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THE BURKHARDT GROUP AND MODULAR FORMS

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Abstract. In this paper we investigate the ring of Siegel modular forms of genus two and level 3. We determine the structure of this ring. It is generated by 10 modular forms (5 of weight 1 and 5 of weight 3) and there are 20 relations (5 in weight 5 and 15 in weight 6). The proof consists of two steps. In a first step we prove that the Satake compactification of the modular variety of genus 2 and level 3 is the normalization of the dual of the Burkhardt quartic. The second part consists in the normalization of the Burkhardt dual. Our basic tool is the representation theory of the Burkhardt group $G = G_{25\,920}$, which acts on our varieties.

1. Introduction

In this paper we investigate the ring of Siegel modular forms of genus 2 and level 3. We determine the structure of this ring. It is generated by 10 modular forms (5 of weight 1 and 5 of weight 3) and there are 20 relations (5 in weight 5 and 15 in weight 6). The proof consists of two steps. In a first step we prove that the Satake compactification of the modular variety of genus 2 and level 3 is the normalization of the dual of the Burkhardt quartic.

This is a hypersurface of degree 18 which we describe explicitly. We give also a description of the normalization map and we prove that it is bijective.

The second part consists of the normalization of the Burkhardt dual. Our basic tool is the Burkhardt group $G = G_{25\,920}$, which acts on our varieties. The representation theory of the Burkhardt group was a strong leading guide.

In fact, we first construct an element in the normalization; then using the action of G, we are able to construct other elements, and at the end we get a ring contained in the field of fractions that satisfies Serre's criterion of normality. This ring will also be Gorenstein.

Several complicated polynomial identities will occur. It is easy to verify them using a computer, but it is very tedious to verify them by hand. But we want to point out that in principle this is possible because no really expensive algorithms such as Gröbner bases have to be used. (We used them to find identities, but after one has them they can be easily verified.) Similarly, for representation-theoretic questions and polynomial computations, respectively, we used the computer algebra systems GAP, cf. [Ga] and SINGULAR, cf. [Si]. As before it would be possible to manage the calculations by hand with the help of the known character tables ATLAS, see [At].

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We did not reproduce any program in this paper. A reader, who does not rely on our statements could write his own programs or can consult [Fr3].

The ring of modular forms with respect to the group $\Gamma_0[3]$ has been determined by Ibukiyama, cf. [Ib]. His ring is a subring of our ring. We do not need results from Ibukiyama's paper and can reprove his main results. But we want to point out that his paper was extremely useful for us.

2. A five-dimensional space of Siegel modular forms of genus 2, weight 1, and level 3

Let $\Gamma_n = \mathrm{Sp}(2n, \mathbb{Z})$ be the full Siegel modular group of genus n. We denote by

$$\Gamma_n[q] = \mathrm{Kernel}\,(\mathrm{Sp}(2n,\mathbb{Z}) \to \mathrm{Sp}(2n, \mathbb{Z}/q\mathbb{Z}))$$

the principal congruence group of level q and by

$$\Gamma_{n,0}[q] := \left\{ M \in \Gamma_n \mid M \equiv \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \bmod q \right\} \tag{1}$$

the subgroup of Γ_n defined by $C \equiv 0 \bmod q$. The group

$$\Gamma_2/\Gamma_2[3] \cong \mathrm{Sp}(4, \mathbb{Z}/3\mathbb{Z})$$

contains the negative unit matrix $-E$ in its center. The so-called *Burkhardt group*

$$G := \mathrm{Sp}(4, \mathbb{Z}/3\mathbb{Z})/\{\pm E\}$$

is the finite simple group of order $25,920$.

For a subgroup of finite index $\Gamma \subset \Gamma_n$, an integer r and a character χ, we consider the space of modular forms $[\Gamma,\, r,\, \chi]$. Its elements are holomorphic functions on the Siegel upper half plane

$$\mathbf{H}_n = \{ Z \in M_n(\mathbb{Z}) \mid Z = Z',\ \mathrm{Im}(Z) > 0\}$$

with the property

$$f(MZ) = \chi(M) \det(CZ+D)^r f(Z), \quad \text{where } MZ = (AZ+B)(CZ+D)^{-1}.$$

Here M is divided into four $n \times n$-blocks as usual. In the case $n = 1$ the standard regularity condition at the cusps has to be added.

We write $[\Gamma\,, r]$ if the character is trivial.

Let $\Gamma_0 \subset \Gamma$ be a normal subgroup of finite index. The group Γ acts on $[\Gamma_0, r]$ by

$$f \mapsto f|M, \quad (f|M)(Z) := \det(CZ+D)^{-r} f(MZ),$$

and this action factors through Γ/Γ_0.

We notice that for even n the negative unit-matrix acts trivially.

We consider the graded algebra

$$A(\Gamma) := \bigoplus\nolimits_{r \in \mathbb{Z}} [\Gamma, r],$$

which is related to the Satake compactification

$$X(\Gamma) := \overline{\mathbf{H}_n/\Gamma} \cong \operatorname{Proj}(A(\Gamma)).$$

Here we use the fundamental theorem of Baily, see [BB], which states that $A(\Gamma)$ is finitely generated and that its associated projective variety is biholomorphic equivalent to the complex space $X(\Gamma)$.

We are interested in the ring $A(\Gamma_2[3])$. The Burkhardt group G acts on this ring as well as on $X(\Gamma_2[3])$. We recall that set theoretically we have

$$X(\Gamma_2[3]) = \mathbf{H}_2/\Gamma_2[3] \cup \textstyle\bigcup_{i=1}^{40} \mathcal{C}_i \cup \bigcup_{i=1}^{40} P_i. \tag{2}$$

Here the union is disjoint and we denote with $\mathcal{C}_i$, P_i, respectively, the 1-dimensional and the 0-dimensional cusps. We recall that each $\mathcal{C}_i$ is biholomorphic to $\mathbf{H}_1/\Gamma_1[3]$ and we have a $(40\,,40)_4$ relation among the cusps. This means that each copy of $\overline{\mathcal{C}_i} \cong X(\Gamma_1[3]) \cong \mathbb{P}^1$ contains 4 cusps P_i and each P_i is contained in 4 copies of $X(\Gamma_1[3])$.

We introduce the group

$$\Gamma_{1,1}[3] := \left\{ M \in \Gamma_2 \mid M \equiv \begin{bmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \bmod 3 \right\}. \tag{3}$$

Both are subgroups of index 40 in Γ_2 and we have that the 0-dimensional cusps are stabilized by subgroups conjugate to $\Gamma_{2,0}[3]$ and the 1-dimensional cusps are stabilized by subgroups conjugate to $\Gamma_{1,1}[3]$.

We want to introduce certain theta series with respect to the root lattice A_2. We use the Gram matrix

$$S := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Next we introduce 5 characteristics

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ P_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ P_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ P_5 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

They are isotropic in the sense $SP \equiv 0 \bmod 3$. We define the theta series

$$\vartheta_i(Z) := \sum_{G \text{ integral}} \mathbf{e}\left(\operatorname{tr}(S[G + P_i/3]Z)\right) \tag{4}$$

(using the notations $S[P] = P'SP$ and $\mathbf{e}(t) = \exp(\pi i\, t)$). Easily, from [An, p. 24], we deduce the following.

Proposition 1. *The theta series $\vartheta_1, \ldots, \vartheta_5$ are contained in $[\Gamma_2[3], 1]$. They span a five-dimensional space V which is invariant under the Burkhardt group and with irreducible* *action. The action is given by the following* 4 *matrices which generate* Γ_2*:*

$$I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

These four matrices act on the ϑ_i *by the following* 5×5*-matrices in the same ordering:*

$$\widetilde{I} := \begin{bmatrix} -1/3 & -2/3 & -2/3 & -2/3 & -2/3 \\ -1/3 & 1/3 & -2/3 & 1/3 & 1/3 \\ -1/3 & -2/3 & 1/3 & 1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 & 1/3 & -2/3 \\ -1/3 & 1/3 & 1/3 & -2/3 & 1/3 \end{bmatrix}, \quad \widetilde{S}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \omega \end{bmatrix},$$

$$\widetilde{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{bmatrix}, \quad \widetilde{S}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \omega \end{bmatrix}$$

with $\omega := \exp(2\pi i/3)$.

The four tilde-matrices generate $\widetilde{G} \subset \mathrm{GL}(5, \mathbb{C})$, a copy of the Burkardt group. We mention that this group acts on V from the right: Applying an element $g \in \widetilde{G}$ to $\alpha_1\vartheta_1 + \ldots + \alpha_5\vartheta_5 \in V$ is the same as multiplying the row $(\alpha_1, \ldots, \alpha_5)$ from the right with g.

We mention some simple properties of $\widetilde{G}$. Complex conjugation defines an outer automorphism of $\widetilde{G}$. It transforms a representation into its dual representation. We need the formula

$$g \longrightarrow \bar{g}'^{-1} = TgT^{-1} \text{ for } g \in \widetilde{G}, \qquad T = \mathrm{diag}\,[2, 1, 1, 1, 1].$$

3. The Burkhardt–Coble invariants

We consider the polynomial ring $\mathbb{C}[A_1, \ldots, A_5]$ in five indeterminates. We define the action of the Burkhardt group such that the natural homomorphism

$$\mathbb{C}[A_1, \ldots, A_5] \longrightarrow \mathbb{C}[\vartheta_1, \ldots, \vartheta_5], \quad A_i \mapsto \vartheta_i,$$

is equivariant. The ring of invariants $\mathbb{C}[A_1, \ldots, A_5]^G$ has been determined by Burkhardt. We use Coble's construction for these invariants, cf. [Co]. The form ϑ_1 belongs to $[\Gamma_{2,0}[3]\,, 1\,, \chi_S]$. Here χ_S is the real Dirichlet character associated to the quadratic form S; we recall, see [An, p. 26], that $\chi_S^2 = 1$ and $\chi_S = 1$ if we restrict to $\Gamma_2[3]$. This means that the variable A_1 is "quasi-invariant" under the image of $B \subset G$ of $\Gamma_{2,0}[3]$. Following Coble, we define the invariants $g_m \in \mathbb{C}[A_1, \dots, A_5]$ by

$$g_m = \gamma_m \sum_{g \in B \backslash G} (A_1^m) \mid g, \qquad m = 2, 4, 6, \dots .$$

The normalizing constants used here have the effect that the constant Fourier coefficients of the corresponding modular forms are 1 if they are different from 0. (Otherwise we need no normalizing constant).

The invariants g_4, g_6, g_{10}, g_{12}, g_{18} are algebraically independent. They are the so-called primary invariants, i.e., the full ring of invariants is a finitely generated free module over $\mathbb{C}[g_4, g_6, g_{10}, g_{12}, g_{18}]$. We are interested in the corresponding modular forms G_4, G_6, G_{10}, G_{12}, G_{18}. With some patience or the aid of a computer, some Fourier coefficients can be computed. We write the Fourier coefficient of a modular form as

$$f(Z) = \sum_T a_f(T) \mathbf{e}(\mathrm{tr}(TZ)).$$

Here T runs through a certain lattice of rational symmetric matrices. In the case of the full modular group or $\Gamma_{2,0}[3]$ this lattice consists of all integral matrices with even diagonal (called even matrices). Recall that $a(T)$ vanishes if T is not semi-positive.

We start with a list of Fourier coefficients of the thetas ϑ_i, $1 \le i \le 5$. Here the matrices T have the property that $3T$ is even. For the following table we define

$$a_i(T) = a_{\vartheta_i}(T/3), \quad T \text{ even}.$$

The following Fourier coefficients are in the range

$$3t_0 \le 12, \quad t_2 \le 6, \quad T = \begin{bmatrix} t_0 & t_1 \\ t_1 & t_2 \end{bmatrix}.$$

Fourier coefficients in this range vanish when they do not occur in the following list:

$$a_1(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = 1,\ a_1(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 6 \end{smallmatrix}\right]) = 6,\ a_1(\left[\begin{smallmatrix} 6 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = 6,\ a_1(\left[\begin{smallmatrix} 6 & -6 \\ -6 & 6 \end{smallmatrix}\right]) = 6,$$
$$a_1(\left[\begin{smallmatrix} 6 & -3 \\ -3 & 6 \end{smallmatrix}\right]) = 12,\ a_1(\left[\begin{smallmatrix} 6 & 3 \\ 3 & 6 \end{smallmatrix}\right]) = 12,\ a_1(\left[\begin{smallmatrix} 6 & 6 \\ 6 & 6 \end{smallmatrix}\right]) = 6.$$

$$a_2(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right]) = 3,\ a_2(\left[\begin{smallmatrix} 6 & -3 \\ -3 & 2 \end{smallmatrix}\right]) = 6,\ a_2(\left[\begin{smallmatrix} 6 & 0 \\ 0 & 2 \end{smallmatrix}\right]) = 6,\ a_2(\left[\begin{smallmatrix} 6 & 3 \\ 3 & 2 \end{smallmatrix}\right]) = 6.$$

$$a_3(\left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = 3,\ a_3(\left[\begin{smallmatrix} 2 & -3 \\ -3 & 6 \end{smallmatrix}\right]) = 6,\ a_3(\left[\begin{smallmatrix} 2 & 0 \\ 0 & 6 \end{smallmatrix}\right]) = 6,\ a_3(\left[\begin{smallmatrix} 2 & 3 \\ 3 & 6 \end{smallmatrix}\right]) = 6,$$
$$a_3(\left[\begin{smallmatrix} 8 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = 3,\ a_3(\left[\begin{smallmatrix} 8 & -6 \\ -6 & 6 \end{smallmatrix}\right]) = 6,\ a_3(\left[\begin{smallmatrix} 8 & 0 \\ 0 & 6 \end{smallmatrix}\right]) = 6,\ a_3(\left[\begin{smallmatrix} 8 & 6 \\ 6 & 6 \end{smallmatrix}\right]) = 6.$$

$$a_4(\left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]) = 6,\ a_4(\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]) = 3,\ a_4(\left[\begin{smallmatrix} 8 & -4 \\ -4 & 2 \end{smallmatrix}\right]) = 3,\ a_4(\left[\begin{smallmatrix} 8 & 2 \\ 2 & 6 \end{smallmatrix}\right]) = 6.$$

$$a_5(\left[\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right]) = 3,\ a_5(\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]) = 6,\ a_5(\left[\begin{smallmatrix} 8 & -2 \\ -2 & 2 \end{smallmatrix}\right]) = 6,\ a_5(\left[\begin{smallmatrix} 8 & 4 \\ 4 & 2 \end{smallmatrix}\right]) = 3.$$

This table can be used to compute some Fourier coefficients $a_{g_i}(T)$ of the invariants:

T	$\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&-1\\-1&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}4&-1\\-1&2\end{smallmatrix}\right]$
G_4	1	240	13440	30240	2160	138240
G_6	1	-504	44352	166320	-16632	2128896
G_{10}	1	-264	$-\frac{26304}{13}$	$\frac{965520}{13}$	-135432	$\frac{92233728}{13}$
G_{12}	1	$\frac{16272}{61}$	$\frac{16873344}{2501}$	$\frac{199422432}{2501}$	$\frac{11599632}{61}$	$\frac{20120196096}{2501}$
G_{18}	1	$\frac{16632}{205}$	$-\frac{105153984}{224065}$	$\frac{1446026256}{224065}$	$-\frac{26449416}{205}$	$-\frac{60302112768}{44813}$

4. The Burkhardt dual

The Fourier coefficients of G_4 and G_6 are integral. Actually they are Eisenstein series. Inspection of the Fourier coefficients shows that the forms G_4G_6, G_{10} as well as G_4^3, G_6^2, G_{12} are linearly independent. By a well-known theorem of Igusa, see [Ig] or [Fr2], the ring of even-weight modular forms of genus two with respect to the full modular group is generated by the forms of weight 4,6,10,12. This means that this ring is contained in $\mathbb{C}[\vartheta_1,\dots,\vartheta_5]$. As a consequence the ring $A(\Gamma_2[3])$ is integral over $\mathbb{C}[\vartheta_1,\dots,\vartheta_5]$. We claim that both rings have the same field of fractions. It is sufficient to show that they have the same homogeneous field of fractions (=the field generated by quotients of homogeneous elements of the same degree). The homogeneous field of fractions $K(\Gamma)$ of $A(\Gamma)$ is nothing but the field of modular functions. We have the inclusions

$$K(\Gamma_2)\subset K(\vartheta_i/\vartheta_j)\subset K(\Gamma_2[3]).$$

It is known that the degree of $K(\Gamma_2[3])$ over $K(\Gamma_2)$ equals the index of $\Gamma_2[3]$ in Γ_2 which is the order of the Burkhardt group (25,920). The Burkhardt group also acts nontrivially on $K(\vartheta_i/\vartheta_j)$ and this action is faithful because the Burkhardt group is simple. We obtain that the degree of $K(\vartheta_i/\vartheta_j)$ over $K(\Gamma_2)$ is greater or equal to $25,920$. This implies the claimed equality. We have proved:

Proposition 2. *The graded algebra $A(\Gamma_2[3])$ is the normalization of the algebra*

$$\mathbb{C}[\vartheta_1,\dots,\vartheta_5].$$

We want to construct this normalization. The first task in this direction is to determine the kernel of

$$\psi:\mathbb{C}[A_1,\dots,A_5]\longrightarrow\mathbb{C}[\vartheta_1,\dots,\vartheta_5]. \tag{5}$$

This is a principal ideal. The modular form G_{18} must be expressible as a polynomial in the forms G_4,G_6,G_{10},G_{12}. This gives a nontrivial relation and there is no relation of smaller degree. Using the Fourier coefficients above one can compute this relation and obtain:

Proposition 3. *The kernel of the natural homomorphism*

$$\psi:\mathbb{C}[A_1,\dots,A_5]\longrightarrow\mathbb{C}[\vartheta_1,\dots,\vartheta_5]$$

is generated by the following polynomial of degree 18*:*

$$\begin{aligned}716696985600\,P_{18}:=\;&173650375g_{18}-67489485g_6g_{12}\\&-205937433g_4^2g_{10}+4148960g_6^3+95627583g_4^3g_6.\end{aligned}$$

This polynomial is normalized such that it has integral coprime coefficients.

This will turn out to be a very interesting polynomial, a good reason to write it in expanded form:

$$\begin{aligned}
P_{18}(A_1,\ldots,A_5) \;=\;& 2A_4^3A_5^9A_3^6 - 2A_1^3A_4^9A_2^6 - 2A_1^3A_5^6A_4^9 - 18A_1A_5^2A_4^8A_2^5A_3^2 \\
& - 18A_1A_2^5A_5^8A_4^2A_3^2 + 2A_3^3A_4^9A_2^6 + A_5^{12}A_2^6 + A_2^{12}A_4^6 + 2A_2^3A_4^9A_3^6 - 2A_1^3A_5^9A_4^6 \\
& - 6A_1^2A_4^{10}A_5A_2^4A_3 - 2A_1^3A_2^6A_5^9 - 2A_5^9A_4^9 - 2A_2^3A_5^3A_3^{12} - 6A_5^6A_2^6A_3^6 + 2A_5^3A_4^6A_3^9 \\
& + 2A_2^6A_5^3A_3^9 + 2A_2^3A_5^9A_3^6 + 2A_5^3A_4^9A_3^6 + 8A_1^3A_2^3A_5^9A_3^3 + 4A_5^3A_4^3A_1^9A_3^3 \\
& - 18A_2^5A_5^2A_4^2A_3^8A_1 - 10A_5^3A_4^3A_1^6A_3^6 + A_1^6A_4^6A_5^6 + 12A_1A_4^2A_5^2A_2^{11}A_3^2 - 2A_5^3A_4^3A_3^{12} \\
& - 6A_4^6A_5^6A_3^6 - 6A_2^6A_5^6A_4^6 + 2A_5^6A_4^3A_3^9 - 8A_3^3A_2^3A_4^6A_5^6 - 6A_1^2A_4A_5A_2^{10}A_3^4 \\
& - 10A_2^3A_5^3A_1^6A_3^6 - 2A_3^3A_5^3A_4^{12} + 4A_2^3A_5^3A_1^9A_3^3 + 2A_3^3A_5^9A_4^6 + 8A_2^3A_5^3A_3^9A_1^3 \\
& - 12A_1^2A_2A_5^4A_4^7A_3^4 - 108A_1^4A_4^5A_5^2A_2^5A_3^2 + 8A_5^3A_4^3A_3^9A_1^3 - 12A_1^2A_5^7A_4A_2^4A_3^4 \\
& - 18A_1A_2^8A_5^5A_4^2A_3^2 - 12A_5A_2^4A_4^7A_1^2A_3^4 + 18A_1^7A_2^2A_4^2A_5^5A_3^2 + 72A_5^5A_4^5A_2^2A_3^5A_1 \\
& - 2A_5^9A_3^6A_1^3 - 12A_1^2A_4A_5^4A_2^7A_3^4 + 2A_3^3A_2^9A_5^6 + 168A_5^3A_4^6A_2^3A_1^3A_3^3 - 8A_5^6A_4^3A_2^3A_3^6 \\
& - 8A_4^3A_5^3A_2^6A_3^6 - 8A_5^3A_4^6A_2^3A_3^6 - 6A_2^6A_4^6A_3^6 + 168A_5^6A_4^3A_2^3A_1^3A_3^3 + 168A_4^3A_5^3A_2^6A_1^3A_3^3 \\
& + A_5^6A_3^{12} - 2A_2^9A_1^3A_3^6 - 18A_4^2A_5^2A_2^8A_1A_3^5 - 18A_4^2A_2^2A_5^8A_3^5A_1 + 72A_4^2A_5^5A_2^5A_3^5A_1 \\
& - 315A_4^4A_5^4A_2^4A_1^2A_3^4 - 108A_1^4A_5^5A_4^5A_2^2A_3^2 - 2A_4^6A_1^3A_3^9 + 8A_3^3A_4^9A_2^3A_5^3 - 8A_3^3A_4^6A_5^3A_2^6 \\
& + A_4^6A_1^6A_3^6 - 2A_3^3A_2^3A_5^{12} - 6A_4A_1^2A_2^{10}A_5^4A_3 + 2A_5^3A_2^9A_3^6 - 2A_4^9A_3^9 - 2A_2^3A_4^3A_3^{12} \\
& + 12A_1A_4^{11}A_5^2A_2^2A_3^2 + A_2^{12}A_3^6 + A_4^6A_3^{12} + A_4^{12}A_3^6 + 8A_2^9A_5^3A_4^3A_3^3 - 18A_2^2A_1A_4^8A_5^5A_3^2 \\
& - 2A_3^3A_2^{12}A_5^3 + 24A_2A_1^2A_5^7A_4^7A_3 - 12A_1^2A_4^4A_5^4A_2^7A_3 - 108A_2^5A_5^2A_4^2A_1^4A_3^5 \\
& + 90A_1^5A_4^4A_5^4A_2^4A_3 - 4A_5^6A_2^6A_1^3A_3^3 - 6A_1^2A_5A_2^{10}A_4^4A_3 + 24A_1^2A_4A_2^7A_5^7A_3 \\
& - 6A_4^4A_2A_5A_3^{10}A_1^2 + 8A_2^3A_4^9A_1^3A_3^3 - 6A_2^4A_4A_5A_1^8A_3^4 + 72A_4^5A_5^2A_2^5A_3^5A_1 \\
& + 8A_1^3A_5^9A_4^3A_2^3 + 12A_4^4A_2A_5A_1^5A_3^7 + 12A_1A_5^{11}A_4^2A_2^2A_3^2 + 2A_5^6A_2^3A_3^9 \\
& + 4A_1^9A_2^3A_4^3A_5^3 + 12A_1^5A_5A_2^4A_4^7A_3 + 2A_4^3A_5^9A_2^6 - 4A_2^6A_4^6A_1^3A_3^3 - 27A_1^4A_4^2A_5^2A_2^8A_3^2 \\
& - 27A_1^4A_4^2A_2^2A_5^8A_3^2 - 108A_1^4A_4^2A_5^5A_2^5A_3^2 + 8A_1^3A_4^9A_2^3A_5^3 + 2A_4^6A_2^3A_3^9 - 2A_2^3A_4^{12}A_3^3 \\
& - 2A_5^9A_3^9 - 2A_2^9A_3^9 + A_2^6A_1^6A_3^6 + 8A_3^3A_5^9A_4^3A_2^3 - 2A_5^6A_3^9A_1^3 + A_5^{12}A_3^6 \\
& + 8A_1^3A_2^3A_4^3A_3^9 + 18A_1^7A_2^5A_5^2A_4^2A_3^2 - 18A_1A_2^2A_4^5A_5^8A_3^2 + 8A_4^3A_2^9A_1^3A_3^3 \\
& - 12A_1^2A_5^7A_4^4A_2^4A_3 - 10A_2^3A_4^3A_1^6A_3^6 - 6A_1^8A_2^4A_4A_5^4A_3 + 8A_2^3A_4^3A_5^3A_3^9 - 18A_3^2A_5^2A_2^8A_4^5A_1 \\
& + A_5^6A_4^{12} - 6A_4A_2A_5^{10}A_1^2A_3^4 + 12A_1^5A_4A_5^4A_2^7A_3 + 12A_1^5A_5^7A_4A_2A_3^4 \\
& + 90A_1^5A_2^4A_4A_5^4A_3^4 + 12A_1^5A_5^7A_4A_2^4A_3 + 90A_1^5A_5^4A_4^4A_2A_3^4 + 2A_5^3A_4^9A_2^6 - 2A_5^3A_4^{12}A_2^3 \\
& - 6A_1^8A_5^4A_4^4A_2A_3 + A_1^6A_2^6A_4^6 - 2A_5^{12}A_4^3A_3^3 - 4A_1^3A_5^6A_4^3A_3^6 + 12A_1^5A_2A_5^4A_4^7A_3 \\
& + 2A_2^9A_5^6A_4^3 + 24A_5A_1^2A_2^7A_4^7A_3 + 2A_3^3A_2^9A_4^6 - 6A_1^2A_3A_2A_4^{10}A_5^4 - 10A_1^6A_2^6A_4^3A_3^3 \\
& + 18A_1^7A_4^5A_5^2A_2^2A_3^2 - 10A_1^6A_5^3A_4^6A_3^3 + 12A_1^5A_4^4A_5A_2^7A_3 + 12A_1^5A_2A_4^4A_5^7A_3 \\
& + 12A_1^5A_2^7A_5A_3^4A_4 - 2A_1^3A_2^9A_4^6 - 3A_1^{10}A_5^2A_2^2A_4^2A_3^2 - 6A_1^8A_4^4A_2^4A_5A_3 - 6A_1^2A_5^{10}A_2^4A_4A_3 \\
& - 27A_1^4A_5^2A_2^2A_4^2A_3^8 - 6A_1^8A_5^4A_2A_4A_3^4 + 2A_2^9A_5^3A_4^6 - 4A_5^6A_2^3A_3^6A_1^3 - 12A_5^4A_4^4A_2A_3^7A_1^2 \\
& + A_5^6A_1^6A_3^6 - 4A_2^6A_5^3A_3^6A_1^3 - 2A_4^9A_3^6A_1^3 + 18A_5^2A_2^2A_4^2A_1^7A_3^5 - 4A_5^3A_4^6A_1^3A_3^6 + A_2^6A_3^{12} \\
& + 8A_1^3A_2^9A_5^3A_4^3 - 8A_2^6A_5^6A_4^3A_3^3 - 10A_1^6A_5^6A_4^3A_3^3 - 2A_2^{12}A_4^3A_3^3 + 8A_1^3A_5^3A_2^9A_3^3 \\
& + 24A_2^7A_5A_3^7A_4A_1^2 - 18A_2^2A_5^2A_4^8A_3^5A_1 - 12A_2A_4^4A_5^7A_1^2A_3^4 - 6A_2A_5A_4^{10}A_1^2A_3^4 \\
& + 12A_5^4A_2A_4A_1^5A_3^7 + 12A_2^4A_4A_5A_1^5A_3^7 - 18A_5^5A_2^2A_4^2A_3^8A_1 - 10A_1^6A_2^6A_5^3A_3^3 \\
& + 24A_4^7A_5A_2A_3^7A_1^2 - 4A_4^6A_2^3A_3^6A_1^3 + 24A_5^7A_4A_2A_3^7A_1^2 - 12A_2^4A_4A_5^4A_3^7A_1^2 + 2A_5^6A_4^9A_3^3 \\
& - 6A_2^4A_4A_5A_3^{10}A_1^2 - 18A_4^5A_5^2A_2^2A_1A_3^8 - 108A_4^5A_5^2A_2^2A_1^4A_3^5 - 6A_5^4A_2A_4A_1^2A_3^{10} \\
& + 168A_2^3A_4^3A_5^3A_1^3A_3^6 - 10A_1^6A_5^6A_2^3A_3^3 - 12A_4^4A_2^4A_5A_1^2A_3^7 + 90A_4^4A_2^4A_5A_1^5A_3^4 \\
& - 10A_1^6A_4^6A_2^3A_3^3 + 4A_1^9A_2^3A_4^3A_3^3 - 4A_2^6A_4^3A_3^6A_1^3 + A_5^6A_2^{12} + A_1^6A_5^6A_2^6 + 8A_1^3A_5^3A_4^9A_3^3 \\
& - 12A_4^4A_5A_2^7A_1^2A_3^4 - 108A_1^4A_5^5A_2^2A_4^2A_3^5 + 12A_5^2A_2^2A_4^2A_3^{11}A_1 + 12A_1^5A_4^7A_5A_2A_3^4 \\
& - 2A_4^3A_5^3A_2^{12} - 4A_1^3A_4^6A_5^3A_2^6 - 2A_2^9A_5^9 - 4A_1^3A_2^3A_4^6A_5^6 + A_4^{12}A_2^6 - 2A_2^9A_4^9 + A_5^{12}A_4^6 \\
& - 4A_1^3A_4^3A_2^6A_5^6 - 4A_1^3A_4^6A_5^6A_3^3 + 2A_2^3A_4^9A_5^6 + 2A_5^9A_4^6A_2^3 - 2A_1^3A_2^9A_5^6 - 2A_4^3A_5^{12}A_2^3 \\
& - 10A_1^6A_4^3A_5^3A_2^6 - 10A_1^6A_5^6A_4^3A_2^3 - 10A_1^6A_5^3A_4^6A_2^3 - 6A_1^2A_3A_2A_5^{10}A_4^4 - 96A_1^6A_2^3A_4^3A_5^3A_3^3
\end{aligned}$$

$$+ 2A_2^6A_4^3A_3^9 + 2A_3^3A_2^6A_5^9 - 27A_1^4A_2^2A_5^2A_4^8A_3^2 - 2A_1^3A_2^6A_3^9 - 6A_1^8A_4^4A_2A_5A_3^4$$
$$+ 8A_1^3A_4^3A_5^9A_3^3 + 2A_4^3A_2^9A_3^6 + 72A_1A_2^5A_5^5A_4^5A_3^2 - 12A_1^2A_3A_2^4A_4^7A_5^4.$$

There is a much better description of the polynomial P_{18}. It is connected with the famous Burkhardt quartic polynomial

$$g_4 = A_1^4 + 8A_1A_2^3 + 8A_1A_3^3 + 8A_1A_4^3 + 8A_1A_5^3 + 48A_2A_3A_4A_5 \tag{6}$$

defining the Burkhardt quartic $\mathcal{B}$. We recall that this variety has 45 double points, and it is rational and birational to $X(\Gamma_2[3])$; for these and other data we refer to [Ba], [Hu], and [Ge].

The polynomial P_{18} is essentially the dual of the quartic. Recall that the dual Q of an irreducible homogeneous polynomial $P(X_1, \dots, X_n)$ is defined by

$$Q(\operatorname{grad} P) \equiv 0 \mod P.$$

Geometrically it describes the set of all tangent hyperplanes. Up to some exceptional cases, the dual of a polynomial is unique up to a constant factor, and the dual of the dual is the polynomial itself. The dual of g_4 is also an invariant polynomial, but invariant under the dual representation, i.e., invariant under the group of all $(g')^{-1}$, $g \in G$. It is a real polynomial. Therefore it is also invariant under all $\bar{g}'^{-1}$. But as we mentioned, this group is conjugate to G, where the conjugation map is given by doubling the first variable A_1.

Proposition 4. *If one takes the dual polynomial of Burkhardt's quartic polynomial g_4 and replaces A_1 by $2A_1$ one obtains* (*up to a constant factor*) *the polynomial P_{18}.*

Proof. The proof is done by straightforward calculation. $\square$

We set

$$\mathcal{A} := \mathbb{C}[A_1, \dots, A_5]/(P_{18}) \tag{7}$$

and we shall denote the projective variety defined by the vanishing of P_{18} with $\mathcal{B}^\vee$ and call it the Burkhardt dual,

$$\mathcal{B}^\vee := \operatorname{Proj}(\mathcal{A}). \tag{8}$$

Hunt [Hu] predicted, using general results of invariant theory, that the degree of the dual of the Burkhardt quartic is 18. He also made some comments about the singular locus of the dual. Actually this singular locus can be determined completely.

5. The singular locus of the Burkhardt dual

We denote by Δ the diagonal in $\mathbf{H}_2$, i.e.,

$$\Delta := \left\{ Z \in \mathbf{H}_2; \quad Z = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} \right\}.$$

For a subgroup $\Gamma \subset \Gamma_2$ of finite index denote the natural projection by

$$\pi_\Gamma : \mathbf{H}_2 \longrightarrow \mathbf{H}_2/\Gamma.$$

The image $\pi_\Gamma(\Delta)$ is an irreducible subvariety of $\mathbf{H}_2/\Gamma$. We use also the notations

$$\Delta[3] := \pi_{\Gamma_2[3]}(\Delta) \quad \text{and} \quad \Delta_0[3] := \pi_{\Gamma_{2,0}[3]}(\Delta).$$

We are interested in

$$G\Delta[3] = \bigcup_{g\in G} g(\Delta[3])$$

and in its closure $G\overline{\Delta[3]}$.

We have to determine the normalizer of $\Delta[3]$ in G. This normalizer $N_G(\Delta[3])$ contains the direct product of two copies of $\mathrm{SL}(2\mathbb{Z}/3\mathbb{Z}/\{\pm E\}$. The whole normalizer is an extension of index two of the direct product of the two copies. This shows that

$$[G : N_G(\Delta[3])] = 45.$$

Therefore $G\Delta[3] \subset X(\Gamma_2[3])$ has 45 irreducible components. They are therefore disjoint.

We denote by H the image of $\Gamma_{1,1}[3]$ in G. The subgroups $N_G(\Delta[3]), B, H$ are maximal in G; their orders are respectively 576, 648, 648 and they have been extensively studied by Burkhardt, see [Bu].

We recall that $\overline{\Delta[3]}$ contains $[N_G(\Delta[3]) : (N_G(\Delta[3]) \cap H)] = 8$ boundary components of the type $\mathcal{C}_i$, and vice versa each $\mathcal{C}_i$ is contained in $[H : (N_G(\Delta[3]) \cap H)] = 9$ irreducible boundary components. Similarly, $\overline{\Delta[3]}$ contains $[N_G(\Delta[3]) : (N_G(\Delta[3]) \cap B)] = 16$, and vice versa each cusp P_i is contained in $B : (N_G(\Delta[3]) \cap B)] = 18$ irreducible boundary components.

We know that the intersection of two closed components of the diagonal locus consists either of two of the 45 $\mathbb{P}^1$ or of the four intersection points of two boundary components. (There are no intersection points away from the boundary.)

Proposition 5. *The singular locus of the Burkhardt dual has* 45 *irreducible components. Each of them has codimension one. The Burkhardt group permutes the components transitively. One of the components is defined* (*as set*) *by the zeros of the ideal*

$$\mathfrak{p} := (A_5 - A_4, A_1A_4 - A_2A_3) \subset \mathcal{A}. \tag{9}$$

This component equals the closure of the image of the diagonal under the natural maps

$$\mathbf{H}_2 \longrightarrow X(\Gamma_2[3]) \longrightarrow \mathrm{Proj}(\mathcal{A}).$$

The whole singular locus can be described also (*as set*) *as the zero locus of the principal ideal*

$$(P_{10}) \subset \mathcal{A}, \quad \textit{where } P_{10} := 13(g_4g_6 - g_{10})/777600.$$

The normalizing constant has been chosen such that the Fourier coefficient of the corresponding modular form with respect to $\begin{bmatrix} 21 \\ 12 \end{bmatrix}$ *is* 1.

Proof. By a classical result, cf. [Ig1] or [Fr1], the modular form χ_{10}, corresponding to P_{10}, vanishes at the 45 diagonal components and has no other zeros. Under the normalization map

$$\phi : X(\Gamma_2[3]) \longrightarrow \mathrm{Proj}(\mathcal{A}) \tag{10}$$

the 45 diagonal components map to 45 different irreducible subvarieties in the variety $\mathrm{Proj}(\mathcal{A})$. This follows from the maximality of B in the Burkhardt group. (The only alternative that all 45 collapse to one can be excluded easily).

A direct computation shows that P_{10} and P_{18} are contained in the ideal $\mathfrak{p}$ and, consequently, also in the 45 conjugate ideals. They all are prime ideals. Hence they define in $\mathcal{A}$ the minimal prime ideals containing P_{10}. Another direct computation shows that the squares of the partial derivatives of P_{18} are contained in the ideal $(P_{18}\,, P_{10})$ and finally that P_{10}^3 is contained in the ideal generated by P_{18} and the first of the partial derivatives. This proves the proposition. $\square$

The proof showed that the first partial derivative of P_{18} has the same zero sets (but possibly with different multiplicities) as P_{10} (considered in $\mathcal{A}$). We will use this later.

So the map ϕ is therefore be biholomorphic outside the locus in the variety $\mathrm{Proj}(\mathcal{A})$ defined by $P_{10} = 0$. We want to study in detail the map along this locus.

The maximality of the subgroups $N_G(\Delta[3]), B, H$ and the G-equivariance of the map ϕ imply that the singular locus of the Burkhardt dual not only has 45 irreducible components, but it also has 40 one-dimensional subvarieties corresponding to the boundary components $\mathcal{C}_i$ and 40 points corresponding to the cusps P_i (also in these cases, the only alternatives would be that all components collapse to one, but this can be excluded easily).

In a first step we claim that the restriction of ϕ to one of the 45 irreducible components is injective. For example, we restrict to $\overline{\Delta[3]}$, the closure of the image of the diagonal in the Satake compactification.

For an accurate description of the map ϕ, we need to recall some facts about modular forms of genus 1. We introduce 2 characteristics:

$$v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and we define the theta series of genus 1 to be

$$\theta_i(\tau) := \sum_{G\,\text{integral}} \mathbf{e}\left(\mathrm{tr}(S[(G+v_i)/3]\tau)\right).$$

We recall that θ_1 and θ_2 generate the ring of modular forms $A(\Gamma_1[3])$. This is a consequence of the dimension formula given, for example, in [Mi, p. 60]. If we denote by $\widetilde{\phi}$ the restriction of ϕ to $\overline{\Delta[3]}$, we get that

$$\widetilde{\phi} : \overline{\Delta[3]} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^4, \tag{11}$$

$$(\tau_1,\, \tau_2) \mapsto (\theta_1(\tau_1)\theta_1(\tau_2),\, \theta_1(\tau_1)\theta_2(\tau_2),\, \theta_2(\tau_1)\theta_1(\tau_2),\, \theta_2(\tau_1)\theta_2)(\tau_2),\, \theta_2(\tau_1)\theta_2)(\tau_2))$$

is bijective.

We have proved now that $\overline{\Delta[3]}$ maps bijectively to the zero locus of $\mathfrak{p}$ in the Burkhardt dual. To get the bijectivity we need some information about their intersection behavior.

Let us consider the standard 1-dimensional boundary component $\mathcal{C}$, which contains

$$\lim_{y\to\infty}\begin{bmatrix}\tau\, 0\\ 0\ iy\end{bmatrix};$$

we have the equation of its image is $A_3 = A_4 = A_5 = 0$.

Moreover, the equation of the image of the so-called cusp

$$\infty := \lim_{y\to\infty}\begin{bmatrix}iy\, 0\\ 0\ iy\end{bmatrix}$$

is $A_2 = A_3 = A_4 = A_5 = 0$.

We have to prove that the 45 components of the singular locus of the Burkhardt dual have the same intersection behavior as the corresponding components in the Satake compactification. It is easy to compute the 45 ideals conjugate to $\mathfrak{p}$ and to verify the intersection behavior since the images of the boundary components in the Burkhardt dual are given by linear equations. We will not reproduce this calculation here. We only recall that the polynomial P_{18} is already contained in $(A_4 - A_5, A_1A_4 - A_2A_3) \subset \mathbb{C}[A_1, \dots, A_5]$. Hence the calculations have to be performed in the polynomial ring $\mathbb{C}[A_1, \dots, A_5]$ (and not in the factor ring $\mathcal{A}$).

Theorem 1. *The normalization map $\phi : X(\Gamma_2[3]) \longrightarrow \mathrm{Proj}(\mathcal{A})$ is bijective.*

6. A basic modular form of weight 3

We have to investigate the action of B in the Burkhardt group G on the 45 diagonal components. From the results of the previous section, it does not make a difference whether we consider it in $\mathrm{Proj}(\mathcal{A})$ or in the normalization.

Proposition 6. *The set of the 45 components of $G\Delta[3]$ decomposes into two orbits under B. The orbit $B\Delta[3]$ contains 18 components. Consequently the other orbit contains 27 components.*

Proof. This is an immediate consequence of the following well-known facts:

$$G = B \cdot N_G(\Delta[3] \cup B \cdot I \cdot N_G(\Delta[3], \quad \text{and} \quad [B : N_G(\Delta[3]) \cap B] = 18.$$

Thus $B\Delta[3]$ contains 18 components and the other orbit contains 27 components. □

We now consider the polynomial of degree 17:

$$\frac{\partial P_{18}}{\partial A_1} \in \mathbb{C}[A_1, \dots, A_5].$$

This is a B-invariant polynomial. As we mentioned after the proof of Proposition 5 this polynomial has the same zero set as P_{10}. This leads us to consider

$$F := \frac{P_{10}^2}{\partial P_{18}/\partial A_1}$$

(in the field of fractions of $\mathcal{A}$). What we found is an element of the normalization!

Theorem 2. *One has the following integral equations in $\mathcal{A}$:*

$$F^2 = P_6, \quad F^3 = P_9$$

with the following polynomials:

$$\begin{aligned}
P_6(A_1,\dots,A_5) := 2^{-4}3^{-18}\big(&A_1^6-2A_1^3A_2^3+A_2^6-2A_1^3A_3^3-A_2^3A_3^3+A_3^6-2A_1^3A_4^3-A_2^3A_4^3\\
&- A_3^3A_4^3 + A_4^6 + 9A_1^2A_2A_3A_4A_5 - 2A_1^3A_5^3 - A_2^3A_5^3 - A_3^3A_5^3 - A_4^3A_5^3 + A_5^6\big),
\end{aligned}$$

$$\begin{aligned}
P_9(A_1,\dots,A_5) := -2^{-7}3^{-27}\big(&2A_1^9 - 6A_1^6A_2^3 + 6A_1^3A_2^6 - 2A_2^9 - 6A_1^6A_3^3 + 3A_1^3A_2^3A_3^3\\
&+ 3A_2^6A_3^3 + 6A_1^3A_3^6 + 3A_2^3A_3^6 - 2A_3^9 - 6A_1^6A_4^3 + 3A_1^3A_2^3A_4^3 + 3A_2^6A_4^3 + 3A_1^3A_3^3A_4^3\\
&- 12A_2^3A_3^3A_4^3 + 3A_3^6A_4^3 + 6A_1^3A_4^6 + 3A_2^3A_4^6 + 3A_3^3A_4^6 - 2A_4^9 + 27A_1^5A_2A_3A_4A_5\\
&- 27A_1^2A_2^4A_3A_4A_5 - 27A_1^2A_2A_3^4A_4A_5 - 27A_1^2A_2A_3A_4^4A_5 + 81A_1A_2^2A_3^2A_4^2A_5^2\\
&- 6A_1^6A_5^3 + 3A_1^3A_2^3A_5^3 + 3A_2^6A_5^3 + 3A_1^3A_3^3A_5^3 - 12A_2^3A_3^3A_5^3 + 3A_3^6A_5^3\\
&+ 3A_1^3A_4^3A_5^3 - 12A_2^3A_4^3A_5^3 - 12A_3^3A_4^3A_5^3 + 3A_4^6A_5^3 - 27A_1^2A_2A_3A_4A_5^4 + 6A_1^3A_5^6\\
&+ 3A_2^3A_5^6 + 3A_3^3A_5^6 + 3A_4^3A_5^6 - 2A_5^9\big).
\end{aligned}$$

There is another representation of F with the G-invariant denominator P_{10}, namely $F = P_{13}/P_{10}$, where

$$\begin{aligned}
P_{13}(A_1,\dots,A_5) := 2^{-3}3^{-13}\big(&A_1^7A_2^3A_3^3 - 2A_1^4A_2^6A_3^3 + A_1A_2^9A_3^3 - 2A_1^4A_2^3A_3^6\\
&- 4A_1A_2^6A_3^6 + A_1A_2^3A_3^9 + A_1^7A_2^3A_4^3 - 2A_1^4A_2^6A_4^3 + A_1A_2^9A_4^3 + A_1^7A_3^3A_4^3\\
&- 15A_1^4A_2^3A_3^3A_4^3 + 2A_1A_2^6A_3^3A_4^3 - 2A_1^4A_3^6A_4^3 + 2A_1A_2^3A_3^6A_4^3 + A_1A_3^9A_4^3\\
&- 2A_1^4A_2^3A_4^6 - 4A_1A_2^6A_4^6 - 2A_1^4A_3^3A_4^6 + 2A_1A_2^3A_3^3A_4^6 - 4A_1A_3^6A_4^6 + A_1A_2^3A_4^9\\
&+ A_1A_3^3A_4^9 - A_1^9A_2A_3A_4A_5 + 3A_1^3A_2^7A_3A_4A_5 - 2A_2^{10}A_3A_4A_5 + 33A_1^3A_2^4A_3^4A_4A_5\\
&+ 3A_2^7A_3^4A_4A_5 + 3A_1^3A_2A_3^7A_4A_5 + 3A_2^4A_3^7A_4A_5 - 2A_2A_3^{10}A_4A_5\\
&+ 33A_1^3A_2^4A_3A_4^4A_5 + 3A_2^7A_3A_4^4A_5 + 33A_1^3A_2A_3^4A_4^4A_5 - 12A_2^4A_3^4A_4^4A_5\\
&+ 3A_2A_3^7A_4^4A_5 + 3A_1^3A_2A_3A_4^7A_5 + 3A_2^4A_3A_4^7A_5 + 3A_2A_3^4A_4^7A_5\\
&- 2A_2A_3A_4^{10}A_5 - 54A_1^2A_2^5A_3^2A_4^2A_5^2 - 54A_1^2A_2^2A_3^5A_4^2A_5^2 - 54A_1^2A_2^2A_3^2A_4^5A_5^2\\
&+ A_1^7A_2^3A_5^3 - 2A_1^4A_2^6A_5^3 + A_1A_2^9A_5^3 + A_1^7A_3^3A_5^3 - 15A_1^4A_2^3A_3^3A_5^3 + 2A_1A_2^6A_3^3A_5^3\\
&- 2A_1^4A_3^6A_5^3 + 2A_1A_2^3A_3^6A_5^3 + A_1A_3^9A_5^3 + A_1^7A_4^3A_5^3\\
&- 15A_1^4A_2^3A_4^3A_5^3 + 2A_1A_2^6A_4^3A_5^3 - 15A_1^4A_3^3A_4^3A_5^3 + 93A_1A_2^3A_3^3A_4^3A_5^3\\
&+ 2A_1A_3^6A_4^3A_5^3 - 2A_1^4A_4^6A_5^3 + 2A_1A_2^3A_4^6A_5^3 + 2A_1A_3^3A_4^6A_5^3 + A_1A_4^9A_5^3\\
&+ 33A_1^3A_2^4A_3A_4A_5^4 + 3A_2^7A_3A_4A_5^4 + 33A_1^3A_2A_3^4A_4A_5^4 - 12A_2^4A_3^4A_4A_5^4\\
&+ 3A_2A_3^7A_4A_5^4 + 33A_1^3A_2A_3A_4^4A_5^4 - 12A_2^4A_3A_4^4A_5^4 - 12A_2A_3^4A_4^4A_5^4\\
&+ 3A_2A_3A_4^7A_5^4 - 54A_1^2A_2^2A_3^2A_4^2A_5^5 - 2A_1^4A_2^3A_5^6 - 4A_1A_2^6A_5^6 - 2A_1^4A_3^3A_5^6\\
&+ 2A_1A_2^3A_3^3A_5^6 - 4A_1A_3^6A_5^6 - 2A_1^4A_4^3A_5^6 + 2A_1A_2^3A_4^3A_5^6 + 2A_1A_3^3A_4^3A_5^6\\
&- 4A_1A_4^6A_5^6 + 3A_1^3A_2A_3A_4A_5^7 + 3A_2^4A_3A_4A_5^7 + 3A_2A_3^4A_4A_5^7 + 3A_2A_3A_4^4A_5^7\\
&+ A_1A_2^3A_5^9 + A_1A_3^3A_5^9 + A_1A_4^3A_5^9 - 2A_2A_3A_4A_5^{10}\big).
\end{aligned}$$

An easy computation tells us the following.

Proposition 7. *The zero locus of F consists of the B-orbit of 27 diagonal components. The multiplicities — counted in $\mathbf{H}_2$ — are one.*

Proof. Since we have only two possibilities, it is enough to check that the zero locus of F does not contain the point $(1, 0, \dots, 0)$. This is easily verified taking the representation $F = P_9/P_6$. □

We obtain more elements of the normalization when we apply the Burkhardt group to F (and we will show that they generate the normalization). Because F is invariant under B, it is one of 40 conjugate elements. It can be shown that they are linearly independent. Computations of this type can be easily managed using of a list of special points of the Burkhardt dual. Such a list can be constructed using the classical unirationalization of the Burkhardt quartic (zero locus of g_4), see [Co],

$$\begin{aligned} y_1 &= 3x_1x_2x_3x_4 \\ y_2 &= x_1(x_2^3+x_3^3-x_4^3) \\ y_3 &= -x_2(x_1^3+x_3^3+x_4^3) \\ y_4 &= x_3(-x_1^3+x_2^3+x_4^3) \\ y_5 &= x_4(x_1^3+x_2^3-x_3^3). \end{aligned} \tag{12}$$

The partial derivatives of g_4 evaluated at points $(2y_1, y_2, y_3, y_4, y_5)$ define points of the Burkhardt dual (zero locus of P_{18}). It is not difficult to compute the values of the 40 conjugate forms at such special points. It is possible to find 40 different points such that the corresponding 40×40-matrix is non-singular. Hence the conjugates of F span a 40-dimensional space which is invariant under G. But this representation is not irreducible. Its character can be computed, and it turns out that it contains a 5-dimensional subrepresentation. Both 5-dimensional representations of the Burkhardt group contain a B-invariant element. This element defines a modular form in $[\Gamma_{2,0}[3], 3, \chi_S]$. It is known that the space of these forms has dimension 3. Fortunately we have already such three forms, namely $\psi_1 := F(\vartheta_1,\dots,\vartheta_5)$, ϑ_1^3 and $\vartheta_2^3+\vartheta_3^3+\vartheta_4^3+\vartheta_5^3$.

What we have seen is that there exist constants α,β such that the 40 conjugate forms of

$$F+\alpha A_1^3+\beta(A_2^3+A_3^4+A_4^3+A_5^3)$$

span a five-dimensional space. A numerical calculation shows that there is exactly one pair of numbers α,β with this property. In this way we obtain:

Proposition 8. *The element*

$$C_1 := -2^5 3^9 F - 7A_1^3 + 4(A_2^3+A_3^3+A_4^3+A_5^3) \tag{13}$$

is contained in a five-dimensional representation space of G. It is the unique (*up to a constant factor*) *B-invariant element of this space.*

Here are some Fourier coefficients of $C_1(\vartheta_1,\dots,\vartheta_5)$:

T	$\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&-1\\-1&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}4&-1\\-1&2\end{smallmatrix}\right]$
C_1	1	−90	3060	2160	−216	4320

We conclude this section observing that we can express the above modular forms as linear combinations of theta series. In fact we can consider the theta series related to the lattices E_6 and $E_6^*(3)$. Looking at the Fourier coefficients we get

$$\vartheta_{E_6} = \vartheta_1^3 + 2(\vartheta_2^3+\vartheta_3^3+\vartheta_4^3+\vartheta_5^3)$$

and

$$27\vartheta_{E_6^*(3)} = 25\vartheta_1^3 + 2C_1(\vartheta_1,\dots,\vartheta_5) - 10(\vartheta_2^3+\vartheta_3^3+\vartheta_4^3+\vartheta_5^3).$$

7. A five-dimensional space of modular forms of weight 3

We already mentioned that G has two 5-dimensional representations, and they are dual. Hence the five-dimensional space, which contains C_1, could be isomorphic to the space generated by the A-s or isomorphic to its dual. One computes that the first case happens and obtains:

Proposition 9. *The element C_1 is part of a tuple $(C_1, \dots, C_5)$ with the same transformation law as the elements $(A_1, \dots, A_5)$. This defines $C_2, \dots, C_5$.*

We derive an explicit representation for the C_i as linear combinations of transformed F. Using the notation of the above theorem, we introduce 5 elements of the Burkhardt group

$$g_1 = \text{unit matrix}, \quad g_2 = \widetilde{I}, \quad g_3 = \widetilde{I}\widetilde{S}_0, \quad g_4 = \widetilde{I}\widetilde{S}_1, \quad g_5 = \widetilde{I}\widetilde{S}_2.$$

We use

$$C := \begin{bmatrix} C_1 \\ \vdots \\ C_5 \end{bmatrix}, \quad G = \begin{bmatrix} C_1^{g_1} \\ \vdots \\ C_1^{g_5} \end{bmatrix}.$$

Finally we introduce the matrix

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2\omega - 1/2 & 0 & -1 - 1/2\omega & 0 \\ -1/2 & 1/2\omega - 1/2 & -1 - 1/2\omega & 0 & 0 \\ 1/2 + 1/2\omega & 1 + 1/2\omega & 1/2 + \omega & 1/2 + \omega & -1/2 - \omega \\ -1/2\omega & -3/2 - 3/2\omega & -1/2\omega + 1/2 & -1/2\omega + 1/2 & 1/2 + \omega \end{bmatrix}.$$

Remark. One has

$$C = \mathrm{T} \cdot G. \tag{14}$$

8. Relations

We have to treat the relations between $A_1, \dots, C_5$. There must be a B- invariant relation in weight 5 as follows from the dimension formula $\dim[\Gamma_{2,0}[3], 5, \chi_S] = 4$.

This relation can be computed by means of the technique of special points and it can be transformed under the Burkhardt group. It turns out that it generates a five-dimensional G-invariant space of relations.

Here is a basis:

Proposition 10. *The space of relations in weight 5 is generated by*

$$\begin{bmatrix} A_1^2 & 2A_2^2 & 2A_3^2 & 2A_4^2 & 2A_5^2 \\ A_2^2 & 2A_1A_2 & 2A_4A_5 & 2A_3A_5 & 2A_3A_4 \\ A_3^2 & 2A_4A_5 & 2A_1A_3 & 2A_2A_5 & 2A_2A_4 \\ A_4^2 & 2A_3A_5 & 2A_2A_5 & 2A_1A_4 & 2A_2A_3 \\ A_5^2 & 2A_3A_4 & 2A_2A_4 & 2A_2A_3 & 2A_1A_5 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix}$$

$$= \begin{bmatrix} A_1^5 + 120A_1A_2A_3A_4A_5 - 10A_1^2A_2^3 - 10A_1^2A_3^3 - 10A_1^2A_4^3 - 10A_1^2A_5^3 \\ -4A_2^5 - 5A_1^3A_2^2 + 20A_2^2A_3^3 + 20A_2^2A_5^3 + 20A_2^2A_4^3 + 30A_1^2A_3A_4A_5 \\ -4A_3^5 - 5A_1^3A_3^2 + 20A_3^2A_4^3 + 20A_3^2A_5^3 + 20A_2^3A_3^2 + 30A_1^2A_2A_4A_5 \\ -4A_4^5 - 5A_1^3A_4^2 + 20A_3^3A_4^2 + 20A_2^3A_4^2 + 20A_4^2A_5^3 + 30A_1^2A_2A_3A_5 \\ -5A_1^3A_5^2 + 20A_3^3A_5^2 + 20A_2^3A_5^2 + 20A_4^3A_5^2 - 4A_5^5 + 30A_1^2A_2A_3A_4 \end{bmatrix}. \tag{15}$$

It is worthwhile to mention that the determinant of the above matrix up to a constant equals P_{10}. This is another explanation for the formula $F = P_{13}/P_{10}$ from Section 6. We remark that if we express $C_1, \dots, C_5$ as rational function in the A-s, using (13) and $F = P_{13}/P_{10}$, the above equality gives an identity in the polynomial ring $\mathbb{C}[A_1, A_2, A_3, A_4, A_5]$ and not for $\mathcal{A}$ alone.

We describe now relations in weight 6. The relation $F^2 = P_6$ is one. If we apply the Burkhardt group to this relation, we obtain 40 linearly independent relations. The 40-dimensional space generated by these relations splits into a 25-dimensional and an irreducible 15-dimensional G-invariant space of relations. Multiplying the 5 relations in weight 5 with $A_1, \dots, A_5$, we obtain 25 relations. Actually they span the mentioned 25-dimensional space. We mention that this 25-dimensional representation contains the trivial representation and a 24-dimensional irreducible one. We describe the remaining 15-dimensional space:

Proposition 11. *There are 15 relations of the form*

$$C_iC_j = Q_1^{(ij)}C_1 + Q_2^{(ij)}C_2 + Q_3^{(ij)}C_3 + Q_4^{(ij)}C_4 + Q_5^{(ij)}C_5 + Q^{(ij)}, \quad 1 \le i \le j \le 5$$

with polynomials $Q_k^{(ij)}$ of degree three and $Q^{(ij)}$ of degree six. They are given explicitly in the following tables:

$$\begin{aligned}
Q^{(11)} = {} & 32A_5^6 + 5A_1^6 + 32A_4^6 + 32A_2^6 + 32A_3^6 + 8A_1^3A_5^3 + 64A_2^3A_5^3 + 8A_1^3A_2^3 \\
& + 64A_5^3A_4^3 + 8A_1^3A_3^3 + 64A_3^3A_4^3 + 8A_1^3A_4^3 + 64A_2^3A_4^3 + 64A_2^3A_3^3 \\
& + 64A_3^3A_5^3 - 144A_1^2A_2A_4A_5A_3,
\end{aligned}$$

$$\begin{aligned}
Q^{(12)} = {} & 288A_3^2A_4^2A_5^2 - 24A_5^3A_1^2A_2 - 24A_4^3A_1^2A_2 - 3A_1^5A_2 + 48A_1^2A_2^4 - 24A_3^3A_1^2A_2 \\
& + 144A_3A_4A_5A_1A_2^2,
\end{aligned}$$

$$\begin{aligned}
Q^{(13)} = {} & 48A_3^4A_1^2 + 288A_4^2A_5^2A_2^2 - 24A_3A_5^3A_1^2 - 24A_3A_1^2A_2^3 - 24A_3A_4^3A_1^2 - 3A_3A_1^5 \\
& + 144A_3^2A_4A_5A_1A_2,
\end{aligned}$$

$$\begin{aligned}
Q^{(14)} = {} & -24A_4A_5^3A_1^2 - 24A_4A_1^2A_2^3 - 24A_3^3A_4A_1^2 + 288A_3^2A_5^2A_2^2 - 3A_4A_1^5 \\
& + 48A_4^4A_1^2 + 144A_3A_4^2A_5A_1A_2,
\end{aligned}$$

$$\begin{aligned}
Q^{(15)} = {} & 48A_5^4A_1^2 - 3A_5A_1^5 - 24A_4^3A_5A_1^2 - 24A_5A_1^2A_2^3 - 24A_3^3A_5A_1^2 + 288A_3^2A_4^2A_2^2 \\
& + 144A_3A_4A_5^2A_1A_2,
\end{aligned}$$

$$\begin{aligned}
Q^{(22)} = {} & 21A_1^4A_2^2 + 96A_3A_5^4A_4 + 96A_3A_4^4A_5 + 24A_4^3A_1A_2^2 + 24A_5^3A_1A_2^2 + 24A_1A_2^5 \\
& + 48A_3A_4A_5A_1^3 - 48A_3A_4A_5A_2^3 + 96A_3^4A_4A_5 + 24A_3^3A_1A_2^2,
\end{aligned}$$

$$\begin{aligned}
Q^{(23)} = {} & -24A_3A_1A_2^4 - 3A_3A_1^4A_2 + 72A_4^2A_5^2A_1^2 - 24A_3^4A_1A_2 + 120A_3A_4^3A_1A_2 \\
& + 120A_3A_5^3A_1A_2 + 144A_3^2A_4A_5A_2^2,
\end{aligned}$$

$$\begin{aligned}
Q^{(24)} = {} & -24A_4A_1A_2^4 - 3A_4A_1^4A_2 + 72A_3^2A_5^2A_1^2 - 24A_4^4A_1A_2 + 120A_4A_5^3A_1A_2 \\
& + 120A_3^3A_4A_1A_2 + 144A_3A_4^2A_5A_2^2,
\end{aligned}$$

$$
\begin{aligned}
Q^{(25)} =& -24A_5A_1A_2^4 + 72A_3^2A_4^2A_1^2 - 24A_5^4A_1A_2 - 3A_5A_1^4A_2 + 120A_4^3A_5A_1A_2 \\
&+ 120A_3^3A_5A_1A_2 + 144A_3A_4A_5^2A_2^2, \\
Q^{(33)} =& 21A_3^2A_1^4 + 96A_4^4A_5A_2 + 24A_3^2A_4^3A_1 + 96A_4A_5^4A_2 + 96A_4A_5A_2^4 + 24A_3^5A_1 \\
&- 48A_3^3A_4A_5A_2 + 24A_3^2A_5^3A_1 + 24A_3^2A_1A_2^3 + 48A_4A_5A_1^3A_2, \\
Q^{(34)} =& -24A_3^4A_4A_1 - 3A_3A_4A_1^4 + 72A_5^2A_1^2A_2^2 - 24A_3A_4^4A_1 + 120A_3A_4A_1A_2^3 \\
&+ 120A_4A_5^3A_3A_1 + 144A_3^2A_4^2A_5A_2, \\
Q^{(35)} =& -3A_3A_5A_1^4 + 72A_4^2A_1^2A_2^2 - 24A_3A_5^4A_1 + 120A_3A_5A_1A_2^3 + 120A_4^3A_5A_3A_1 \\
&+ 144A_3^2A_4A_5^2A_2 - 24A_3^4A_5A_1, \\
Q^{(44)} =& 21A_4^2A_1^4 + 96A_3A_5^4A_2 + 24A_4^2A_1A_2^3 + 24A_3^3A_4^2A_1 + 96A_3A_5A_2^4 + 24A_4^5A_1 \\
&- 48A_4^3A_5A_3A_2 + 96A_3^4A_5A_2 + 24A_4^2A_5^3A_1 + 48A_3A_5A_1^3A_2, \\
Q^{(45)} =& 72A_3^2A_1^2A_2^2 - 3A_4A_5A_1^4 - 24A_4^4A_5A_1 + 120A_4A_5A_1A_2^3 + 144A_4^2A_5^2A_3A_2 \\
&+ 120A_3^3A_4A_5A_1 - 24A_4A_5^4A_1, \\
Q^{(55)} =& 24A_5^5A_1 + 21A_5^2A_1^4 + 24A_5^2A_1A_2^3 + 96A_3A_4A_2^4 + 24A_4^3A_5^2A_1 \\
&+ 96A_3A_4^4A_2 + 96A_3^4A_4A_2 + 24A_3^3A_5^2A_1 - 48A_4A_5^3A_3A_2 + 48A_3A_4A_1^3A_2.
\end{aligned}
$$

The polynomials $Q_k^{(ij)}$ are given in the same ordering in the following table:

$4A_2^3-4A_1^3+4A_4^3$ $+4A_3^3+4A_5^3$	$12A_1A_2^2$ $-24A_3A_4A_5$	$12A_1A_3^2$ $-24A_2A_4A_5$	$12A_1A_4^2$ $-24A_2A_3A_5$	$12A_1A_5^2$ $-24A_2A_3A_4$
$3A_1^2A_2$	$-4A_3^3-4A_4^3-4A_5^3$ $+A_1^3+8A_2^3$	$-12A_2A_3^2$	$-12A_2A_4^2$	$-12A_2A_5^2$
$3A_1^2A_3$	$-12A_2^2A_3$	$-4A_5^3-4A_2^3-4A_4^3$ $+A_1^3+8A_3^3$	$-12A_3A_4^2$	$-12A_3A_5^2$
$3A_1^2A_4$	$-12A_2^2A_4$	$-12A_3^2A_4$	$-4A_5^3-4A_3^3-4A_2^3$ $+A_1^3+8A_4^3$	$-12A_4A_5^2$
$3A_1^2A_5$	$-12A_2^2A_5$	$-12A_3^2A_5$	$-12A_4^2A_5$	$-4A_2^3-4A_3^3-4A_4^3$ $+A_1^3+8A_5^3$
$-12A_3A_4A_5$ $+6A_1A_2^2$	$6A_1^2A_2$	$-12A_1A_4A_5$	$-12A_1A_3A_5$	$-12A_1A_3A_4$
$-6A_1A_2A_3$	$-3A_1^2A_3$	$-3A_1^2A_2$	$-12A_4A_5^2$	$-12A_4^2A_5$
$-6A_1A_2A_4$	$-3A_1^2A_4$	$-12A_3A_5^2$	$-3A_1^2A_2$	$-12A_3^2A_5$
$-6A_1A_2A_5$	$-3A_1^2A_5$	$-12A_3A_4^2$	$-12A_3^2A_4$	$-3A_1^2A_2$
$6A_1A_3^2$ $-12A_2A_4A_5$	$-12A_1A_4A_5$	$6A_1^2A_3$	$-12A_1A_2A_5$	$-12A_1A_2A_4$
$-6A_1A_3A_4$	$-12A_2A_5^2$	$-3A_1^2A_4$	$-3A_1^2A_3$	$-12A_2^2A_5$
$-6A_1A_3A_5$	$-12A_2A_4^2$	$-3A_1^2A_5$	$-12A_2^2A_4$	$-3A_1^2A_3$
$6A_1A_4^2$ $-12A_2A_3A_5$	$-12A_1A_3A_5$	$-12A_1A_2A_5$	$6A_1^2A_4$	$-12A_1A_2A_3$
$-6A_1A_4A_5$	$-12A_2A_3^2$	$-12A_2^2A_3$	$-3A_1^2A_5$	$-3A_1^2A_4$
$6A_1A_5^2$ $-12A_2A_3A_4$	$-12A_1A_3A_4$	$-12A_1A_2A_4$	$-12A_1A_2A_3$	$6A_1^2A_5$

We shall need a certain relation of weight 8, which is a consequence of the relations of weight 5 and 6. One obtains this relation if one multiplies the first of the 5 relations in weight 5 by C_1 and the replaces all the occurring C_1C_i by linear functions in the C-s (relations of weight 6). The result is

Proposition 12. *The polynomial*

$$
\begin{aligned}
R_6 := {} & 5C_1A_1^5 - 5A_1^8 - 20C_1A_1^2A_2^3 - 20C_1A_1^2A_3^3 - 20C_1A_1^2A_4^3 - 20C_1A_1^2A_5^3 \\
& - 14A_1^3A_2^2C_2 - 14A_1^3A_3^2C_3 - 14A_1^3A_5^2C_5 - 14A_1^3A_4^2C_4 + 32A_1^2A_2^3A_5^3 \\
& + 32A_1^2A_5^3A_4^3 + 32A_1^2A_3^3A_4^3 + 32A_1^2A_2^3A_4^3 + 32A_1^2A_2^3A_3^3 + 32A_1^2A_3^3A_5^3 \\
& + 32A_2^3A_5^2C_5 + 32A_2^3A_3^2C_3 + 32A_2^3A_4^2C_4 + 32A_2^2A_3^3C_2 + 32A_2^2A_4^3C_2 \\
& + 32A_2^2A_5^3C_2 + 32A_3^2A_5^3C_3 + 32A_3^3A_5^2C_5 + 32A_3^2A_4^3C_3 + 32A_3^3A_4^2C_4 \\
& + 32A_4^2A_5^3C_4 + 32A_4^3A_5^2C_5 - 128A_1^2A_5^6 + 120C_1A_1A_2A_3A_4A_5 - 128A_1^2A_4^6 \\
& - 128A_1^2A_2^6 - 128A_1^2A_3^6 - 2A_1^5A_5^3 - 2A_1^5A_2^3 - 2A_1^5A_3^3 - 2A_1^5A_4^3 \\
& - 16A_3^5C_3 - 16A_4^5C_4 - 16A_5^5C_5 + 24A_1^2A_2A_4A_5C_3 + 144A_1^4A_2A_4A_5A_3 \\
& + 24A_1^2A_2A_3A_5C_4 + 24A_1^2A_2A_3A_4C_5 + 24A_1^2A_3A_4A_5C_2 - 2304A_2^2A_3^2A_4^2A_5^2 \\
& - 288A_2^4A_3A_4A_5A_1 - 288A_3^4A_4A_5A_1A_2 - 288A_4^4A_3A_5A_1A_2 - 16A_2^5C_2 \\
& - 288A_5^4A_3A_4A_1A_2
\end{aligned}
$$

is contained in the ideal generated by the relations of weight 5 and 6.

9. The ring of modular forms

In this section we will see that the elements and relations constructed in the previous sections are enough to describe the ring of modular forms; in fact, in several steps we shall prove:

Theorem 3. *The graded algebra $A(\Gamma_2[3])$ is generated by the forms $A_1, \dots, A_5$ of weight one and forms $C_1, \dots, C_5$ of weight three.*

Both span five-dimensional irreducible representations of the Burkhardt group. There are 5 relations in weight 5 and 15 relations in weight 6. These relations generate the ideal of all relations. The Hilbert function is

$$
\begin{aligned}
\sum_{r=0}^{\infty} \dim[\Gamma_2(3), r] t^r &= \frac{(1 + t + t^2 + 6t^3 + 6t^4 + t^5 + t^6 + t^7)}{(1-t)^4} \\
&= 1 + 5t + 15t^2 + 40t^3 + 95t^4 + 196t^5 + 360t^6 + 605t^7 + 949t^8 + 1410t^9 + \dots .
\end{aligned}
$$

Proof. We take $A_1, \dots, C_5$ as 10 independent variables and consider the polynomial ring

$$\mathbb{C}[A_1, \dots, A_5, C_1, \dots, C_5].$$

Let $\mathfrak{I}$ be the ideal generated by the 20 relations. We denote the factor ring by

$$\mathcal{R} := \mathbb{C}[A_1, \dots, A_5, C_1, \dots, C_5]/\mathfrak{I}.$$

The relations of weight 5 are polynomials of degree ≤ 1 in the C-s. We denote them by $R_1, \dots, R_5$. Recall that there is an additional relation R_6 of weight 8.

We also consider the polynomial ring

$$\mathbb{C}[A_1, \dots, A_5]$$

and its free module of rank 6

$$\mathcal{F} := \mathbb{C}[A_1,\dots,A_5] \oplus \mathbb{C}[A_1,\dots,A_5]C_1 + \ldots + \mathbb{C}[A_1,\dots,A_5]C_5 \cong \mathbb{C}[A_1,\dots,A_5]^6.$$

We consider the free submodule

$$\mathcal{G} = \mathbb{C}[A_1,\dots,A_5]\cdot R_1 \oplus \ldots \oplus \mathbb{C}[A_1,\dots,A_5]\cdot R_6 \cong \mathbb{C}[A_1,\dots,A_5]^6 \subset \mathcal{F},$$

where $R_1,\dots,R_6$ are embedded into $\mathcal{F}$ in an obvious way.

Proposition 13. *The sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{R} \longrightarrow 0 \tag{16}$$

is exact.

Proof. The map $\mathcal{F} \to \mathcal{R}$ is surjective because of the relations of weight 6. The module $\mathcal{G}$ is in the kernel of this map. We have to show that it is the full kernel. For this reason it is convenient to equip $\mathcal{F}$ with a ring structure. We can consider $\mathcal{F}$ as the factor ring of $\mathbb{C}[A_1,\dots,C_5]$ by the ideal which is generated by the relations of weight 6. Then the map $\mathcal{F} \to \mathcal{R}$ is a ring homomorphism and the kernel is generated by $\mathcal{G}$ as an ideal. All we have to show is that $\mathcal{G}$ is an ideal (and not only a module over $\mathbb{C}[A_1,\dots,A_5]$). To prove this, one has to consider the 30 products C_iR_j, $i = 1,\dots,5$, $j = 1,\dots,6$. After replacing the occurring C_iC_j by means of the relations of degree 6, one gets elements of $\mathcal{F}$. One has to prove that these are contained in $\mathcal{R}\cdot\mathcal{G}$. This is done by explicit calculations which we will not reproduce here. □

This proposition shows that $\mathcal{R}$ has a free resolution of length one; thus the "Auslander–Buchsbaum formula" implies that it is a Cohen–Macaulay module over the ring $\mathbb{C}[A_1,\dots,A_5]$, cf. [BH][1].

A well-known criterion, cf. [Ei], states: Let $A \to B$ a homomorphism of noetherian rings such that B is a finitely generated A-module and let M be a finitely generated B-module. Then M is a Cohen–Macaulay B-module if and only if it is a Cohen–Macaulay A-module. If especially B is a Cohen–Macaulay A-module, then B is a Cohen–Macaulay ring (i.e., a Cohen-Macaulay B-module). We obtain the following.

Corollary 1. *The ring* $\mathcal{R} = \mathbb{C}[A_1,\dots,A_5,C_1,\dots,C_5]/\mathfrak{I}$ *is a Cohen–Macaulay ring.*

An important consequence of this fact and of Theorem 1 is that the map

$$\operatorname{Proj}(\mathcal{R}) \longrightarrow \operatorname{Proj}(\mathcal{A})$$

is a bijective, and thus we obtain:

The variety $\operatorname{Proj}(\mathcal{R})$ *is irreducible* (*but possibly not reduced*).

This argument shows even more. The localizations of the rings $\mathcal{A}$ and $\mathcal{R}$ by the multiplicative set generated by P_{10} are isomorphic. Especially we obtain:

The variety $\operatorname{Proj}(\mathcal{R})$ *is nonsingular outside the locus of the ideal* $P_{10}\cdot\mathcal{R}$.

We know that the locus $P_{10} = 0$ considered in $\operatorname{Proj}(A(\Gamma_2[3])$ consists of 45 irreducible components. We obtain that the locus of the ideal $P_{10}\cdot\mathcal{R}$ in $\operatorname{Proj}(\mathcal{R})$ also consists of 45 irreducible components. They are transitively permuted under the Burkhardt group.

[1] A finitely generated module M over a noetherian ring R is called a Cohen–Macaulay module if $M_\mathfrak{p}$ is a Cohen–Macaulay module for all prime ideals $\mathfrak{p}$ in R in the sense of [Se].

Lemma 4. *The point* $[0,0,0,1,1,-16,0,0,0,0]$ *is in the zero set of the ideal* $\mathfrak{I}$. *Moreover it is in the locus of* $P_{10}\cdot\mathcal{R}$. *It is a regular point of* $\mathrm{Proj}(\mathcal{R})$.

It is easy to check the equations for the point, and for an individual point it is easy to check that it is regular. We omit the computations.

It follows from the above lemma that the singular locus of $\mathrm{Proj}(\mathcal{R})$ has codimension ≥ 2. The same then is true for $\mathrm{Spec}(\mathcal{R})$. From Serre's normality criterion, [Se], now it follows that $\mathcal{R}$ is a normal ring (and especially an integral domain).

The dimension formula is an easy consequence of the exact sequence $0\to\mathcal{G}\to\mathcal{F}\to\mathcal{R}\to 0$.

We mention that our dimension formula agrees with the formulas obtained by means of the Trace formula or by the Riemann–Roch (for r big enough), cf. [Ch1], [Ch2] (there is a numerical error even in the second corrected form), [Ha], [Ya]. Moreover, independently, Gunji has found the same expression for the Hilbert function, see [Gu], studying in great detail the low dimensional cases. This completes the proof of the theorem. □

The nature of the singular locus can be described in more detail:

Proposition 14. *Let* $\mathcal{A}_{\mathfrak{p}}$ *be the homogeneous localization* (*local ring of the variety* $\mathrm{Proj}(\mathcal{A})$ *at the generic point of* $\mathfrak{p}\subset\mathcal{A}$). *There is an isomorphism of its formal completion*

$$\widehat{\mathcal{A}}_{\mathfrak{p}}\cong K[[X^2,X^3,\dots]].$$

Here K *denotes the field of rational functions of the zero locus of* $\mathfrak{p}$.

Proof. We denote by $\mathfrak{q}$ the prime ideal of the corresponding locus in $\mathrm{Proj}(\mathcal{R})$. We know that $\mathcal{R}$ is regular at the generic point of $\mathfrak{q}$. Hence

$$\widehat{\mathcal{R}}_{\mathfrak{q}}\cong K[[X,X^2,X^3,\dots]]$$

is isomorphic to the ring of formal power series. As variable X we can take a suitable conjugate F' of F. (The latter does not vanish along the diagonal but only along conjugates of the diagonal). We know that F^2 and F^3 are in $\mathcal{A}$. The same is true for F'. Hence $K[[X^2,X^3,\dots]]$ is contained in $\widehat{\mathcal{A}}_{\mathfrak{p}}$. Equality must hold because this ring is not regular. □

We recall that a Cohen–Macaulay graded algebra A is said to be Gorenstein if its canonical module is isomorphic to A. As an immediate consequence of the results of [St] we have:

Corollary 2. *The ring* $A(\Gamma_2[3])$ *is a Gorenstein ring.*

10. Final remark

The central role in our picture plays the dual of the Burkhardt quartic. In the usual algebro-geometric approaches the quartic itself plays a central role. Classically it is known that the modular variety $\mathbf{H}_2/\Gamma_2[3]$ is birational equivalent to the Burkhardt quartic. This is in accordance with our result since a hypersurface and its dual always are birational equivalent.

Nevertheless one may ask whether the classical birational map is visible in our picture. The answer is yes.

There is a modular form X_5 of weight 5 with respect to the full modular group but which picks up the nontrivial character of this group. From the zero locus it is clear that $B_1 := X_5/F$ is a holomorphic modular form of weight two. It can be shown that this form is contained in a five-dimensional G-invariant space of modular forms on $\Gamma_2[3]$, but all with the same nontrivial character. This five-dimensional space defines a Burkhardt quartic and yields the classical birational map between the moduli space and the Burkhardt quartic.

The modular form B_1 can be expressed by means of theta *nullwerte* and also as additive lift in the sense of Borcherds. We intend to come back to this subject in a separate paper.

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