0^2(4))$ and $M(\Gamma_0^2(4),\psi)$ are representable by theta constants.

REMARK 5.6. T. Ibukiyama represented the generators of $A(\Gamma_0^2(4),\psi)$ and $f_{21/2}(Z)$ explicitly by theta constants ([Ib]). Especially $A(\Gamma_0^2(4),\psi)$ is generated by algebraically independent modular forms f_1, X, g_2 and f_3 whose weights are 1, 2, 2 and 3, respectively. $f_{21/2}(Z)$ is divisible by nine theta constants and not divisible by one theta constant. Let $Z\in\mathfrak{S}_2$. Then there exists $M\in\Gamma_2$ such that $M\langle Z\rangle=\begin{pmatrix}Z_1&0\\0&Z_2\end{pmatrix}$, if and only if one of ten theta constants vanishes at Z (J.-I. Igusa, [H]). Hence $f_{21/2}(Z)$ does not belong to $S_{21/2}(\Gamma_2^{\alpha},\psi)$ (cf. Remark 4.7).

Appendix. The generating functions

We list here the generating functions of `SiegelHalf[j,k]` and `SiegelHalfpsi[j,k]`.

TABLE A.1. $\sum_{j,\,k=0}^{\infty}$ `SiegelHalf[j,k]`s^jt^k is a rational function of s and t whose denominator is

$$(1-s^2)^2(1-s^3)^2(1-t)(1-t^2)^2(1-t^3).$$

The coefficients of s^jt^k $(0\le j\le 9,\ 0\le k\le 7)$ in the numerator are given by the following matrix.

0	0	−3	−6	−6	−3	4	3	−3	−4
0	0	1	1	1	3	3	1	1	1
−1	−1	7	17	20	8	−12	−8	8	10
1	1	2	7	7	−2	−9	−4	1	2
2	3	−2	−12	−20	−9	8	4	−8	−8
1	3	−5	−21	−23	−5	12	6	−7	−9
0	0	−1	−1	2	2	1	3	4	2
−2	−3	4	14	13	0	−8	−2	7	7

TABLE A.2. $\sum_{j,\,k=0}^{\infty}$`SiegelHalfpsi[j,k]`s^jt^k is a rational function of s and t whose denominator is

$$(1-s^2)^2(1-s^3)^2(1-t)(1-t^2)^2(1-t^3).$$

The coefficients of $s^j t^k$ $(0 \le j \le 9,\ 0 \le k \le 7)$ in the numerator are given by the following matrix.

−3	0	6	6	−6	−21	−11	3	6	2
2	0	−4	−5	1	12	10	1	−3	−2
6	0	−12	−11	17	47	23	−6	−12	−4
0	0	0	5	10	4	−5	−6	−3	1
−5	0	13	15	−12	−41	−25	−1	9	5
−6	1	15	9	−21	−46	−24	6	14	4
3	2	−6	−12	−3	13	14	6	−2	−3
4	0	−9	−8	8	26	17	0	−6	−2

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