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SIMPLE GRADED RINGS OF SIEGEL MODULAR FORMS, DIFFERENTIAL OPERATORS AND BORCHERDS PRODUCTS

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In this paper, we show that the graded ring of Siegel modular forms of $\Gamma_0(N) \subset Sp(2,\mathbb{Z})$ has a very simple unified structure for $N = 1, 2, 3, 4$, taking Neben-type case (the case with character) for $N = 3$ and 4. All are generated by 5 generators, and all the fifth generators are obtained by using the other four by means of differential operators, and it is also obtained as Borcherds products. As an appendix, examples of Euler factors of L-functions of Siegel modular forms of $Sp(2,\mathbb{Z})$ of odd weight are given.

Keywords: Siegel modular form; differential operator; Borcherds product.

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1. Introduction

In this paper, we give examples of congruence subgroups Γ of $Sp(2,\mathbb{Z})$ such that the graded ring $A(\Gamma)$ of Siegel modular forms of Γ is generated by five modular forms, among which four are algebraically independent and the other is neatly described by using the first four. It is well-known as Igusa's theorem that the ring of Siegel modular forms of degree two belonging to the full modular group $Sp(2,\mathbb{Z})$ are generated by four algebraically independent forms ϕ_4, ϕ_6, χ_{10}, χ_{12} and a modular form χ_{35}, where the suffices are the weights of the forms. It is also well known that χ_{35} is expressed as a Borcherds product (cf. Gritsenko and Nikulin [9]). In Igusa's paper [17], the modular form χ_{35} was given by an average of theta constants over azygous triples and this is rather complicated. Here we give another very simple way to construct χ_{35} from ϕ_4, ϕ_6, χ_{10}, χ_{12} by using a differential operator. Now we explain the result for other groups. For any natural number N and n, we put

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z});\ C \equiv 0 \bmod N \right\}.$$

When $n = 2$, we write $\Gamma_0(N) = \Gamma_0^{(2)}(N)$ for the sake of simplicity. We denote by ψ_3 the character of $\Gamma_0(3)$ defined by $\psi_3(\gamma) = \left(\frac{-3}{\det(D)}\right)$ and by ψ_4 the character of $\Gamma_0(4)$ defined by $\psi_4(\gamma) = \left(\frac{-1}{\det(D)}\right)$. We put $\Gamma_0^{\psi_3}(3) = \{\gamma \in \Gamma_0(3); \psi_3(\gamma) = 1\}$ and $\Gamma_0^{\psi_4}(4) = \{\gamma \in \Gamma_0(4); \psi_4(\gamma) = 1\}$.

Theorem 1.1. *For each $\Gamma = Sp(2, \mathbb{Z})$, $\Gamma_0(2)$, $\Gamma_0^{\psi_3}(3)$, or $\Gamma_0^{\psi_4}(4)$, the graded ring $A(\Gamma)$ of Siegel modular forms is generated by five Siegel modular forms consisting of four algebraically independent forms and another one constructed by the first four by using a differential operator. The fifth generator is also expressed as a Borcherds product.*

As for precise generators and the differential operators, see Sec. 2. We note that the generators of the ring for $Sp(2, \mathbb{Z})$, $\Gamma_0(2)$ and $\Gamma_0^{\psi_4}(4)$ have been already known in [17, 12, 11]. So for these groups, a new point here is that the fifth generator is obtained by a simple differential operator and has a Borcherds product expression. The result on generators for $\Gamma_0^{\psi_3}(3)$ is newly obtained, and we can reprove our former result on $\Gamma_0(3)$ in [12] more easily and satisfactorily by our new result. Of course this kind of theorem is in a sense accidental, and there are some other discrete subgroups for which the same theorem holds even among subgroups containing the principal congruence subgroup $\Gamma(2)$, for example, the unique subgroup $\Gamma_e(1)$ of index two in $Sp(2, \mathbb{Z})$, but we omit further results here.

Since our expression of χ_{35} is so simple, it is now very easy to calculate many Fourier coefficients of χ_{35}. Hence in Appendix A, we give a table of some Fourier coefficients. We also give Euler factors at 2 of spinor L function of Siegel cusp forms of $Sp(2, \mathbb{Z})$ of odd weight which have never appeared in any literature as far as the authors know.

Finally we would like to say a few words on our differential operators. A general theory of holomorphic differential operators on Siegel modular forms which produce new modular forms from given several modular forms was studied in Ibukiyama [13] in full generality. Our differential operator in the above theorem is one of these operators. In one variable case, this kind of operators are called Rankin–Cohen differential operators. Starting from modular forms of one variable of weight k and l, we obtain a new modular form of weight $k + l + 2\nu$ by this operator where ν is a non-negative integer. We can also consider differential operators on more than two modular forms of one variable but we cannot get a new result, namely, this kind of operator is obtained by combination of those on two forms. As for Siegel modular forms, the situation is slightly different. For example, the Rankin–Cohen type differential operators on two Siegel modular forms of weight k and l are defined for degree two in [5] and for general degree in [7], and the weight of constructed Siegel modular form is $k+l+2\nu$ where ν is a non-negative integer. So, for example, if we start from modular forms of even weight, then we get only a modular form of even weight in this case. But as we see in the theorem, if we start from several modular forms more than two, it happens that we get odd weight from even weights. This is a very interesting point and a trick for our construction. The same sort of construction is also valid for vector valued Siegel modular forms which will appear elsewhere (cf. [14–16]).

2. Differential Operators

We review Siegel modular forms to fix notation. For any natural number n, we denote by H_n the Siegel upper half plane of degree n

$$H_n = \{Z \in M_n(\mathbb{C});\ {}^tZ = Z,\ Im(Z) > 0\}.$$

For any commutative ring R, we define the symplectic group over R by

$$Sp(n,R) = \{g \in M_{2n}(R);\ {}^tgJg = J\}$$

where $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$ and 1_n is the unit matrix. For a discrete subgroup $\Gamma \subset Sp(n,\mathbb{R})$ and a character χ of Γ, we say that a holomorphic function $F(Z)$ on H_n is a Siegel modular form of weight k with character χ if

$$F((AZ+B)(CZ+D)^{-1}) = \chi(\gamma)\det(CZ+D)^k F(Z) \quad \text{for any } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$$

and bounded at cusps. We denote by $A_k(\Gamma,\chi)$ the space of all Siegel modular forms of weight k of Γ with character χ. When χ is the identity character, we write $A_k(\Gamma) = A_k(\Gamma,\chi)$ and call $f \in A_k(\Gamma)$ just a Siegel modular forms of weight k. We put $A(\Gamma) = \bigoplus_{k=0}^{\infty} A_k(\Gamma)$. The space $A(\Gamma)$ is a graded ring.

For $Z \in H_n$, we write the (i,j) components of Z by z_{ij}. We put $\langle n\rangle = n(n+1)/2+1$. For $\langle n\rangle$ numbers of Siegel modular forms $f_i \in A_{k_i}(\Gamma)$ of weight k_i $(1 \le i \le \langle n\rangle)$, we define a new function $\{f_1,\dots,f_{\langle n\rangle}\}_{n+1}$ by

$$\{f_1,\dots,f_{\langle n\rangle}\}_{n+1} = \begin{vmatrix} k_1f_1 & k_2f_2 & \cdots & k_{\langle n\rangle-1}f_{\langle n\rangle-1} & k_{\langle n\rangle}f_{\langle n\rangle} \\ \frac{\partial f_1}{\partial z_{11}} & \frac{\partial f_2}{\partial z_{11}} & \cdots & \frac{\partial f_{\langle n\rangle-1}}{\partial z_{11}} & \frac{\partial f_{\langle n\rangle}}{\partial z_{11}} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \frac{\partial f_1}{\partial z_{nn}} & \frac{\partial f_2}{\partial z_{nn}} & \cdots & \frac{\partial f_{\langle n\rangle-1}}{\partial z_{nn}} & \frac{\partial f_{\langle n\rangle}}{\partial z_{nn}} \end{vmatrix}.$$

Proposition 2.1. (1) *The above function* $\{f_1,\dots,f_{\langle n\rangle}\}_{n+1}$ *is a Siegel cusp form of weight* $k_1+\cdots+k_{\langle n\rangle}+n+1$.

(2) *If* $f_1,\dots,f_{\langle n\rangle}$ *are algebraically independent, then* $\{f_1,\dots,f_{\langle n\rangle}\}_{n+1} \neq 0$.

Proof. For $\nu \geq 2$, put $F_\nu = f_\nu^{k_1}/f_1^{k_\nu}$. Then $F_\nu(\gamma Z) = F_\nu(Z)$, namely, these are automorphic functions on H_n invariant by Γ. Define $F(Z)$ by the Jacobian.

$$F(Z) = \frac{\partial(F_2, \ldots, F_{\langle n\rangle})}{\partial(z_{11}, \ldots, z_{nn})}.$$

Then we have

$$\begin{aligned}F(Z) &= \frac{\partial(F_2(\gamma Z), \ldots, F_{\langle n\rangle}(\gamma Z))}{\partial(z_{11}, z_{12}, \ldots, z_{nn})}\\ &= \frac{\partial(F_2(\gamma Z), \ldots, F_{\langle n\rangle}(\gamma Z))}{\partial((\gamma Z)_{11}, (\gamma Z)_{12}, \ldots, (\gamma Z)_{nn})} \times \frac{\partial((\gamma Z)_{11}, \ldots, (\gamma Z)_{nn})}{\partial(z_{11}, z_{12}, \ldots, z_{nn})}\\ &= F(\gamma Z)\det(CZ+D)^{-n-1}.\end{aligned}$$

Hence $F(Z)$ is a meromorphic Siegel modular form of weight $n+1$ of Γ. We also have

$$\begin{aligned}\frac{\partial}{\partial z_{ij}}(f_\nu^{k_1}/f_1^{k_\nu}) &= k_1(f_\nu^{k_1-1}/f_1^{k_\nu})\frac{\partial f_\nu}{\partial z_{ij}} - k_\nu(f_\nu^{k_1}/f_1^{k_\nu+1})\frac{\partial f_1}{\partial z_{ij}}\\ &= \left(\frac{k_1 f_\nu^{k_1-1}}{f_1^{k_\nu}}\right)\left(\frac{\partial f_\nu}{\partial z_{ij}} - \frac{k_\nu f_\nu}{k_1 f_1} \times \frac{\partial f_1}{\partial z_{ij}}\right).\end{aligned}$$

This implies that

$$\{f_1, \ldots, f_{\langle n\rangle}\}_{n+1} = \frac{f_1^{k_2+\cdots+k_{\langle n\rangle+1}}}{k_1^{\langle n\rangle-1}(f_2 f_3 \cdots f_{\langle n\rangle})^{k_1-1}} F(Z).$$

This means that $\{f_1, \ldots, f_{\langle n\rangle}\}_{n+1}$ is a holomorphic Siegel modular form of weight $k_1+\cdots+k_{\langle n\rangle}+n+1$. In particular, if f_ν $(1 \leq \nu \leq \langle n\rangle)$ are algebraically independent, then $f_\nu^{k_1}/f_1^{k_\nu}$ are local parameters of $\langle n\rangle - 1$ dimensional variety $\Gamma\backslash H_n$, so the functional determinant does not vanish identically. Hence we get (2). □

When $n = 2$, for any $Z \in H_2$, we write

$$Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}.$$

For any Siegel modular form f of degree two, we denote by $a(t_1, t_2, t; f) = a(t_1, t_2, t)$ the Fourier coefficient of f at

$$\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}.$$

Now first we apply this to level 1 case. We denote by ϕ_4 or ϕ_6 the unique modular form of $Sp(2, \mathbb{Z})$ of weight 4 or 6 such that $a(0,0,0;\phi_4) = a(0,0,0;\phi_6) = 1$.

We denote by χ_{10} or χ_{12} the unique cusp form of $Sp(2,\mathbb{Z})$ of weight 10 or 12 such that $a(1,1,1;\chi_{10}) = a(1,1,1;\chi_{12}) = 1$. Then the function

$$\chi = \frac{1}{(2\pi i)^3}\{\phi_4,\phi_6,\chi_{10},\chi_{12}\}_3 = \frac{1}{(2\pi i)^3}\begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \frac{\partial\phi_4}{\partial\tau} & \frac{\partial\phi_6}{\partial\tau} & \frac{\partial\chi_{10}}{\partial\tau} & \frac{\partial\chi_{12}}{\partial\tau} \\ \frac{\partial\phi_4}{\partial z} & \frac{\partial\phi_6}{\partial z} & \frac{\partial\chi_{10}}{\partial z} & \frac{\partial\chi_{12}}{\partial z} \\ \frac{\partial\phi_4}{\partial\omega} & \frac{\partial\phi_6}{\partial\omega} & \frac{\partial\chi_{10}}{\partial\omega} & \frac{\partial\chi_{12}}{\partial\omega} \end{vmatrix}$$

is a modular form of weight $4+6+10+12+3 = 35$. This does not vanish, since ϕ_4, ϕ_6, χ_{10}, χ_{12} are algebraically independent. We denote by E_4 or E_6 the Eisenstein series of $SL_2(\mathbb{Z})$ of weight 4 or 6 with constant term 1. Let $\phi_{10,1}$ or $\phi_{12,1}$ be the Jacobi cusp form of weight 10 or 12 of index one given in Eichler–Zagier. The Fourier–Jacobi expansion of χ is given by

$$\begin{aligned}(p^2)&\frac{1}{2\pi i}\left(4E_4\frac{\partial E_6}{\partial\tau} - 6E_6\frac{\partial E_4}{\partial\tau}\right) \times \frac{1}{2\pi i}\left(\frac{\partial\phi_{10,1}}{\partial z}\phi_{12,1} - \phi_{10,1}\frac{\partial\phi_{12,1}}{\partial z}\right) + O(p^3) \\ &= -3456\Delta_{12}\phi_{23,2}(p)^2 + O(p^3),\end{aligned}$$

where $q = \exp(2\pi i\tau), \zeta = \exp(2\pi iz), p = \exp(2\pi i\omega)$ and

$$\begin{aligned}\phi_{23,2} &= 12q(\zeta - \zeta^{-1}) + O(q^2) \\ &= 12(4\pi i)(\Delta(\tau))^2 z + O(z^2)\end{aligned}$$

is a non-zero Jacobi form of weight 23 of index 2. We see that $a(2,3,1;\chi) = 2^9\cdot 3^4$. We can see $\chi \neq 0$ also by this calculation. We put $\chi_{35} = \chi/(2^9\cdot 3^4)$. Then χ_{35} is the unique Siegel modular form of weight 35 of degree 2 of $Sp(2,\mathbb{Z})$ such that $a(2,3,1;\chi_{35}) = 1$. Denote by $\Gamma_e(1)$ is the unique index two subgroup of $Sp(2,\mathbb{Z})$. We have a cusp form χ_5 of $\Gamma_e(1)$ of weight 5 such that $\chi_5^2 = \chi_{10}$ [17]. Then $\{\phi_4,\phi_6,\chi_5,\chi_{12}\}_3$ is the modular form of weight 30. We denote this by χ_{30}. Then χ_{35} is equal to $\chi_5\chi_{30}$ up to constant. It is known that $A(Sp(2,\mathbb{Z})) = \mathbb{C}[\phi_4,\phi_6,\chi_{10},\chi_{12},\chi_{35}]$ and $A(\Gamma_e(1)) = \mathbb{C}[\phi_4,\phi_6,\chi_5,\chi_{12},\chi_{30}]$.

We note that the Eisenstein series ϕ_4, ϕ_6 are the same function as those in Igusa [18, p. 848]. But our normalization of χ_{10}, χ_{12}, χ_{35} are different from Igusa's notation. In fact, Igusa's function χ_{10}, χ_{12} or χ_{35} in [18, p. 848] is $-1/4$, $1/12$ or $i/4$ times of out χ_{10}, χ_{12} or χ_{35}.

Remark. χ_{35} is an odd function with respect to z. In the above $\frac{\partial F}{\partial z}$ for any Siegel modular forms F of even weights are odd function with respect to z and all the other derivatives are even. Hence this fits the fact. The Fourier coefficients $a(t_1,t_2,t)$ at $\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2\end{pmatrix}$ satisfies $a(t_1,t_2,-t) = -a(t_1,t_2,t)$.

3. Siegel Modular Forms of Odd Weights of Level 2

The ring of modular forms of $\Gamma_0(2)$ was determined in Ibukiyama [12], and all the generators are given by theta constants. First we review this theorem here, and then we show that in this case also the Siegel modular form of weight 19, which is the fundamental generator of odd weights forms, is obtained by differentiating even weights forms.

As usual, we define theta constants of characteristic $m = {}^t({}^tm', {}^tm'') \in \mathbb{Z}^4$ ($m', m'' \in \mathbb{Z}^2$) by

$$\theta_m(Z) = \sum_{p\in\mathbb{Z}^2} e\left(\frac{1}{2}{}^t\left(p+\frac{m'}{2}\right)Z\left(p+\frac{m'}{2}\right)+{}^t\left(p+\frac{m'}{2}\right)\frac{m''}{2}\right),$$

where we put $e(x) = e^{2\pi ix}$ and $Z \in H_2$. We put

$$\begin{aligned}
X &= (\theta_{0000}^4+\theta_{0001}^4+\theta_{0010}^4+\theta_{0011}^4)/4,\\
Y &= (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2,\\
Z &= (\theta_{0100}^4-\theta_{0110}^4)^2/16384,\\
K &= (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096.
\end{aligned}$$

Each X, Y, Z, or K is a modular form of $\Gamma_0(2)$ of weight 2, 4, 4, 6 respectively, and these four forms are algebraically independent. There also exists a cusp form χ_{19} of weight 19 belonging to $\Gamma_0(2)$, given explicitly by theta constants as follows.

$$\begin{aligned}
\theta &= \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111},\\
\theta' &= (\theta_{1000}^{12}-\theta_{1001}^{12}-\theta_{1100}^{12}+\theta_{1111}^{12})/1536,\\
T &= (\theta_{0100}\theta_{0110})^4/256,\\
\chi_{19} &= \theta\theta'(8YZ-X^2T+YT+1024ZT+96T^2-8XK)/32.
\end{aligned}$$

Here $a(2,3,1;\chi_{19}) = -1$.

We know that

Proposition 3.1 [12, p. 34].

$$A(\Gamma_0(2)) = \mathbb{C}[X,Y,Z,K] \oplus \chi_{19}\mathbb{C}[X,Y,Z,K].$$

where $\oplus$ means a direct sum as modules.

The only thing we would like to claim here is the following proposition.

Proposition 3.2.

$$\chi_{19} = \frac{1}{512(2\pi i)^3} \times \begin{vmatrix} 2X & 4Y & 4Z & 6K \\ \frac{\partial X}{\partial\tau} & \frac{\partial Y}{\partial\tau} & \frac{\partial Z}{\partial\tau} & \frac{\partial K}{\partial\tau} \\ \frac{\partial X}{\partial z} & \frac{\partial Y}{\partial z} & \frac{\partial Z}{\partial z} & \frac{\partial K}{\partial z} \\ \frac{\partial X}{\partial\omega} & \frac{\partial Y}{\partial\omega} & \frac{\partial Z}{\partial\omega} & \frac{\partial K}{\partial\omega} \end{vmatrix}.$$

The constant is obtained by calculation of the Fourier coefficient of this determinant at $(2,3,1)$. We see that χ_{19} divides χ_{35}. This is obvious since $\Gamma_0(2) \subset Sp(2,\mathbb{Z})$ but also clear from the relation

$$\{\phi_4,\phi_6,\chi_{10},\chi_{12}\}_3 = \{X,Y,Z,K\}_3 \times \frac{\partial(\phi_4,\phi_6,\chi_{10},\chi_{12})}{\partial(X,Y,Z,K)},$$

where we put

$$\frac{\partial(\phi_4,\phi_6,\chi_{10},\chi_{12})}{\partial(X,Y,Z,K)} = \begin{pmatrix} \frac{\partial\phi_4}{\partial X} & \frac{\partial\phi_6}{\partial X} & \frac{\partial\chi_{10}}{\partial X} & \frac{\partial\chi_{12}}{\partial X} \\ \frac{\partial\phi_4}{\partial Y} & \frac{\partial\phi_6}{\partial Y} & \frac{\partial\chi_{10}}{\partial Y} & \frac{\partial\chi_{12}}{\partial Y} \\ \frac{\partial\phi_4}{\partial Z} & \frac{\partial\phi_6}{\partial Z} & \frac{\partial\chi_{10}}{\partial Z} & \frac{\partial\chi_{12}}{\partial Z} \\ \frac{\partial\phi_4}{\partial K} & \frac{\partial\phi_6}{\partial K} & \frac{\partial\chi_{10}}{\partial K} & \frac{\partial\chi_{12}}{\partial K} \end{pmatrix}.$$

The last functional determinant is calculated explicitly if we use the relations

$$\begin{aligned} \phi_4 &= 4\,X^2 - 3\,Y + 12288\,Z, \\ \phi_6 &= -8\,X^3 + 9\,XY + 73728\,XZ - 27648\,K, \\ \chi_{10} &= YK, \\ \chi_{12} &= 3\,Y^2Z - 2\,XYK + 3072\,K^2. \end{aligned}$$

The result is

$$\begin{aligned} &81\big(-67108864X^2K^2 + 65536X^2Y^2Z - 16384XKY^2 \\ &\quad - Y^4 - 16384Y^3Z + 68719476736ZK^2 - 67108864Y^2Z^2\big). \end{aligned}$$

4. Siegel Modular Forms of Level 3

We put

$$\psi_3(\gamma) = \left(\frac{-3}{\det D}\right) \quad \text{for } \gamma = \begin{pmatrix} A & B \\ 3C & D \end{pmatrix} \in \Gamma_0(3).$$

We put $\Gamma_0^{\psi_3}(3) = \{\gamma \in \Gamma_0(3); \psi_3(\gamma) = 1\}$. Then the group $\Gamma_0^{\psi_3}(3)$ is index 2 in $\Gamma_0(3)$. We have already given the structure of $A^{(\text{even})} = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma_0(3))$ in [12], but in this section, we describe $\bigoplus_{k=0}^{\infty} A_k(\Gamma_0^{\psi_3}(3))$ explicitly and reprove the result for $A^{(\text{even})}$ in [12] fairly easily.

For any even integral symmetric matrix S of size n and $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in H_2$, we define a theta function by

$$\theta_S(Z) = \sum_{x,y \in \mathbb{Z}^n} e^{\pi i(({}^t xSx)\tau + 2({}^t xSy)z + ({}^t ySy)\omega)}.$$

We take the following three even symmetric matrices

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$E_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

and

$$E_6^* = 3E_6^{-1} = \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}.$$

We have $\det(A_2) = 3$, $\det(E_6) = 3$ and $\det(E_6^*) = 243 = 3^5$ and these have level 3. By these conditions, we see $\theta_{A_2}(Z) \in A_1(\Gamma_0(3), \psi_3)$ and $\theta_{E_6}(Z)$, $\theta_{E_6^*}(Z) \in A_3(\Gamma_0(3), \psi_3)$ (cf. [2]). We define another theta function with spherical function. We put

$$S_4 = \begin{pmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 3/2 \\ 3/2 & 0 & 3 & 0 \\ 0 & 3/2 & 0 & 3 \end{pmatrix}$$

and for $x = (x_i)$, $y = (y_i) \in \mathbb{R}^4$, we put

$$Q(x,y) = \begin{pmatrix} {}^t x S_4 x & {}^t x S_4 y \\ {}^t y S_4 x & {}^t y S_4 y \end{pmatrix}.$$

As in [12], we put

$$\theta_4(Z) = \sum_{x,y \in \mathbb{Z}^4} (c^2 - d^2) \exp\left(2\pi i tr(Q(x,y)Z)\right)$$

where $c = (x_1y_3 - x_3y_1) + (x_2y_4 - y_2x_4)$, $d = (x_1y_4 - y_1x_4) + (x_3y_2 - y_3x_2) + (x_1y_2 - y_1x_2)$. We know that $\theta_4 \in S_4(\Gamma_0(3))$ (cf. [12]).

To make symbols consistent with those in [12], we put

$$\begin{aligned} \alpha_1 &= \theta_{A_2}, \\ \beta_3 &= \theta_{E_6} - 10\theta_{A_2}^3 + 9\theta_{E_6^*}, \\ \delta_3 &= \theta_{E_6} - 9\theta_{E_6^*}, \\ \gamma_4 &= \theta_4. \end{aligned}$$

We put

$$\chi_{14} = \frac{1}{2^9 \cdot 3^{10}} \times \frac{1}{(2\pi i)^3} \begin{vmatrix} \alpha_1 & 3\beta_3 & 4\gamma_4 & 3\delta_4 \\ \frac{\partial \alpha_1}{\partial \tau} & \frac{\partial \beta_3}{\partial \tau} & \frac{\partial \gamma_4}{\partial \tau} & \frac{\partial \delta_4}{\partial \tau} \\ \frac{\partial \alpha_1}{\partial z} & \frac{\partial \beta_3}{\partial z} & \frac{\partial \gamma_4}{\partial z} & \frac{\partial \delta_4}{\partial z} \\ \frac{\partial \alpha_1}{\partial \omega} & \frac{\partial \beta_3}{\partial \omega} & \frac{\partial \gamma_4}{\partial \omega} & \frac{\partial \delta_4}{\partial \omega} \end{vmatrix}.$$

Then $\chi_{14} \in A_{14}(\Gamma_0(3), \psi_3)$ with $a(2,3,1;\chi_{14}) = 1$.

We put

$$B = \mathbb{C}[\alpha_1, \beta_3, \gamma_4, \delta_3],$$
$$C = \mathbb{C}[\alpha_1^2, \beta_3^2, \delta_3^2, \gamma_4],$$

and denote by $B^{(\text{odd})}$ or $B^{(\text{even})}$ the submodule of the graded ring B consisting of odd degree or even degree elements, respectively. Namely, we have

$$B^{(\text{even})} = \mathbb{C}[\alpha_1^2, \alpha_1\beta_3, \alpha_1\delta_3, \gamma_4, \beta_3^2, \delta_3^2, \beta_3\delta_3].$$

Theorem 4.1. *The four modular forms α_1, β_3, γ_4, δ_3 are algebraically independent. We have*

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_0^{\psi_3}(3)) = B \oplus B\chi_{14},$$
$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3)) = B^{(\text{even})} \oplus C\alpha_1\chi_{14} \oplus C\beta_3\chi_{14} \oplus C\delta_3\chi_{14} \oplus C\alpha_1\beta_3\delta_3\chi_{14},$$
$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_0(3), \psi_3) = B^{(\text{odd})} \oplus B^{(\text{even})}\chi_{14}.$$

Proof. For any function $F(Z)$ on H_2, the Witt operator W is defined by

$$(W(F))(\tau, \omega) = F\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix},$$

where $W(F)$ is a function on $H_1 \times H_1$. In order to describe the images of generators under W, we define functions on $\tau \in H_1$ by

$$f_1(\tau) = \sum_{x \in \mathbb{Z}^2} e(({}^t x A_2 x)\tau) = 1 + 6q + 6q^3 + \cdots,$$
$$f_2(\tau) = \sum_{x \in \mathbb{Z}^6} e(({}^t x E_6 x)\tau) = 1 + 72q + 270q^2 + 720q^3 + \cdots,$$
$$f_3(\tau) = \sum_{x \in \mathbb{Z}^6} e(({}^t x E_6^* x)\tau) = 1 + 54q^2 + 72q^3 + \cdots.$$

We have $f_1 \in A_1(\Gamma_0^{(1)}(3), \psi)$, f_2, $f_3 \in A_3(\Gamma_0^{(1)}(3), \psi)$, where we put $\psi(\gamma) = \psi(d) = \left(\frac{-3}{d}\right)$ for $\gamma = \begin{pmatrix} a & b \\ 3c & d \end{pmatrix} \in \Gamma_0^{(1)}(3)$. We know that $A_k(\Gamma_0^{(1)}(3)) = A_{k+1}(\Gamma_0^{(1)}(3), \psi) = 0$

for odd k. We see by usual dimension formula that

$$\sum_{k=0}^{\infty}(\dim A_k(\Gamma_0^{(1)}(3)) + \dim A_k(\Gamma_0^{(1)}(3),\psi))t^k = \frac{1}{(1-t)(1-t^3)}.$$

Since $f_1^3 - f_3$ vanish at $i\infty$ and f_1 does not, two forms f_1 and f_3 are algebraically independent. Comparing the dimensions we have

$$\mathbb{C}[f_1, f_3] = \bigoplus_{k=0}^{\infty}\left(A_{2k}(\Gamma_0^{(1)}(3)) \oplus A_{2k+1}(\Gamma_0^{(1)}(3),\psi)\right).$$

Also we have $f_3 = (4f_1^3 - f_2)/3$. We write $f_i = f_i(\tau)$ and $f_i^{'} = f_i(\omega)$. Then since $W(\alpha_1) = f_1 f_1^{'}$, $W(\theta_{E_6}) = f_2 f_2^{'}$ and $W(\theta_{E_6^*}) = f_3 f_3^{'}$, we have

$$\begin{aligned}
W(\alpha_1) &= f_1 f_1^{'},\\
W(\beta_3) &= 6(f_1 f_1^{'})^3 + 2f_2 f_2^{'} - 4(f_1^3 f_2^{'} + (f_1^{'})^3 f_2,\\
W(\delta_3) &= -16(f_1 f_1^{'})^3 + 4(f_1^3 f_2^{'} + (f_1^{'})^3 f_2),\\
W(\gamma_4) &= 0.
\end{aligned}$$

Since it is obvious that $W(\alpha_1)$, $W(\beta_3)$, $W(\delta_3)$ are algebraically independent, we see the four forms α_1, β_3, δ_3, γ_4 are algebraically independent by a standard induction as in Igusa [17, I]. It is known that the dimension formula of modular forms belonging to $\Gamma_0(3)$ is given by

$$\sum_{k=0}^{\infty}\dim A_k(\Gamma_0(3))t^k = \frac{(1+2t^4+t^6)+t^{15}(1+2t^2+t^6)}{(1-t^2)(1-t^4)(1-t^6)^2}$$

(cf. [12, p. 23]). Since $B^{(\mathrm{even})} = C \oplus C\alpha_1\beta_3 \oplus C\alpha_1\delta_3 \oplus C\beta_3\delta_3$, we see $B^{(\mathrm{even})} = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma_0(3))$ by comparing the dimensions. We show that $\alpha_1\chi_{14}$, $\beta_3\chi_{14}$, $\delta_3\chi_{14}$, $\alpha_1\beta_3\delta_3\chi_{14}$ are free generators of the module over C. This is equivalent to say that α_1, β_3, δ_3, $\alpha_1\beta_3\delta_3$ are free over C. Since $C = \mathbb{C}[\alpha_1^2, \beta_3^2, \delta_3^2, \gamma_4]$, the proof is easily obtained by comparing parity of the degree of each variable. Hence we see that

$$\bigoplus_{k=1}^{\infty} A_{2k-1}(\Gamma_0(3)) = C\alpha_1\chi_{14} \oplus C\beta_3\chi_{14} \oplus C\delta_3\chi_{14} \oplus C\alpha_1\beta_3\delta_3\chi_{14},$$

comparing the dimensions. Next, we prove the assertion for $\Gamma_0^{\psi_3}(3)$. Assume that $f \in A_k(\Gamma_0(3), \psi_3)$. Then we see $\alpha_1 f \in A_{k+1}(\Gamma_0(3))$. If k is even, then $\alpha_1 f$ is divisible by χ_{14}. If χ_{14}/α_1 is holomorphic, then $\chi_{14}/\alpha_1 \in A_{13}(\Gamma_0(3))$. But we have $A_{13}(\Gamma_0(3)) = \{0\}$ by dimension formula and this is a contradiction. Put $g = \alpha_1 f/\chi_{14}$. Then $g \in B$. Then we have $\alpha_1^2 f^2 = g^2\chi_{14}^2$. We have α_1, f^2, g, $\chi_{14}^2 \in B$ and B is a weighted polynomial ring. Since $(\chi_{14}/\alpha_1)^2$ is not holomorphic, χ_{14}^2 is not divisible by α_1^2 as an element of the weight polynomial ring. So α_1 divide g^2. But since $g \in B$, this means that g is divisible by α_1 and $g/\alpha_1 \in A_{k-14}(\Gamma_0(3)) \subset B$. Hence $f \in B\chi_{14}$. If k is odd, then $\alpha_1 f \in A_{k+1}(\Gamma_0(3)) \subset B^{(\mathrm{even})}$. Again, we see $\alpha_1^2 f^2 \in B^{(\mathrm{even})}$, $f^2 \in B^{(\mathrm{even})}$. So if we put $\alpha_1 f = P(\alpha_1, \beta_3, \delta_3, \gamma_4)$ where P is a polynomial of even (weighted) total degree, then P^2 is divided by α_1^2. This means that α_1 divides P in B and hence we have $f \in B$. The assertion for $\Gamma_0^{\psi_3}(3)$ is obvious from these. $\square$

In [12], we needed seven generators for $\bigoplus_{k=0}^{\infty} A_{2k}(\Gamma_0(3))$. Notation being as in [12, p. 21], (but see correction of several misprints in [12] in the last page of this paper,) we can show that $t_2 = \alpha_1^2$, $u_4 = \gamma_4$, $v_6 = \delta_3^2$, $w_6 = \beta_3^2$, $x_4 = \alpha_1\beta_3$, $y_4 = \alpha_1\delta_3$, and $z_6 = \beta_3\delta_3$. Also, by comparing the images under the Witt operator and Fourier coefficients of both sides, we can show that

$$\begin{aligned}
\phi_4 &= 8\,x_4 + 41\,t_2^2 - 162\,u_4 + 5\,y_4,\\
\phi_6 &= 277\,t_2^3 - 2187\,t_2u_4 + 80\,t_2x_4 + \frac{11}{8}\,v_6 + w_6 + \frac{7}{2}\,z_6 + \frac{91}{2}\,t_2y_4,\\
\chi_{10} &= (64\,t_2^3u_4 + 32\,x_4t_2u_4 - 16\,y_4t_2u_4 - 4\,z_6u_4 + u_4v_6 + 4\,u_4w_6)/6144,\\
\chi_{12} &= (-124416\,t_2^4u_4 - 192\,t_2y_4w_6 - 768\,t_2^3z_4 + 16\,t_2y_4v_6 + 256\,t_2x_4w_6\\
&\quad + 4096\,t_2^4x_4 + 1536\,t_2^3w_6 - 1024\,t_2^4y_4 + 16\,w_6^2 - v_6^2 - 16\,w_6z_6\\
&\quad + 4\,v_6z_6 + 4096\,t_2^6 + 7776\,t_2z_6u_4 + 10077696\,t_2^2u_4^2\\
&\quad - 62208\,t_2^2x_4u_4 + 31104\,t_2^2y_4u_4 + 2519424\,x_4u_4^2 - 1944\,t_2v_6u_4\\
&\quad - 7776\,t_2w_6u_4 - 68024448\,u_4^3 - 1259712\,y_4u_4^2)/3981312.
\end{aligned}$$

In particular, we have

$$\chi_{10} = \gamma_4(8\alpha_1^3 + 2\beta_3 - \delta_3)^2/6144.$$

For the sake of simplicity, from now on we put $a = \alpha_1$, $b = \beta_3$, $c = \gamma_4$, $d = \delta_3$. To write down the formula for χ_{14}^2, we define two modular forms f_{18} of weight 18 and f_{24} of weight 24 of $\Gamma_0(3)$ as follows.

$$\begin{aligned}
f_{18} &= 634894848\,ab^3c^2 + 201216\,a^6d^3b - 5178816\,a^5d^3c - 60949905408\,a^7c^2b\\
&\quad + 27264\,a^3b^3d^2 - 5441955840\,a^7c^2d - 79501824\,a^5b^3c - 23040\,a^6b^2d^2\\
&\quad - 124032\,a^3b^4d - 11156009472\,a^4c^2b^2 - 7938048\,a^{12}bd\\
&\quad - 5462016\,a^9b^2d + 279936\,a^2d^4c + 2516904576\,a^4c^2d^2 - 5598720\,a^2b^4c\\
&\quad - 949542912\,a^{11}cb + 152285184\,a^{11}cd - 416544768\,a^8b^2c + 16796160\,a^8d^2c\\
&\quad + 1344435190272\,a^3c^3b - 44079842304\,c^3bd - 1365504\,a^6b^3d\\
&\quad + 28065792\,a^{12}b^2 - 4096\,b^6 - 143259487488\,a^3c^3d\\
&\quad + 15073280\,a^{18} + 125\,d^6 + 174680064\,a^8bdc + 6158592\,a^2b^3dc\\
&\quad - 979776\,a^2d^3bc - 839808\,a^2b^2d^2c - 24670199808\,a^{10}c^2\\
&\quad - 34012224\,ad^2bc^2 - 32134205039616\,a^2c^4 + 476171136\,ab^2dc^2\\
&\quad + 58786560\,a^5b^2dc + 839808\,a^5d^2bc + 7890835968\,a^4c^2bd
\end{aligned}$$

$$\begin{aligned}
&- 181398528\, ad^3c^2 + 42528\, a^3b^2d^3 - 9120\, a^3d^4b \\
&- 559104\, a^9d^2b + 26112\, a^3b^5 + 3648\, b^4d^2 + 952\, b^3d^3 - 150\, d^5b \\
&- 1572864\, a^{15}d + 38535168\, a^{15}b + 1132032\, a^6b^4 - 788299776\, a^{14}c \\
&- 22039921152\, c^3b^2 - 2544\, a^3d^5 + 8691712\, a^9b^3 - 1056768\, a^{12}d^2 \\
&+ 185344\, a^9d^3 - 22039921152\, c^3d^2 - 1536\, b^5d - 1140\, d^4b^2 \\
&+ 2556630853632\, a^6c^3 + 5088\, a^6d^4,
\end{aligned}$$

$$\begin{aligned}
f_{24} = &- 1970749440\, a^7d^2bc^2 - 165888\, a^2b^6c - 43008\, a^6b^4d^2 \\
&- 16926659444736\, abc^5 + 4478976\, ac^2b^5 + 8411086946304\, a^5c^4b \\
&+ 27895062528\, a^6c^3d^2 + 15881803923456\, a^8c^4 + 201553920\, c^3b^2d^2 \\
&+ 3604480\, a^{12}b^4 + d^8 + 16777216\, a^{24} + 103195607040\, a^{16}c^2 \\
&+ 28311552\, a^{18}b^2 - 1048576\, a^{18}d^2 + 589824\, a^9b^5 \\
&- 2208301056\, a^{20}c + 8324176896\, a^3c^3bd^2 - 340402176\, a^4c^2b^2d^2 \\
&+ 101559956668416\, c^6 - 256\, a^6d^6 - 40310784\, c^3b^4 + 2048\, a^3b^7 \\
&+ 13107200\, a^{15}b^3 - 64532889133056\, a^4c^5 - 1867840487424\, a^{12}c^3 \\
&+ 24576\, a^{12}d^4 - 12317184\, ac^2b^3d^2 - 64\, b^6d^2 + 48\, d^4b^4 \\
&- 77760\, a^2cb^2d^4 + 20155392\, c^3d^4 - 12\, d^6b^2 + 217728\, a^2cb^4d^2 \\
&+ 33554432\, a^{21}b + 53248\, a^6b^6 + 103514112\, a^{11}d^2bc - 3072\, a^3b^5d^2 \\
&- 1617408\, a^8d^4c - 594542592\, a^{11}b^3c + 35831808\, a^8b^2d^2c \\
&- 93560832\, a^8b^4c + 1110158991360\, a^2c^4b^2 - 71833817088\, a^2d^2c^4 \\
&+ 8424\, a^2d^6c - 808704\, a^5d^4bc - 404397785088\, a^6c^3b^2 - 3312451584\, a^{17}cb \\
&+ 8448\, a^6d^4b^2 - 6967296\, a^5b^5c + 4976640\, a^5b^3d^2c + 2799360\, ad^4bc^2 \\
&- 1572864\, a^{15}d^2b - 933888\, a^{12}b^2d^2 + 1152\, a^3d^4b^3 + 636014592\, a^4b^4c^2 \\
&- 128\, a^3d^6b - 278528\, a^9b^3d^2 + 24576\, a^9d^4b + 25194240\, a^4d^4c^2 \\
&- 36199084032\, a^3c^3b^3 + 114661785600\, a^{13}c^2b + 103514112\, a^{14}d^2c \\
&- 1507945807872\, a^9c^3b + 8814624768\, a^7b^3c^2 + 47584641024\, a^{10}c^2b^2 \\
&- 3224862720\, a^{10}c^2d^2 - 1974730752\, a^{14}b^2c.
\end{aligned}$$

By Igusa [18, p. 849], we get

$$(\chi_{35})^2 = -\chi_{10} f_{18}^2 f_{24}/(2^{70} \cdot 3^{24}).$$

(Note that our notations are slightly different from those in Igusa [18].)

To obtain a relation between χ_{35} and χ_{14}, we calculate the following functional determinant

$$\mathrm{jac}_{35} = \frac{\partial(\phi_4, \phi_6, \chi_{10}, \chi_{12})}{\partial(a, b, c, d)}.$$

By computer calculation we see easily that

$$\mathrm{jac}_{35} = (8a^3 + 2b - d)f_{18}/(2^{25} \cdot 3^6).$$

Since $2^9 3^4 \chi_{35} = \mathrm{jac}_{35}(2^9 3^{10} \chi_{14})$, we get

$$\chi_{35} = \chi_{14} \times f_{18}(8a^3 + 2b - d)/2^{25}.$$

Incidentally, we have $a(0,0,0; f_{18}) = 2^{21}$, $a(0,0,0; 8a^3 + 2b - d) = 2^4$, and the product is 2^{25}. We have $a(2,3,1;\chi_{35}) = a(2,3,1;\chi_{14}) = 1$ and $a(t_1,t_2,t_{12};\chi_{35}) = a(t_1,t_2,t_{12};\chi_{14}) = 0$ if $t_1 < 2$ and $t_2 \le 3$ or $t_1 \le 2$ and $t_2 < 3$. This fits the above relation.

By virtue of the relation $\chi_{10} = c(8a + 2b - d)^2/(2^{11} \cdot 3)$ and the above, we get

$$\chi_{14}^2 = -c \cdot f_{24}/(2^{31} \cdot 3^{25})$$

where c is the cusp form of weight 4.

5. Siegel Modular Forms of Level 4

First we review the results in [11]. We define a character ψ_4 of $\Gamma_0(4)$ by $\psi_4(\gamma) = \left(\frac{-1}{\det D}\right)$ for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$. We define a subgroup of $\Gamma_0(4)$ with index two by

$$\Gamma_0^{\psi_4}(4) = \{\gamma \in \Gamma_0(4); \psi_4(\gamma) = 1\}.$$

We put

$$\begin{aligned}
f_{1/2} &= \theta_{0000}(2Z),\\
f_1 &= f_{1/2}^2,\\
g_2 &= \theta_{0000}(2Z)^4 + \theta_{0100}(2Z)^4 + \theta_{1000}(2Z)^4 + \theta_{1100}(2Z)^4,\\
h_2 &= \theta_{0000}(2Z)^4 + \theta_{0001}(2Z)^4 + \theta_{0010}(2Z)^4 + \theta_{0011}(2Z)^4,\\
f_3 &= (\theta_{0001}(2Z)\theta_{0010}(2Z)\theta_{0011}(2Z))^2,\\
\chi_{11} &= \chi_5(2Z)(\theta_{0001}(2Z)^4 - \theta_{0010}(2Z)^4)\\
&\qquad \times (\theta_{0001}(2Z)^4 - \theta_{0011}(2Z)^4)(\theta_{0010}(2Z)^4 - \theta_{0011}(2Z)^4)/4096.
\end{aligned}$$

Then we have $f_1 \in A_1(\Gamma_0(4), \psi_4)$, g_2, $h_2 \in A_2(\Gamma_0(4))$, $f_3 \in A_3(\Gamma_0(4), \psi_4)$ and $\chi_{11} \in A_{11}(\Gamma_0(4))$. We know that $\dim A_{11}(\Gamma_0(4)) = 1$ and $a(3,2,1;\chi_{11}) = -1$ (cf. [11]).

Theorem 5.1 (Hayashida–Ibukiyama [11]). *Four forms f_1, g_2, h_2, f_3 are algebraically independent and we have*

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_0^{\psi_4}(4)) = \mathbb{C}[f_1, g_2, h_2, f_3, \chi_{11}].$$

Again we see that

$$\chi_{11} = -\frac{1}{2^{18}\cdot 3(2\pi i)^3} \times \begin{vmatrix} f_1 & 2g_2 & 2h_2 & 3f_3 \\ \frac{\partial f_1}{\partial \tau} & \frac{\partial g_2}{\partial \tau} & \frac{\partial h_2}{\partial \tau} & \frac{\partial f_3}{\partial \tau} \\ \frac{\partial f_1}{\partial z} & \frac{\partial g_2}{\partial z} & \frac{\partial h_2}{\partial z} & \frac{\partial f_3}{\partial z} \\ \frac{\partial f_1}{\partial \omega} & \frac{\partial g_2}{\partial \omega} & \frac{\partial h_2}{\partial \omega} & \frac{\partial f_3}{\partial \omega} \end{vmatrix}.$$

Indeed, since f_1 and f_3 are Siegel modular forms with character and g_2 and h_2 are without character, we see that $\{f_1, g_2, h_2, f_3\}_3$ is a Siegel modular form without character of weight 11. Comparing a Fourier coefficient, we get the result.

6. Borcherds Product

In this section, we give a construction of χ_{19}, χ_{14} and χ_{11} by Borcherds product. As for Siegel paramodular groups, a concrete theory of Borcherds product is treated in Gritsenko and Nikulin [9]. But the Borcherds product for congruence subgroups $\Gamma_0(N)$ is not well-known, since $\Gamma_0(N)$ is not an automorphism group of a lattice in general. So first we explain the essence of our way to construct Borcherds product shortly. Since we use a weaker version of Jacobi forms for construction of Borcherds product, we explain this first.

For any function ϕ on $H_1 \times \mathbb{C}$ and $g \in SL_2(\mathbb{R})$ and $X = (\lambda, \mu) \in \mathbb{R}^2$, $\kappa \in \mathbb{R}$, we put

$$\phi|_{k,m}[g] = (c\tau + d)^{-k} e^m\left(\frac{-cz^2}{c\tau + d}\right)\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$
$$\phi|_m[X, \kappa] = e^m(\lambda^2\tau + 2\lambda z + \lambda\mu + \kappa)\phi(\tau, z + \lambda\tau + \mu),$$

where we put $e^m(x) = e^{2\pi i m x}$ for any $x \in \mathbb{C}$. For simplicity, we write $\phi|_m[X, 0] = \phi|_m[X]$. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index. We denote by Γ^J the semi direct product $\Gamma \ltimes \mathbb{Z}^2$ where the product is defined as in Eichler–Zagier [6, p. 9]. Then the above operations of $\gamma \in \Gamma$ and $X \in \mathbb{Z}^2$ define an action of Γ^J. For any integers k and m with $m \geq 0$, a holomorphic function ϕ on $H_1 \times \mathbb{C}$ is called a weak Jacobi form of weight k and index m of Γ^J when it satisfies the following conditions:

(1) $\phi|_{k,m}[\gamma] = \phi$ for any $\gamma \in \Gamma$.
(2) $\phi|_m[X] = \phi$ for any $X \in \mathbb{Z}^2$.
(3) ϕ has the Fourier expansion

$$\phi(\tau, z)|_{k,m}[\gamma] = \sum_{n,l} c_\gamma(n, l) e(n\tau + lz)$$

for all $\gamma \in SL_2(\mathbb{Z})$ and besides $c_\gamma(n, l) = 0$ unless $n \geq 0$ (cf. [6]). (In the usual definition of Jacobi forms we assume that $c_\gamma(n, l) = 0$ unless $4nm - l^2 \geq 0$.) Instead of the condition (3), we sometimes use weaker condition.

(3*) There is n_0 such that $c_\gamma(n, l) = 0$ for all $n < n_0$ and all $\gamma \in SL_2(\mathbb{Z})$.

If ϕ satisfies (1), (2) and (3*), we say that ϕ is a very weak Jacobi form. We denote by $J_{k,m}^{\text{weak}}(\Gamma^J)$ or $J_{k,m}^{vw}(\Gamma^J)$ the space of weak, or very weak Jacobi forms, respectively. Sometimes we consider Jacobi forms with character χ of Γ, taking the condition $\phi|_{k,m}[\gamma] = \chi(\gamma)\phi$ instead of the condition (1). The space of such Jacobi forms is denoted by $J_{k,m}(\Gamma^J, \chi)$ and so on. It is obvious that if $\phi(\tau, z)$ is very weak then $\alpha(\tau)\phi(\tau, z)$ is a weak Jacobi form for some modular form $\alpha(\tau)$ of Γ. (For example, take α to be some power of Δ where Δ is the Ramanujan delta function.)

Also we denote by $A(\Gamma)$ the graded ring of all holomorphic modular forms on H_1 belonging to Γ and by $A_k(\Gamma, \chi)$ the space of holomorphic modular forms of Γ of weight k with character χ. When χ is trivial, we write $A_k(\Gamma) = A_k(\Gamma, \chi)$. We define $\phi_{0,1} \in J_{0,1}^{\text{weak}}(SL_2(\mathbb{Z})^J)$, $\phi_{-2,1} \in J_{-2,1}^{\text{weak}}(SL_2(\mathbb{Z})^J)$ and $\phi_{-1,2} \in J_{-1,2}^{\text{weak}}(SL_2(\mathbb{Z})^J)$ as in Eichler–Zagier [6, pp. 108 and 110]. Namely we put

$$\begin{aligned}
\phi_{-2,1} &= \frac{\phi_{10,1}}{\Delta} \\
&= (-2 + \zeta + \zeta^{-1}) + q(-12 + 8\zeta + 8\zeta^{-1} - 2\zeta^2 - 2\zeta^{-2}) + \cdots, \\
\phi_{0,1} &= \frac{\phi_{12,1}}{\Delta} \\
&= (10 + \zeta + \zeta^{-1}) + q(108 - 64\zeta - 64\zeta^{-1} + 10\zeta^2 + 10\zeta^{-2}) + \cdots, \\
\phi_{-1,2} &= \frac{\phi_{11,2}}{\Delta} \\
&= (\zeta - \zeta^{-1}) + q(3(\zeta - \zeta^{-1}) + \zeta^3 - \zeta^{-3}) + \cdots
\end{aligned}$$

where $\phi_{10,1}$, $\phi_{12,1}$ or $\phi_{11,2}$ is a unique element in $J_{10,1}(SL_2(\mathbb{Z})^J)$, $J_{12,1}(SL_2(\mathbb{Z})^J)$, $J_{11,2}(SL_2(\mathbb{Z})^J)$ up to constant.

By the way, we have the following product expansion (cf. [3, p. 184, Theorem 6.5]).

$$\phi_{-2,1}(\tau, z) = \zeta^{-1}(1-\zeta)^2 \prod_{n=1}^{\infty} \frac{(1-q^n\zeta)^2(1-q^n\zeta^{-1})^2}{(1-q^n)^4},$$

$$\phi_{-1,2}(\tau, z) = \zeta^{-1}(1-\zeta^2) \prod_{n=1}^{\infty} \frac{(1-q^n\zeta^2)(1-q^n\zeta^{-2})}{(1-q^n)^2}.$$

We denote by $J_{\text{even},*}^{\text{weak}}(\Gamma^J)$ the graded ring of weak Jacobi forms of even weight of arbitrary index. We also denote by $J_{\text{odd},*}^{\text{weak}}(\Gamma^J)$ the vector space of weak Jacobi forms of arbitrary odd weight, and by $J_{*,*}^{\text{weak}}(\Gamma^J)$ the space of arbitrary weight and index.

Proposition 6.1. *We have*

$$J_{*,*}^{\text{weak}}(\Gamma^J) = A(\Gamma)[\phi_{-2,1}, \phi_{0,1}] \oplus \phi_{-1,2} A(\Gamma)[\phi_{-2,1}, \phi_{0,1}].$$

When $-1_2 \in \Gamma$, *more precisely we have*

$$\begin{aligned}
J_{\text{even},*}^{\text{weak}}(\Gamma^J) &= A(\Gamma)[\phi_{-2,1}, \phi_{0,1}], \\
J_{\text{odd},*}^{\text{weak}}(\Gamma^J) &= \phi_{-1,2} J_{\text{even},*}^{\text{weak}}(\Gamma).
\end{aligned}$$

Proof. Since our proof is almost the same as in [6], we sketch an outline shortly here. First we assume that $-1_2 \in \Gamma$. Assume that k is even. For any $\phi \in J_{k,m}^{\mathrm{weak}}(\Gamma^J)$, we see $\phi(\tau,-z) = \phi(\tau,z)$ so take the Taylor expansion $\phi = \sum_{t=0}^{\infty} f_t(\tau)z^{2t}$. We denote by $J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r)$ the space of those $\phi \in J_{k,m}^{\mathrm{weak}}(\Gamma^J)$ such that $f_t = 0$ unless $t \geq r$. By the mapping from ϕ to f_r, we get the exact sequence

$$0 \to J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r+1) \to J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r) \to A_{k+2r}(\Gamma).$$

Also we can show that $\phi(\tau,z) = 0$ has exactly $2m$ zeros counting multiplicity inside the fundamental parallelotope and we get $J_{k,m}^{\mathrm{weak}}(\Gamma^J)(m+1) = \{0\}$. Hence we get

$$\dim J_{k,m}^{\mathrm{weak}}(\Gamma^J) \leq \sum_{r=0}^{m} \dim A_{k+2r}(\Gamma).$$

Now $\phi_{0,1}$, $\phi_{-2,1} \in J_{\mathrm{even},*}^{\mathrm{weak}}(\Gamma^J)$ and since $\phi_{0,1} = 12 + O(z^2)$ and $\phi_{-2,1} = (2\pi i z)^2 + O(z^4)$, the functions $\phi_{0,1}^a \phi_{-2,1}^b$ $(0 \leq a,\ 0 \leq b)$ are linearly independent over $A(\Gamma)$. Hence the maximum of the above inequality is attained.

The case for odd k is similarly proved. We see $\phi(\tau,z) = -\phi(\tau,-z)$, so take the Taylor expansion $\phi = \sum_{t=0}^{\infty} f_t(\tau)z^{2t+1}$. We denote by $J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r)$ the space of those $\phi \in J_{k,m}^{\mathrm{weak}}(\Gamma^J)$ such that $f_t = 0$ unless $t \geq r$. By the mapping from ϕ to f_r, we get the exact sequence

$$0 \to J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r+1) \to J_{k,m}^{\mathrm{weak}}(\Gamma^J)(r) \to A_{k+2r+1}(\Gamma).$$

Also we can show $J_{k,m}^{\mathrm{weak}}(\Gamma^J)(m-1) = \{0\}$. Hence we get

$$\dim J_{k,m}^{\mathrm{weak}}(\Gamma^J) \leq \sum_{r=0}^{m-2} \dim A_{k+2r+1}(\Gamma)$$

and

$$J_{\mathrm{odd},*}^{\mathrm{weak}}(\Gamma^J) = \phi_{-1,2} A(\Gamma)[\phi_{-2,1}, \phi_{0,1}].$$

Now we assume that $-1_2 \notin \Gamma$. For any Jacobi weak form $\phi(\tau,z) \in J_{k,m}^{\mathrm{weak}}(\Gamma^J)$ we consider two Jacobi forms f and g defined by

$$f = \phi(\tau,z) + \phi(\tau,z)|_{k,m}[-1_2],$$
$$g = \phi(\tau,z) - \phi(\tau,z)|_{k,m}[-1_2].$$

We denote by $\tilde{\Gamma}$ the group generated by Γ and -1_2 and by χ the character of $\tilde{\Gamma}$ which is defined by the natural projection to $\tilde{\Gamma}/\Gamma \cong \{\pm 1\}$. Then $f \in J_{k,m}(\tilde{\Gamma}^J)$. Since $A(\tilde{\Gamma}) = A_{\mathrm{even}}(\Gamma) = \bigoplus_{k=0}^{\infty} A_{2k}(\Gamma)$, we see that $f \in A_{\mathrm{even}}(\Gamma)[\phi_{-2,1}, \phi_{0,1}] + \phi_{-1,2} A_{\mathrm{even}}(\Gamma)[\phi_{-2,1}, \phi_{0,1}]$. On the other hand, we have $g \in J_{k,m}^{\mathrm{weak}}(\tilde{\Gamma}^J, \chi)$. We have the following exact sequence as before.

$$0 \to J_{k,m}^{\mathrm{weak}}(\tilde{\Gamma}^J,\chi)(s+1) \to J_{k,m}^{\mathrm{weak}}(\tilde{\Gamma}^J,\chi)(s) \to A_{k+2s+1}(\tilde{\Gamma},\chi).$$

For even k we get

$$\dim J_{k,m}^{\mathrm{weak}}(\tilde{\Gamma}^J,\chi) \leq \sum_{s=0}^{m-2} \dim A_{k+2s+1}(\tilde{\Gamma},\chi)$$

and for odd k we get

$$\dim J_{k,m}^{\mathrm{weak}}(\tilde{\Gamma}^J,\chi) \le \sum_{s=0}^{m} \dim A_{k+2s}(\tilde{\Gamma},\chi).$$

Since $A_k(\tilde{\Gamma},\chi) = A_k(\Gamma)$ if k is odd, we see that

$$g \in A_{\mathrm{odd}}(\Gamma)[\phi_{-2,1},\phi_{0,1}] + \phi_{-1,2}A_{\mathrm{odd}}(\Gamma)[\phi_{-2,1},\phi_{0,1}].$$

By these we get the conclusion of the proposition. □

In a sense the Borcherds product is obtained as a log version of Saito–Kurokawa lifting, so we need the result that the discrete group we consider is generated by Jacobi group and an elementary change of variables. We shall show this now.

We consider four kinds of elements of $Sp(2,\mathbb{Z})$.

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad u(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C(a,b,c,d) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ cN & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u(S) = \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix},$$

where $x \in \mathbb{Z}$, $\left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0^{(1)}(N)$, ${}^tS = S \in M_2(\mathbb{Z})$.

Lemma 6.2. *For any natural number N, the group $\Gamma_0(N)$ is generated by the above four kinds of matrices.*

Proof. Take $\gamma = \left(\begin{smallmatrix} A & B \\ NC & D \end{smallmatrix}\right) \in \Gamma_0(N)$ and put $A = (a_{ij})$, $C = (c_{ij})$. We can assume a_{11}, $a_{21} \ge 0$ by multiplying $C(-1,0,0,-1)$ or $TC(-1,0,0,-1)T$ if necessary. Multiplication of T and $u(x)$ gives the Euclid algorithm to the pair (a_{11},a_{21}), so we can assume that $a_{21} = 0$. Then, since $\gamma \in SL_4(\mathbb{Z})$, a_{11} and N are coprime. Denote by m the g.c.d. of a_{11} and c_{11}. Then if we put $c_0 = c_{11}/m$ and $d_0 = -a_{11}/m$, then Nc_0 and d_0 are coprime. Hence there exists a_0, b_0 such that $a_0d_0 - b_0c_0N = 1$. Multiplying $C(a_0,b_0,c_0,d_0)$ from left for a suitable choice of a_0, b_0, we may assume that the first column of γ is $(m,0,0,Nc_{22})$. Here m and Nc_{22} are coprime. Multiplying $u(1)$ from left, this is changed into $(m,m,Nc_{22},-Nc_{22})$. We take a pair of coprime integers c_1, d_1 such that $c_1m + d_1c_{22} = 0$ and a pair a_1, b_1 such that $a_1d_1 - Nb_1c_1 = 1$. Then multiplying $C(a_1,b_1,c_1,d_1)$ and $TC(a_1,-b_1,-c_1,d_1)T$ from left, the first column becomes $(a_2,a_2,0,0)$ for some $a_2 \in \mathbb{Z}$. Since $\gamma \in SL_4(\mathbb{Z})$, we have $a_2 = \pm 1$. By multiplying $u(-1)$ from left, we get $(1,0,0,0)$. Since $\gamma \in Sp(2,\mathbb{R})$, we have $c_{12} = d_{12} = 0$ and $d_{11} = 1$. Then $\left(\begin{smallmatrix} a_{22} & b_{22} \\ c_{22}N & d_{22} \end{smallmatrix}\right) \in \Gamma_0^{(1)}(N)$. Hence by taking $TC(d_{22},-b_{22},-c_{22}N,a_{22})T\gamma$, we can assume that $C = 0$, $d_{12} = 0$, and $a_{11} = a_{22} = d_{11} = d_{22} = 1$. This is a product of $u(x)$ and $u(S)$ for suitable x and S. □

To construct a kind of Saito–Kurokawa lifting of weight 0, we define a Hecke operator shifting indices as in [6]. We put

$$\Delta_N(t) = \left\{ g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix};\ a, b, c, d \in \mathbb{Z}, \det(g) = t \right\}.$$

For any $\phi \in J^{vw}_{k,m}(\Gamma_0^{(1)}(N)^J)$, we define

$$(\phi|_{k,m}V_t)(\tau, z) = t^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)\backslash\Delta_N(t)} (c\tau+d)^{-k} e^{mt}\left(\frac{-cz^2}{c\tau+d}\right) \times \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{tz}{c\tau+d}\right).$$

This operator is a mapping from $J^{vw}_{k,m}\big(\Gamma_0^{(1)}(N)^J\big)$ to $J^{vw}_{k,mt}\big(\Gamma_0^{(1)}(N)^J\big)$.

Now to calculate everything more concretely, we should describe representatives of $\Gamma_0^{(1)}(N)\backslash\Delta_N(t)$. We take a complete set of cusps of $\Gamma_0^{(1)}(N)$. Namely, we take a complete set of representatives $\{g_s\}$ of double coset $\Gamma_0^{(1)}(N)\backslash SL_2(\mathbb{Q})/P(\mathbb{Q})$, where $P(\mathbb{Q}) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right); a,\ b,\ \in \mathbb{Q} \right\}$. Since $SL_2(\mathbb{Q}) = SL_2(\mathbb{Z})P(\mathbb{Q})$, we may assume $g_i \in SL_2(\mathbb{Z})$. Now take $g \in \Delta_N(t)$. Since $g\left(\begin{smallmatrix} 1 & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$, we have

$$g \in \Gamma_0^{(1)}(N) g_s \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

for some s, $a > 0$, b, $d > 0$ with $ad = t$. If

$$\Gamma_0^{(1)}(N) g_1 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \Gamma_0^{(1)}(N) g_2 \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

for $a_2 > 0$, $d_2 > 0$, then obviously $g_1 = g_2$, $a = a_2$, $d = d_2$. Define a natural number h_s by

$$g_s^{-1}\Gamma_0^{(1)}(N) g_s \cap P(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} 1 & h_s n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z} \right\},$$

where we put $P(\mathbb{Z}) = P(\mathbb{Q}) \cap SL_2(\mathbb{Z})$. Then $b \equiv b_2 \bmod h_s d$. We put $g_s = \left(\begin{smallmatrix} x_s & y_s \\ z_s & w_s \end{smallmatrix}\right)$. Then the condition for $g_s\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_N(t)$ is that $az_s \equiv 0 \bmod N$ and $ad = t$. As a whole we see that

$$\Gamma_0^{(1)}(N)\backslash\Delta_N(t) = \bigcup_s \left\{ g_s \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a, b, d \in \mathbb{Z}, ad = t, az_s \equiv 0 \bmod N, b = 0, \ldots, h_s d - 1 \right\}.$$

Now we take $\phi(\tau,z) \in J_{0,1}^{vw}(\Gamma_0^{(1)}(N)^J)$. We put

$$\phi(\tau,z)|_{0,1}[g_s] = \sum_{n,l} c_s(n,l)e(n\tau + lz).$$

Here we note that $(\phi(\tau,z)|_{0,1}[g_s])|_1[(\lambda,\mu)] = \phi(\tau,z)|_{0,1}[g_s]$ for any integers λ, $\mu \in \mathbb{Z}$. So we get $n \in h_s^{-1}\mathbb{Z}$, $l \in \mathbb{Z}$ and it is shown as usual that $c_s(n,l)$ depends only on $4n - l^2$ and $l \bmod 2$, so we write $c_{s,l}(4n-l^2) = c_s(n,l)$. Here $n \in h_s^{-1}\mathbb{Z}$ might not be an integer. But if $4 \nmid h_s$ then $l \bmod 2$ is determined only by $4n - l^2$ and in this case we write $c_s(4n-l^2) = c_{s,l}(4n-l^2)$ for simplicity.

We define an operator L by

$$L\phi = \sum_{t=1}^{\infty} (\phi|_{0,1}V_t)(\tau,z)e^{2\pi i t\omega}.$$

For each cusp, we put $\phi_s(\tau,z) = \phi(\tau,z)|_{0,1}[g_s]$ and $n_s = N/(z_s,N)$. Then we have

$$\begin{aligned} L\phi &= \sum_s \sum_{t=1} \frac{1}{t} \sum_{ad=t, az_s \equiv 0 \bmod N} \sum_{b=0}^{h_s d-1} \phi_s\left(\frac{a\tau+b}{d}, az\right) e(t\omega) \\ &= \sum_s \sum_{t=1}^{\infty} \sum_{ad=t, a\in n_s\mathbb{Z}} (ad)^{-1} dh_s \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nd-l^2)e(an\tau + alz + t\omega) \\ &= \sum_s h_s \sum_{a=1}^{\infty} \frac{1}{an_s} \sum_{m=1}^{\infty} \sum_{n,l\in\mathbb{Z}} c_{s,l}(4nm-l^2)e(n_s(n\tau+lz+m\omega))^a \\ &= \sum_s \frac{h_s}{n_s} \log\left(\prod_{l,m,n\in\mathbb{Z}, m\geq 1} (1 - e(n_s(n\tau+lz+m\omega)))^{c_{s,l}(4mn-l^2)}\right). \end{aligned}$$

We note that $L\phi$ converges only for those $Z \in H_2$ such that $\mathrm{Im}(Z)$ is big enough, namely $\mathrm{Im}(Z) - Y_0$ is positive definite for some fixed Y_0. But by analytic continuation, $\exp(L\phi)$ becomes multi-valued meromorphic function on H_2 according to the general theory of Borcherds product (cf. [3, p. 177, Theorem 5.1] or [4, p. 88, Theorem 3.22]). Its singularities and zeros are on so called rational quadratic divisors. The behaviour on rational quadratic divisors is determined by exponents, and in our case if $n_s^{-1}h_s c_{s,l}(N)$ are integers for $N < 0$, then the product is single valued (here if $4mn - l^2 \geq 0$ we see $|e(n\tau + lz + m\omega)| < 1$ on H_2 and we can take the principal value as a branch of $(1 - e(n\tau + lz + m\omega)^{n_s^{-1}h_s c_{s,l}(4mn-l^2)})$, and if $c_{s,l}(N) > 0$ for $N < 0$ besides, then it is holomorphic).

Now we see that $L\phi$ is invariant by the action of $\Gamma_0^{(1)}(N)^J$. Since ϕ is of weight 0, this is also true for $\exp(L\phi)$ which is actually given by the following infinite product

$$\prod_{l,m,n\in\mathbb{Z}, m>0} (1 - (q^n\zeta^l p^m)^{n_s})^{c_{s,l}(4mn-l^2)h_s/n_s},$$

where we put $p = e(\omega)$. If this is invariant by exchange of τ and ω, then by Lemma 6.2, we see that this is a Siegel modular form belonging to $\Gamma_0(N)$. But actually this is false since we have terms with $n \leq 0$ while all $m \geq 1$ in the above product. So to get invariance for this exchange, we must multiply something to $\exp(L\phi)$. To give such a candidate, we prepare notations. For integers m, l, $n \in \mathbb{Z}$, we write $(n,l,m) > 0$ if (1) $m > 0$, n, $l \in \mathbb{Z}$, or (2) $m = 0$, $n > 0$, $l \in \mathbb{Z}$, or (3) $m = n = 0$, $l > 0$. For $\phi(\tau,z) \in J_{1,0}^{vw}$, we denote by $c_{s,l}(N)$ the Fourier coefficients at each cusp as before. We put $d(\phi) = \sum_s \sum_{m>0,n<0,m,n,l\in\mathbb{Z}} n_s^{-1} h_s c_{s,l}(4mn - l^2)$. For some integer b and positive integers a, c, we put

$$F(\tau,z,\omega) = q^a \zeta^b p^c \prod_{(n,l,m)>0} (1 - (q^n \zeta^l p^m)^{n_s})^{n_s^{-1} h_s c_{s,l}(4mn-l^2)},$$

$$f(\tau,z) = q^a \zeta^b \prod_s \prod_{n,l\geq 1} ((1 - (q^n\zeta^l)^{n_s})(1 - (q^n\zeta^l)^{n_s})^{n_s^{-1}h_s c_{s,l}(-l^2)}$$

$$\times \prod_{n=1}^{\infty} (1 - q^{nn_s})^{n_s^{-1}h_s c_{s,l}(0)} \prod_{l\geq 1} (1 - \zeta^{ln_s})^{n_s^{-1}h_s c_{s,l}(-l^2)}.$$

We note here that if $N < 0$, we have $c_{s,l}(N) \neq 0$ only for finitely many N, so in the above product expression of $f(\tau,z)$, only finitely many different l appears. Indeed if ϕ is very weak, there is n_0 such that $c_s(n,r) = 0$ if $n < n_0$. So, if $N < 4n_0 - 1$ for $N \equiv -r^2 \bmod 4$ ($r = 0$ or 1), then $c_{s,l}(N) = c_s((N + r^2)/4, r) = 0$. Also for a fixed $N < 0$, the number of triples (n,m,r) with $m \geq 1$ and $n < 0$ such that $4nm - l^2 = N < 0$ is finite.

We denote by χ a Dirichlet character modulo N and define a character of $\Gamma_0(N)$ or $\Gamma_0^{(1)}(N)$ by $\chi_2(\gamma) = \chi(\det(D))$ or $\chi_1(\gamma) = \chi(d)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ or $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, respectively.

Proposition 6.3.

(1) *If the relation*

$$a - c = \sum_s \sum_{m>0,n<0,l,m,n\in\mathbb{Z}} m h_s c_{s,l}(4mn - l^2)$$

is satisfied, then we have $F(\omega,z,\tau) = (-1)^{d(\phi)} F(\tau,z,\omega)$.

(2) *If* $f(\tau,z) \in J_{k,c}(\Gamma_0^{(1)}(N), \chi_1)$ *and* $(-1)^{d(\phi)} = (-1)^k \chi(-1)$, *then we have* $F(\tau,z,\omega) \in A_k(\Gamma_0(N), \chi_2)$.

Proof. The product $F(\tau,z,\omega)$ was defined by multiplying something to $\exp(L\phi)$ so that it is relatively symmetric for τ and ω. The idea of this definition is explained as follows. The part of the product in F for $m \geq 1$, n, $l \in \mathbb{Z}$ appears already in $\exp(L\phi)$. There if $m \geq 1$, $n \geq 1$, then $(1 - (q^n\zeta^l p^m)^{n_s})$ and $(1 - (q^m\zeta^l p^n)^{n_s})$ have the same exponent $n_s^{-1} h_s c_{s,l}(4mn - l^2)$. So this part is symmetric by exchange of τ and ω without any change. When $m \geq 1$ and $n = 0$, to make the product symmetric, the part $\prod_s \prod_{m\geq 1} (1 - (q^m\zeta^l)^{n_s})^{n_s^{-1}h_s c_{s,l}(-l^2)}$ is multiplied to $\exp(L\phi)$ in $F(\tau,z,\omega)$. So this part becomes also symmetric. Finally when $m \geq 1$ and $n \leq -1$, instead of multiplying a power of $(1-(q^m\zeta^l p^n)^{n_s})$, we multiply $q^a p^c$ to avoid a negative power of p. Since $(1-(q^m\zeta^l p^n)^{n_s}) = (-(q^m\zeta^l p^n)^{n_s})(1-(q^{-m}\zeta^{-l}p^{-n})^{n_s})$, the part coming from $(-q^m\zeta^l p^n)$ and $q^a p^c$ remains. Namely, we have

$$\frac{F(\tau,z,\omega)}{F(\omega,z,\tau)} = q^{a-c}p^{c-a}(-1)^{d(\phi)}(pq^{-1})^A\zeta^B,$$

where

$$A = \sum_s \sum_{m>0,n<0,m,n,l\in\mathbb{Z}} mh_s c_{s,l}(4mn-l^2),$$
$$B = \sum_s \sum_{m>0,n<0,m,n,l\in\mathbb{Z}} lh_s c_{s,l}(4mn-l^2).$$

We have $B=0$ since $c_{s,l}(4mn-l^2) = c_{s,-l}(4mn-l^2)$. By our condition in (1), we also have $a-c=A$. Hence (1) is proved. Since $F(\tau,z,\omega) = f(\tau,z)\exp(L\phi)$ and $\exp(L\phi)$ is invariant by the action of $\Gamma_0^{(1)}(N)^J$, we see that under the action of $\Gamma_0^{(1)}(N)^J$ the function $F(\tau,z,\omega)$ behaves like a Siegel modular form of weight k with character χ_2. By (1) and Lemma 6.2, we get $F(\tau,z,\omega)\in A_k(\Gamma_0(N),\chi_2)$. □

In the above, we explained how to construct Borcherds product from an element of $J_{1,0}^{vw}$. Now conversely, if we are given a Siegel modular form $F\in S_k(\Gamma_0(N),\chi_2)$ and want to express F by Borcherds product, how can one find $\phi\in J_{1,0}^{vw}$ which we start from? About this we explain rought idea which is experimentally valid. We take the Fourier–Jacobi coefficient $f_m(\tau,z)$ of p^m of F for the smallest m such that $f_m\neq 0$. Then if $f_m(\tau,z)$ is expressed as an infinite product and if F can be written as a Borcherds product, then $f_m(\tau,z)$ should appear as a factor of the product. Since there is no p in this product, this part should be the part for $m\le 0$ which should be multiplied to $\exp(L\phi)$ to create symmetry. So $f_m(\tau,z)$ should be equal to $f(\tau,z)$ in the above proposition. Although we do not know which products in $f_m(\tau,z)$ should correspond to which cusps, we just distribute some parts of $(1-(q^n\zeta^l p^m)^r)$ to a cusp with $n_s=r$ and hope that there exists $\phi\in J_{0,1}^{vw}(\Gamma_0^{(1)}(N)^J)$ such that $f(\tau,z) = f_m(\tau,z)p^m$ for some a, b, c satisfying the condition of (2). Concrete calculations that this idea works will be given in the examples below.

Now we shall show that $\chi_{19}\in S_{19}(\Gamma_0(2))$, $\chi_{14}\in S_{14}(\Gamma_0(3),\psi_3)$, $\chi_{11}\in S_{11}(\Gamma_0(4))$ are all obtained by Borcherds products.

For any prime p, $\Gamma_0^{(1)}(p)$ has two cusps represented by $g_1 = 1_2$ and $g_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and we have $n_1=1$, $n_2=p$ and $h_1=1$, $h_2=p$. We have

$$\Gamma_0^{(1)}(p)\backslash\Delta_p(t) = \left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in M_2(\mathbb{Z}); ad=t, b=0,\ldots,d-1\right\}$$
$$\cup\left\{g_2\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in M_2(\mathbb{Z}); ad=t, a\equiv 0 \bmod p, b=0,\ldots,pd-1\right\}.$$

First, we consider Siegel modular forms of level 2. Since

$$\chi_{19} = \eta(\tau)^8\eta(2\tau)^{32}\phi_{-1,2}p^2+\cdots$$

and

$$\phi_{-1,2} = \zeta^{-1}(1-\zeta^2)\prod_{n=1}^{\infty}(1-q^n\zeta^2)(1-q^n\zeta^{-2})(1-q^n)^{-2},$$

and $\dim S_{19}(\Gamma_0(2)) = 1$, if we can find a very weak Jacobi form $\phi(\tau,z) \in J^{vw}_{0,1}(\Gamma_0^{(1)}(2))$ such that

$$c_1(-4) = 1, c_1(0) = 6, \qquad c_1(n) = 0 \quad \text{if } n \le 0,\ n \ne 0,-4,\ n \in \mathbb{Z},$$
$$c_2(0) = 32, \qquad c_2(n) = 0 \quad \text{if } n \le 0,\ n \ne 0,\ n \in \mathbb{Z},$$

its Borcherds lift should be χ_{19}, where $c_1(n)$ or $c_2(n)$ are the Fourier coefficients of ϕ or $\phi_2 = \phi|_{0,1}[g_2]$.

Now we construct this ϕ. Define modular forms of $\Gamma_0^{(1)}(2)$ by

$$\alpha(\tau) = \theta_{00}(2\tau)^4 + \theta_{10}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(2)),$$
$$\beta(\tau) = \frac{1}{16}\theta_{00}(2\tau)^4\theta_{10}(2\tau)^4 \in A_4(\Gamma_0^{(1)}(2)).$$

Their Fourier expansion at cusps $i\infty$ and 0 are given by

$$\begin{aligned}
\alpha(\tau) &= 1 + 24q + 24q^2 + \cdots,\\
\beta(\tau) &= 0 + q + 8q^2 + \cdots,\\
\left(\alpha \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= -\frac{1}{2}\alpha(\tau/2)\\
&= -\frac{1}{2} - 12q^{\frac{1}{2}} - 12q + \cdots,\\
\left(\beta \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= \frac{1}{256}\left(\alpha(\tau/2)^2 - 64\beta(\tau/2)\right)\\
&= \frac{1}{256} - \frac{1}{16}q^{\frac{1}{2}} + \frac{7}{16}q + \cdots.
\end{aligned}$$

We note that $\beta(\tau) = \eta(2\tau)^{16}/\eta(\tau)^8$. Hence β has no zero on H_1. Now $\beta\phi$ should be a weak Jacobi form of weight 4 and index 1. Then by virtue of Proposition 6.1, $\beta\phi$ should be written as

$$\begin{aligned}
\beta(\tau)\phi(\tau,z) = {}&\left(a\alpha(\tau)^3 + b\alpha(\tau)\beta(\tau)\right)\phi_{-2,1}(\tau,z)\\
&+ \left(c\alpha(\tau)^2 + d\beta(\tau)\right)\phi_{0,1}(\tau,z)
\end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. Imposing the condition on the Fourier coefficients we demanded, now we find

$$\begin{aligned}
\phi(\tau,z) = {}&\frac{\alpha(\tau)^2}{\beta(\tau)}\left(\frac{\varphi_{0,1}(\tau,z) - \alpha(\tau)\varphi_{-2,1}(\tau,z)}{12}\right)\\
&+ \frac{32}{3}\alpha(\tau)\varphi_{-2,1}(\tau,z) - \frac{8}{3}\varphi_{0,1}(\tau,z).
\end{aligned}$$

Incidentally we can show that this has integral coefficients. Indeed we easily see $\alpha - 1 \in 24q\mathbb{Z}[[q]]$. Also we see $\phi_{0,1} - \phi_{-2,1} \in 12\mathbb{Z}[[q,\zeta,\zeta^{-1}]]$ since $(E_4\phi_{0,1} - E_6\phi_{-2,1})/12$

is the Eisenstein Jacobi form of weight 4 and of index 1 having integral coefficients where E_4 or E_6 are Eisenstein series of $SL_2(\mathbb{Z})$ of weight 4 or 6, each belonging to $1 + 24\mathbb{Z}[[q]]$. Since we have

$$\begin{aligned}\phi(\tau, z) &= \left(\frac{\alpha(\tau)^2}{\beta(\tau)} - 32\right)\left(\frac{\varphi_{0,1}(\tau, z) - \alpha(\tau)\varphi_{-2,1}(\tau, z)}{12}\right) + 8\alpha(\tau)\varphi_{-2,1}(\tau, z) \\ &= q^{-1} + (6 + \zeta + \zeta^{-1}) + (8 + 128(\zeta + \zeta^{-1}) + 6(\zeta^2 + \zeta^{-2}))q + \cdots,\end{aligned}$$

we see that all Fourier coefficients of $\phi(\tau, z)$ are integers. Also we have

$$\begin{aligned}&\left(\phi\left|\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right.\right)(\tau, z) \\ &\quad= \left(\frac{16\alpha\left(\frac{\tau}{2}\right)^2}{\alpha\left(\frac{\tau}{2}\right)^2 - 64\beta\left(\frac{\tau}{2}\right)} - 8\right)\left(\frac{\varphi_{0,1}(\tau, z) - \alpha\left(\frac{\tau}{2}\right)\varphi_{-2,1}(\tau, z)}{3}\right) \\ &\qquad + \left(\frac{512\alpha\left(\frac{\tau}{2}\right)\beta\left(\frac{\tau}{2}\right)}{\alpha\left(\frac{\tau}{2}\right)^2 - 64\beta\left(\frac{\tau}{2}\right)}\right)\varphi_{-2,1}(\tau, z),\end{aligned}$$

and all Fourier coefficients of $\left(\phi|\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]\right)(\tau, z)$ are integers.

Conclusion. Borcherds lift of

$$\phi = \frac{\alpha^2}{\beta}\left(\frac{\varphi_{0,1} - \alpha\varphi_{-2,1}}{12}\right) + \frac{32}{3}\alpha\varphi_{-2,1} - \frac{8}{3}\varphi_{0,1}$$

is χ_{19}. Namely we have

$$\chi_{19} = q^3\zeta^{-1}p^2 \prod_{(n,l,m)>0} \left(1 - q^n\zeta^l p^m\right)^{c_1(4nm-l^2)} \left(1 - q^{2n}\zeta^{2l}p^{2m}\right)^{c_2(4nm-l^2)}.$$

Secondly, we consider a Siegel modular form of level 3. Since

$$\chi_{14} = \eta(\tau)^9\eta(3\tau)^{21}\phi_{-1,2}(\tau, z)p^2 + \cdots,$$

and $\dim S_{14}(\Gamma_0(3), \psi_3) = 1$, if we can find a very weak Jacobi form $\phi(\tau, z) \in J_{0,1}^{vw}(\Gamma_0^{(1)}(3))$ such that

$$\begin{aligned}c_1(-4) = 1, c_1(0) = 7, \qquad & c_1(n) = 0 \quad \text{if } n \le 0,\ n \ne 0, -4,\ n \in \mathbb{Z}, \\ c_2(0) = 21, \qquad & c_2(n) = 0 \quad \text{if } n < 0,\ n \in \mathbb{Z},\end{aligned}$$

its Borcherds lift should be χ_{14}.

Now we construct this ϕ. Put

$$\alpha(\tau) = \sum_{x,y\in\mathbb{Z}} q^{x^2+3xy+3y^2} \in A_1(\Gamma_0^{(1)}(3), \psi),$$

and

$$\beta(\tau) = \alpha(\tau)^3 - \frac{54\eta(3\tau)^9}{\eta(\tau)^3} \in A_3(\Gamma_0^{(1)}(3), \psi),$$

where $\psi\left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right) = \left(\frac{-3}{d}\right)$ is a character of $\Gamma_0^{(1)}(3)$. Their Fourier coefficients are

$$\begin{aligned}
\alpha(\tau) &= 1 + 6q + 0q^2 + 6q^3 + \cdots, \\
\beta(\tau) &= 1 - 36q - 54q^2 - 252q^3 + \cdots, \\
\left(\alpha \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= -\frac{i}{\sqrt{3}}\alpha\left(\frac{\tau}{3}\right) \\
&= -\frac{i}{\sqrt{3}}\left(1 + 6q^{\frac{1}{3}} + 0q^{\frac{2}{3}} + 6q + \cdots\right), \\
\left(\beta \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= -\frac{i}{3\sqrt{3}}\beta\left(\frac{\tau}{3}\right) \\
&= -\frac{i}{3\sqrt{3}}\left(1 - 36q^{\frac{1}{3}} - 54q^{\frac{2}{3}} - 252q + \cdots\right).
\end{aligned}$$

We note that $\alpha^3 - \beta$ has no zero on H_1 since it is expressed as an infinite product. Because $(\alpha^3 - \beta)\phi$ should be a weak Jacobi form of weight 3 and index 1 with character χ, $(\alpha^3 - \beta)\phi$ should satisfy the condition

$$\begin{aligned}
(\alpha(\tau)^3 - \beta(\tau))\phi(\tau, z) = {} & \left(a\alpha(\tau)^5 + b\alpha(\tau)^2\beta(\tau)\right)\phi_{-2,1}(\tau, z) \\
& + \left(c\alpha(\tau)^3 + d\beta(\tau)\right)\phi_{0,1}(\tau, z)
\end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. Comparing its Fourier coefficients, now we find

$$\begin{aligned}
\phi(\tau, z) = {} & \frac{54\alpha(\tau)^3}{\alpha(\tau)^3 - \beta(\tau)}\left(\frac{\varphi_{0,1}(\tau, z) - \alpha(\tau)^2\varphi_{-2,1}(\tau, z)}{12}\right) \\
& + \frac{15}{2}\alpha(\tau)^2\varphi_{-2,1}(\tau, z) - \frac{1}{2}\varphi_{0,1}(\tau, z).
\end{aligned}$$

Since we have $\alpha(\tau)^2 - 1 \in 12q\mathbb{Z}[[q]]^\times$, and $\alpha(\tau)^3 - \beta(\tau) \in 54\mathbb{Z}[[q]]^\times$, all the Fourier coefficients of ϕ are integers by using $\phi_{0,1} \equiv \phi_{-2,1} \bmod 12$. Also

$$\begin{aligned}
& \left(\phi \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau, z) \\
& = \left(\frac{54\alpha\left(\frac{\tau}{3}\right)^3}{\alpha\left(\frac{\tau}{3}\right)^3 + \beta\left(\frac{\tau}{3}\right)} - 6\right)\left(\frac{\phi_{0,1}(\tau, z) - \alpha\left(\frac{\tau}{3}\right)^2\phi_{-2,1}(\tau, z)}{12}\right) \\
& \quad + 3\left(\frac{\alpha\left(\frac{\tau}{3}\right)^3 - \beta\left(\frac{\tau}{3}\right)}{\alpha\left(\frac{\tau}{3}\right)^3 + \beta\left(\frac{\tau}{3}\right)}\right)\alpha\left(\frac{\tau}{3}\right)^2\phi_{-2,1}(\tau, z),
\end{aligned}$$

Hence all Fourier coefficients of $\left(\phi \middle| \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right)(\tau, z)$ are integers.

Conclusion. Borcherds lift of

$$\phi = \frac{54\alpha^3}{\alpha^3 - \beta}\left(\frac{\varphi_{0,1} - \alpha^2\varphi_{-2,1}}{12}\right) + \frac{15}{2}\alpha^2\varphi_{-2,1} - \frac{1}{2}\varphi_{0,1}$$

is χ_{14}. Namely

$$\chi_{14} = q^3\zeta^{-1}p^2 \prod_{(n,l,m)>0} \left(1 - q^n\zeta^l p^m\right)^{c_1(4nm-l^2)} \left(1 - q^{3n}\zeta^{3l}p^{3m}\right)^{c_2(4nm-l^2)},$$

where the meaning of $(n,l,m) > 0$ is as before.

Thirdly, we consider Siegel modular forms of level 4. Cusps of $\Gamma_0^{(1)}(4)$ are represented by $g_1 = 1_2, g_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $g_3 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Representatives of $\Delta_4(t)$ are given by

$$\begin{aligned} \Gamma_0^{(1)}(4)\backslash\Delta_4(t) = &\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; ad = t,\ b = 0,\ldots,d-1 \right\} \\ &\cup \left\{ g_2 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; ad = t,\ a \in 2\mathbb{Z},\ b = 0,\ldots,d-1 \right\} \\ &\cup \left\{ g_3 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; ad = t,\ a \in 4\mathbb{Z},\ b = 0,\ldots,4d-1 \right\}. \end{aligned}$$

Because

$$\chi_{11} = \eta(\tau)^8\eta(4\tau)^{16}\phi_{-1,2}p^2 + \cdots,$$

and $\dim S_{11}(\Gamma_0(4)) = 1$, if we can find a very weak Jacobi form $\phi(\tau,z) \in J_{0,1}^{vw}(\Gamma_0^{(1)}(4))$ such that

$$\begin{aligned} c_1(-4) = 1, \quad c_1(0) = 6, \qquad & c_1(n) = 0 \quad \text{if } n \le 0,\ n \ne 0,-4,\ n \in \mathbb{Z}, \\ & c_2(n) = 0 \quad \text{if } n \le 0,\ n \in \mathbb{Z}, \\ c_3(0,0) = 16, \qquad & c_3(n,l) = 0 \quad \text{if } n < 0,\ n \in \mathbb{Z}, \end{aligned}$$

its Borcherds lift should be χ_{11}.

Now we construct this ϕ. Define modular forms of $\Gamma_0^{(1)}(4)$ by

$$\begin{aligned} \alpha(\tau) &= \theta_{00}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(4)), \\ \beta(\tau) &= \theta_{10}(2\tau)^4 \in A_2(\Gamma_0^{(1)}(4)). \end{aligned}$$

Their Fourier coefficients are

$$\begin{aligned} \alpha(\tau) &= 1 + 8q + 24q^2 + \cdots, \\ \beta(\tau) &= 0 + 16q + 0q^2 + \cdots, \\ \left(\alpha \,\middle|\, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= -\frac{1}{4}\alpha(\tau/4) \\ &= -\frac{1}{4} - 2q^{\frac{1}{4}} - 6q^{\frac{1}{2}} + \cdots, \\ \left(\beta \,\middle|\, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)(\tau) &= \frac{1}{4}\left(\beta(\tau/4) - \alpha(\tau/4)\right) \\ &= -\frac{1}{4} + 2q^{\frac{1}{4}} - 6q^{\frac{1}{2}} + \cdots \end{aligned}$$

and

$$\left(\alpha\left|\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\right.\right)(\tau)=\beta(\tau),$$
$$\left(\beta\left|\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\right.\right)(\tau)=\alpha(\tau).$$

We remark that $16\eta(\tau)^{24}=\alpha\beta(\alpha-\beta)^4$. Hence β has no zero on H_1. Because $\beta\phi$ should be a weak Jacobi form of weight 2 and index 1, $\beta\phi$ should satisfy the condition

$$\begin{aligned}\beta(\tau)\phi(\tau,z)&=\left(a\alpha(\tau)^2+b\alpha(\tau)\beta(\tau)+c\beta(\tau)^2\right)\phi_{-2,1}(\tau,z)\\&\quad+\left(d\alpha(\tau)+e\beta(\tau)\right)\phi_{0,1}(\tau,z)\end{aligned}$$

for some $a,b,c,d,e\in\mathbb{R}$. Comparing its Fourier coefficients, now we find

$$\begin{aligned}\phi(\tau,z)&=\frac{4\alpha(\tau)}{\beta(\tau)}\left(\frac{\varphi_{0,1}(\tau,z)-\alpha(\tau)\varphi_{-2,1}(\tau,z)}{3}\right)\\&\quad+\frac{20}{3}\alpha(\tau)\varphi_{-2,1}(\tau,z).\end{aligned}$$

Since $\alpha(\tau)+\beta(\tau)-1\in 24\mathbb{Z}[[q]]^{\times}$. It is shown as before that the Fourier coefficients of the following three forms are integers.

$$\begin{aligned}\phi(\tau,z)&=\frac{16\alpha(\tau)}{\beta(\tau)}\left(\frac{\phi_{0,1}(\tau,z)-(\alpha(\tau)+\beta(\tau))\phi_{-2,1}(\tau,z)}{12}\right)\\&\quad+8\alpha(\tau)\phi_{-2,1}(\tau,z),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left(\phi\left|\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\right.\right)(\tau,z)&=\frac{8\beta(\tau)}{\alpha(\tau)}\left(\frac{\phi_{0,1}(\tau,z)-(\alpha(\tau)+\beta(\tau))\phi_{-2,1}(\tau,z)}{12}\right)\\&\quad+4\beta(\tau)\phi_{-2,1}(\tau,z),\end{aligned}$$

$$\begin{aligned}\left(\phi\left|\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right.\right)(\tau,z)&=\left(\frac{16\alpha\left(\frac{\tau}{4}\right)}{\alpha\left(\frac{\tau}{4}\right)-\beta\left(\frac{\tau}{4}\right)}\right)\left(\frac{\phi_{0,1}(\tau,z)-\left(\alpha\left(\frac{\tau}{4}\right)+\beta\left(\frac{\tau}{4}\right)\right)\phi_{-2,1}(\tau,z)}{12}\right)\\&\quad+3\left(\frac{\alpha\left(\frac{\tau}{4}\right)\beta\left(\frac{\tau}{4}\right)}{\alpha\left(\frac{\tau}{4}\right)-\beta\left(\frac{\tau}{4}\right)}\right)\phi_{-2,1}(\tau,z).\end{aligned}$$

Conclusion. Borcherds lift of

$$\phi=\frac{4\alpha}{\beta}\left(\frac{\varphi_{0,1}-\alpha\varphi_{-2,1}}{3}\right)+\frac{20}{3}\alpha\varphi_{-2,1}$$

is χ_{11}. Namely, we have

$$\begin{aligned}\chi_{11}&=q^3\zeta^{-1}p^2\prod_{(n,l,m)>0}\left(1-q^n\zeta^lp^m\right)^{c_1(4nm-l^2)}\\&\quad\times\left(1-q^{2n}\zeta^{2l}p^{2m}\right)^{c_2(4nm-l^2)/2}\left(1-q^{4n}\zeta^{4l}p^{4m}\right)^{c_{3,l}(4nm-l^2)}.\end{aligned}$$

Appendix A. Euler Factors

The simple expression of χ_{35} makes it easy to calculate the Fourier coefficients of Siegel modular forms of odd weights of $Sp(2,\mathbb{Z})$. So, we give here some examples of Euler 2 factors of spinor L functions with some other eigen values.

The Euler p-factor of the spinor L-function is given by

$$(1-\lambda(p)p^{-s}+(\lambda(p)^2-\lambda(p^2)-p^{2k-4})p^{-s}-\lambda(p)p^{2k-3-3s}+p^{4k-6-4s})^{-1}$$

where $\lambda(p^a)$ is the eigenvalue of the Hecke operator $T(p^a)$ (cf. Andrianov [1, p. 88]). We define the Hecke polynomial of a Siegel modular form F at p by

$$H_p(u,F)=u^4-\lambda(p)u^3+(\lambda(p)^2-\lambda(p^2)-p^{2k-4})u^2-\lambda(p)p^{2k-3}u+p^{4k-6}.$$

If we denote $a(t_1,t_2,t)$ the Fourier coefficient of Siegel modular form F of odd weight k at $\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$ and $a(t_1,t_2,t;p^a)$ the coefficient of $T(p^a)F$ at the same matrix, we have

$$\begin{aligned}
a(2,3,1;2) &= a(4,6,2)+2^{k-2}(-a(2,3,1)+a(1,6,1)),\\
a(2,4,1;2) &= a(4,8,2)+2^{k-2}(-a(2,4,1)+a(1,8,1)),\\
a(2,3,1;4) &= a(8,12,4)+2^{k-2}(-a(4,6,2)+a(2,12,2))\\
&\quad -2^{2k-4}(a(2,3,1)+a(1,6,1)),\\
a(2,4,1;4) &= a(8,16,4)+2^{k-2}(a(2,16,2)-a(4,8,2))\\
&\quad -2^{2k-4}(a(1,8,1)+a(2,4,1)),\\
a(2,3,1;3) &= a(6,9,3)-3^{k-2}(a(2,3,1)+a(1,6,1)),\\
a(2,4,1;3) &= a(6,12,3),\\
a(2,3,1;5) &= a(10,15,5),\\
a(2,4,1;5) &= a(10,20,5)-5^{k-2}a(2,4,1),\\
a(3,4,2;2) &= a(6,8,4)-2^{k-2}a(2,6,2).
\end{aligned}$$

By using these relations, we can calculate Hecke polynomials at 2. We put $\chi_{39}=\phi_4\chi_{35}$, $\chi_{41}=\chi_{35}\phi_6$, $\chi_{43}=\chi_{35}\phi_4^2$, $\chi_{45,e}=\chi_{35}\phi_4\phi_6$, $\chi_{45,c}=\chi_{35}\chi_{10}$.

Since we get

$$\begin{aligned}
T(2)\chi_{45,e} &= -10766446755840\chi_{45,e}+6671813516187402240\chi_{45,c},\\
T(2)\chi_{45,c} &= 2052096\chi_{45,e}-11638111469568\chi_{45,c}.
\end{aligned}$$

We get eigenforms

$$\chi_{45,\pm}=\chi_{45,e}+\big(-212384\pm 32\sqrt{3219068329}\big)\chi_{45,c}.$$

The eigenvalues and Hecke polynomials $H_2(u,F)$ at two for the above F are given as follows.

$$\begin{aligned}
\lambda(2,\chi_{35}) &= -25073418240=-2^{17}\cdot 3^3\cdot 5\cdot 13\cdot 109,\\
\lambda(4,\chi_{35}) &= 138590166352717152256=2^{34}\cdot 313\cdot 25773193,
\end{aligned}$$

$$\begin{aligned}
\lambda(2,\chi_{39}) &= -283917680640 = -2^{20}\cdot 3^2\cdot 5\cdot 11\cdot 547,\\
\lambda(4,\chi_{39}) &= -3619320962049810366464 = -2^{40}\cdot 109\cdot 1249\cdot 24179,\\
\lambda(2,\chi_{41}) &= -1478110740480 = -2^{18}\cdot 3^3\cdot 5\cdot 11\cdot 3797,\\
\lambda(4,\chi_{41}) &= 205535162772980871725056 = 2^{36}\cdot 4127\cdot 724722623,\\
\lambda(2,\chi_{43}) &= -4069732515840 = -2^{17}\cdot 3^3\cdot 5\cdot 127\cdot 1181,\\
\lambda(4,\chi_{43}) &= -7025213863729397585936384,\\
&= -2^{34}\cdot 137\cdot 461\cdot 32587\cdot 198689,\\
\lambda(2,\chi_{45,\pm}) &= 2^{17}\cdot 3\big(-67\cdot 425207 \pm 167\sqrt{3219068329}\big),\\
\lambda(4,\chi_{45,\pm}) &= 2^{35}\big(-3563629466346751 \mp 3^2\cdot 1249236899\sqrt{3219068329}\big),
\end{aligned}$$

$$\begin{aligned}
H_2(\chi_{35},u) &= u^4 + 25073418240u^3 + 416299159590409338880u^2\\
&\quad + 2^{67}\cdot 25073418240u + 2^{134},\\
&= \big(u^2 + 196608\big(63765 + \sqrt{931783609}\big)u + 2^{67}\big)\\
&\quad \times \big(u^2 + 196608\big(63765 - \sqrt{931783609}\big)u + 2^{67}\big),\\
H_2(\chi_{39},u) &= u^4 + 283917680640u^3 + 65339104410568260321280u^2\\
&\quad + 283917680640\cdot 2^{75}u + 2^{150}\\
&= (u^2 + (141958840320 + 1572864\sqrt{12276590561}u + 2^{75})\\
&\quad \times (u^2 + (141958840320 - 1572864\sqrt{12276590561}u + 2^{75}),\\
H_2(\chi_{41},u) &= u^4 + 1478110740480u^3 + 1677044743445695745228800u^2\\
&\quad + 1478110740480\cdot 2^{79}u + 2^{158}\\
&= \big(u^2 - \big(-739055370240 + 393216\sqrt{505009125721}\big)u + 2^{79}\big)\\
&\quad \times \big(u^2 - \big(-739055370240 - 393216\sqrt{505009125721}\big)u + 2^{79}\big),\\
H_2(\chi_{43},u) &= u^4 + 4069732515840u^3 + 18752233335756256738017280u^2\\
&\quad + 4069732515840\cdot 2^{83}u + 2^{166}\\
&= \big(u^2 - \big(-2034866257920 + 196608\sqrt{122398046613649}\big)u + 2^{83}\big)\\
&\quad \times\big(u^2 - \big(-2034866257920 - 196608\sqrt{122398046613649}\big)u + 2^{83}\big),\\
H_2(\chi_{45,\pm},u) &= u^4 - 2^{17}\cdot 3\big(-67\cdot 425207 \pm 167\sqrt{3219068329}\big)u^3\\
&\quad + 2^{52}\cdot 13\big(31\cdot 37\cdot 2746651 \mp 3^2\cdot 29\cdot 71\sqrt{3219068329}\big)u^2\\
&\quad - 2^{104}\cdot 3\big(-67\cdot 425207 \pm 167\sqrt{3219068329}\big)u + 2^{174}.
\end{aligned}$$

Here we used the following Fourier coefficients.

det(2T)	*matrix*	χ_{35}	χ_{39}
23	$(1,6,1)$	0	0
31	$(1,8,1)$	0	0
23	$(2,3,1)$	1	1
31	$(2,4,1)$	−69	171
92	$(2,12,2)$	0	0
124	$(2,16,2)$	0	0
92	$(4,6,2)$	−16483483648	−146478727168
124	$(4,8,2)$	1137360371712	−25047862345728
207	$(6,9,3)$	−6265491005023317	−82431566868072467
279	$(6,12,3)$	815894058438939960	−2179565272517644688840
368	$(8,12,4)$	70785096262933807104	−486173799845176383847 6
496	$(8,16,4)$	−4884171642142432690176	−8313571977352516164648 96
575	$(10,15,5)$	947008164231993093750 0	888204068691846492093750 0
775	$(10,20,5)$	−8686092839378577431953125	1396071647771671579615804687 5

matrix	χ_{41}	χ_{43}
$(1,6,1)$	0	0
$(1,8,1)$	0	0
$(2,3,1)$	1	1
$(2,4,1)$	−573	411
$(2,12,2)$	0	0
$(2,16,2)$	0	0
$(4,6,2)$	−928354926592	−1870709260288
$(4,8,2)$	531947372937216	−768861505978368
$(6,9,3)$	276892068422699318 7	−29309429601382173237
$(6,12,3)$	7355225506178063048 40	−270365770771852664120 40
$(8,12,4)$	−26019005688812887080 96	−6303243753020672396230 656
$(8,16,4)$	14908890259689784297390 08	−2590633182491496354850799 616
$(10,15,5)$	−5831121555872266319789062500	−448901104534453028634890625 00
$(10,20,5)$	2298951723283032840588498046875	24028072506765581226004804687 5

matrix	$\chi_{45,e}$	$\chi_{45,c}$
$(1,6,1)$	0	0
$(1,8,1)$	0	0
$(2,3,1)$	1	0
$(2,4,1)$	−333	0
$(2,6,2)$	0	0
$(2,12,2)$	0	0
$(2,16,2)$	0	0
$(3,4,2)$	25408	1
$(4,6,2)$	−1970353733632	2052096
$(4,8,2)$	656127793299456	−683347968
$(6,8,4)$	6398259637015019520	−11585971814400
$(6,9,3)$	−6557206528924901052 93	−206480972614698
$(6,12,3)$	3276645475555800519123 60	68758163880694434
$(8,12,4)$	−649694849951194910199644 16	5978206147637673984
$(8,16,4)$	2163483850337479050964815052 8	−19907426471633454366 72
$(10,15,5)$	−7590347802551939438675390625 00	735523870120782234375000
$(10,20,5)$	−1258185877880017957242445019531 25	−244929448750220484046875000

Some more eigenvalues at other primes are given by

$$
\begin{aligned}
\lambda(3,\chi_{35}) &= -11824551571578840 = -2^3\cdot 3^{11}\cdot 5\cdot 4817\cdot 346429,\\
\lambda(5,\chi_{35}) &= 9470081642319930937500 = 2^2\cdot 3^3\cdot 5^7\cdot 13\cdot 23\cdot 41\cdot 137\cdot 668289019,\\
\lambda(3,\chi_{39}) &= -1274599574571722040 = 2^3\cdot 3^{12}\cdot 5\cdot 59959599211,\\
\lambda(5,\chi_{39}) &= 8882040686918464920937500\\
&= 2^2\cdot 3^2\cdot 5^7\cdot 11\cdot 47\cdot 61\cdot 73\cdot 46141\cdot 29729663,\\
\lambda(3,\chi_{41}) &= -1283634468791983080 = -2^3\cdot 3^{12}\cdot 5\cdot 122321\cdot 493657,\\
\lambda(5,\chi_{41}) &= -5831121555872266319789062500\\
&= -2^2\cdot 3^3\cdot 5^9\cdot 11\cdot 20413117\cdot 123110838817,\\
\lambda(3,\chi_{43}) &= -657824259785529596400 = -2^3\cdot 3^{14}\cdot 5\cdot 343836777839.\\
\lambda(5,\chi_{43}) &= -448901104534453028634890625 00\\
&= -2^2\cdot 3^3\cdot 5^8\cdot 41\cdot 773\cdot 17657\cdot 1901457470879,\\
\lambda(3,\chi_{45,\pm}) &= -940124365399227162888 \mp 6607391123670336\sqrt{3219068329}\\
&= -2^3\cdot 3^{15}(1552657\cdot 5274739 \pm 2^3\cdot 31^2\cdot 7487\sqrt{3219068329}),\\
\lambda(5,\chi_{45,\pm}) &= -91524828188692615793303906 2500\\
&\quad \pm 2353676384386503150000000 0\sqrt{3219068329}\\
&= 2^2\cdot 3\cdot 5^9\cdot 13(-11\cdot 83\cdot 32901334030254317 47\\
&\quad \pm 2^6\cdot 29^2\cdot 751\cdot 3733\cdot 511939\sqrt{3219068329}).
\end{aligned}
$$

Correction of the paper [12]

(1) p. 20, in the definition of f_6, the coefficients of $\theta_2\theta_4$ should be $-2271/3328$ instead of $2271/3328$.
(2) p. 21, in v_6 and w_6, the coefficients of $\theta_4\theta_6$ is $-3321/4$ instead of $3321/4$.
(3) p. 24, the coefficient of $g_2\chi_6$ is -1728 instead of -1729.
(4) p. 27, $\uparrow$ 8, the right-hand side of $W(E_4-81E_4^{'})$ -100 instead of 100 and $+10$ instead of -10.
(5) p. 29, l.17, the coefficient of $\theta_2 f_6$ should be 110592 instead of 432 and that of $\theta_2\theta_6$ is -124416 instead of 124416.
(6) p. 29, $\uparrow$ 4, $E_4+E_4^{'}$ should be replaced by $E_4+81E_4^{'}$. The coefficient of $\theta_2 f_6$ is 172800 instead of 675/256.
(1) (7) p. 30, in Eq. (1), the coefficient of $(E_4+81E_4^{'})f_6$ is -108 instead of 180 and that of $\theta_2^2 f_6$ is -20520 instead of -34200.

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