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L-functions of $S_3(\Gamma(4,8))$

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Abstract

We prove most of B. van Geemen and D. van Straten's conjectures on the explicit description of Andrianov L-functions of Siegel cuspforms of degree 2 of weight 3 for the group $\Gamma(4,8)$, which are contained in [B. van Geemen, D. van Straten, The cuspform of weight 3 on $\Gamma_2(2,4,8)$, Math. Comp. 61 (204) (1993) 849–872]. These L-functions are related to the Galois representations on the Siegel modular threefold $\Gamma(4,8)\setminus\mathfrak{H}_2$ as determined by B. van Geemen and N. Nygaard [B. van Geemen, N.O. Nygaard, On the geometry and arithmetic of some Siegel modular threefolds, J. Number Theory 53 (1995) 45–87].

Keywords: Siegel modular form and modular threefold; Andrianov L-function; Yoshida lift; Jacquet–Langlands theory

1. Introduction and main idea

As a next step of the Eichler–Shimura theory, B. van Geemen and N. Nygaard [3] compare L-functions related to Galois representations on Siegel modular threefolds $\Gamma\setminus\mathfrak{H}_2$ and Andrianov L-functions of cuspforms in $S_3(\Gamma)$. Here, the Γ's are congruence subgroups larger than

$$\Gamma(4,8)=\left\{\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma(4)\;\middle|\;\operatorname{diag}(B)\equiv\operatorname{diag}(C)\equiv 0\pmod 8\right\}.$$

They determined the Galois representations on H_l^3 of the modular threefolds, and give a conjecture relating these to Andrianov L-function of certain cuspforms.

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Further, B. van Geemen and D. Straten [4] analyzed $S_3(\Gamma(4,8))$ and determined all the Hecke eigenforms belonging to $S_3(\Gamma(4,8))$ as follows. Using a theta embedding $\Theta : \Gamma(2,4,8) \setminus \mathfrak{H}_2 \to \mathbf{P}^{13}$, and regarding $M_3(\Gamma(2,4,8))$ as a quotient space of homogeneous polynomials of degree 6 with respect to the theta constants in Θ, they showed that $S_3(\Gamma(4,8))$ is spanned by certain six-fold products of theta constants. Considering the action of $Sp_2(\mathbb{Z})$ on these products due to the transformation formula, they showed $S_3(\Gamma(4,8))$ is divided into direct sums of seven irreducible $Sp_2(\mathbb{Z})$-modules. The seven modules contain the elements in Table 1.

Here, we set the Igusa theta constant associated to a characteristic $m = (m_1, m_2, m_3, m_4)$, with $m_i \in \{0, 1\}$ by

$$\theta_m(Z) = \sum_{a,b\in\mathbb{Z}} \mathbf{e}\left(\left(Z\begin{bmatrix} m_1 + (a/2) \\ m_2 + (b/2)\end{bmatrix} + m_3(m_1+2a)/2 + m_4(m_2+2b)/2\right)/2\right),$$

where we denote $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$, $x \in \mathbb{C}$, and $Z[v] = {}^t vZv$, $Z \in \mathfrak{H}_2$.

For a six-fold product θ, a character χ_θ on $\Gamma(2)$ is determined by $\chi_\theta(\gamma) = \frac{\theta|\gamma}{\theta}$ and satisfies $\chi_\theta^4 = 1$. They showed that χ_θ is characterized by a unique θ. When $\chi_\theta^2 = 1$, the Hecke algebra $\mathcal{H}_{(\check{2})} = \bigotimes_{p\neq 2} \mathcal{H}_v(GSp_2(\mathbb{Q}_p), GSp_2(\mathbb{Z}_p))$ outside of 2 acts on the one-dimensional space $\mathbb{C}\theta$, and thus θ is a Hecke eigenform. When χ_θ is not real-valued, $\mathcal{H}_{(\check{2})}$ acts on the two-dimensional space spanned by θ and θ' which has the complex conjugate character of χ_θ, so an appropriate linear combination of θ and θ' is a Hecke eigenform (cf. [4, Proposition 7.4]).

Computing some Hecke operators for the eigenforms obtained as above, they conjectured that their Andrianov L-functions are as in Table 2.

Here ω_d denotes the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ and $\otimes$ denotes the convolution product. The symbols θ_μ, ρ_i, ψ_1 denote some elliptic eigenforms belonging to the spaces (see Table 3).

Table 1

Space	dim	Theta series
$S_3(\Gamma(4))$	15	$\theta_1 = \theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}\theta_{(0,1,1,0)}\theta_{(1,1,1,1)}(Z)$
$S_3(\Gamma(4,8))$	90	$\theta_2 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}(Z)$
	90	$\theta_3 = \theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,0,1)}(Z)$
	360	$\theta_4 = \theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,0,1)}\theta_{(1,0,0,1)}(Z)$
	180	$\theta_5 = \theta_{(0,0,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}\theta_{(0,1,1,0)}\theta_{(1,1,1,1)}(Z)$
	60	$\theta_6 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,0,1,1)}\theta_{(0,1,1,0)}(Z)$
	360	$\theta_7 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}(Z)$

Table 2

Label	Eigenform	Conjectured Andrianov L-function outside of 2
R_6^-	$F_1 = \theta_1$	$\zeta(s-1)\zeta(s-2)L(s,\rho_1)$
$R_4^-(0;2)$	$F_2 = \theta_2 - 4\theta_2'$	$\zeta(s-1)\zeta(s-2)L(s,\rho_1)$
$R_4(1,1;0)$	$F_3 = \theta_3 + 16\theta_3'$	$\zeta(s-1)\zeta(s-2)L(s,\rho_1\otimes\omega_{-1})$
$R_4^-(1;1)$	$F_4 = \theta_4 + 4\theta_4'$	$L(s-1,\theta_\mu\otimes\omega_{-2})L(s,\rho_3\otimes\omega_{-2})$
R_6^*	$F_5 = \theta_5$	$L(s-1,\theta_\mu)L(s,\rho_2)$
$R_4^-(2;0)$	$F_6 = \theta_6$	$L(s-1,\theta_\mu\otimes\omega_{-2})L(s,\rho_2\otimes\omega_{-2})$
$R_5^*(1;0)$	$F_7 = \theta_7$	$L(s,\theta_\mu\otimes\psi_1)$

Table 3

Elliptic cuspform	Space
θ_μ	$S_2(\Gamma_0(32))$
ψ_1	$S_3(\Gamma_0(32), \omega_{-1})$
ρ_1	$S_4(\Gamma_0(8))$
$\rho_2 = \theta_{\mu^3}$	$S_4(\Gamma_0(32))$
ρ_3	$S_4(\Gamma_0(32))$

In particular, θ_μ is obtained by the Größen-character μ related to the elliptic curve $y^2 = x^3 - x$ with complex multiplication:

$$\theta_\mu(z) = \sum_{\mathfrak{a}} \mu(\mathfrak{a})\mathbf{e}\big(N(\mathfrak{a})z\big), \quad z \in \mathfrak{H},$$

where $\mathfrak{a}$ runs through all integral ideals of $\mathbb{Z}[i]$ prime to 2. For these conjectures, our main result is

Main Theorem. *The conjectures for F_i, $1 \leqslant i \leqslant 6$, are true.*

Our proof is using the Yoshida lift as follows. The conjectured $L(s, F_i)$ for $1 \leqslant i \leqslant 6$ are products of L-functions of elliptic modular forms, and the Yoshida lift [14] can provide a Siegel modular form having such a type of L-function. Indeed, in the $Sp_2(\mathbb{Z})$ module generated by F_i, due to the Yoshida lift, we construct an eigenform having the conjectured $L(s, F_i)$. At this moment, since $L(s, F) = L(s, F|\gamma)$ with $F|\gamma$ translated for $\gamma \in Sp_2(\mathbb{Z})$ (see Proposition 2.2 for a more rigorous discussion), we see that $L(s, F_i)$ is just the conjectured one.

Although we believe that the conjecture for F_7 is true, it seems to need more preparations. By base change, ψ_1 is lifted to an automorphic form on $SL_2(\mathbb{Q}(\sqrt{-1}))$. But, the theta lift from $SO(3,1) \simeq SL_2(\mathbb{C})$ to $Sp_2(\mathbb{R})$ as in [6] cannot provide a Siegel modular form of weight 3. Further, we are interested in the Galois representation related to ψ_1 and that related to the modular threefold $\ker(\chi_{\theta_7}) \setminus \mathfrak{H}_2$.

This paper is organized as follows. In Section 2, we review the definition of Andrianov L-function by Evdokimov [2] for adélic forms. In Section 3, we give a short review of the Yoshida lift. In Section 4, we prove the conjectures.

Notation. For a ring A with norms, the group of units of A is denoted by $A^\times$ and by A^1 the group of elements of norm 1. We denote by $M_k^n(\Gamma, \chi)$ and $S_k^n(\Gamma, \chi)$ the space of Siegel modular forms and that of cuspforms of degree n, of weight k, with a character χ on a congruence subgroup $\Gamma \subset Sp_n(\mathbb{Z})$.

2. Andrianov L-function for adélic forms

We review the definition of the Andrianov L-function by Evdokimov [2] for adélic forms, and see how the L-function changes w.r.t. translations of forms by $\gamma \in Sp_2(\mathbb{Z})$ (Proposition 2.2). Further, using this occasion, we recall the definition of the spinor L-function, and clarify the difference between Andrianov and spinor L-functions. These L-functions are likely to be regarded as the same thing, but they are different things, strictly. Indeed, the spinor L-function is invariant w.r.t. translations by elements of $Sp_2(\mathbb{Z})$.

In [2] originally, the Andrianov L-function is defined for classical Siegel modular forms, using his Hecke operators. The spinor L-function is defined for adélic forms on $GSp_2(\mathbb{A})$ (or for their Whittaker models). We can extend a classical Siegel modular form F to a form $F^\natural$ on $GSp_2(\mathbb{A})$, canonically. Then, the Andrianov L-function of F coincides with the spinor L-function of $F^\natural$. However, when we do not extend F canonically, there may be difference between the L-functions. It is caused by the difference of Hecke operators by which the L-functions are defined. The Hecke operators of the former act on forms globally, but those of the latter act locally.

Now, we treat the Andrianov L-function. Let $\Gamma(N)$ be the principal congruence subgroup of level N. For Dirichlet characters η, ψ defined modulo N, let $M_k(N, \eta, \psi) \subset M_k(\Gamma(N))$ denote the space of all Siegel modular forms F satisfying

$$F|_k\gamma(a,b) = \eta(a)\psi(b)F,$$

for every $\gamma(a,b) \equiv \mathrm{diag}[a, ab, a^{-1}, (ab)^{-1}] \pmod N$ in $Sp_2(\mathbb{Z})$. Here, for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GSp_2(\mathbb{R})$ and F we set

$$F|_k g(z) = \det(cz+d)^{-k} F\big((az+b)(cz+d)^{-1}\big). \tag{2.1}$$

Every $F \in M_k(\Gamma(N))$ can be decomposed as $F = \sum_{\eta,\psi} F_{\eta,\psi}$, $F_{\eta,\psi} \in M_k(N,\eta,\psi)$. We set, for $t \in \mathbb{Q}$,

$$\delta(t) = \mathrm{diag}[1,1,t,t], \qquad \varepsilon(t) = \mathrm{diag}\big[1,t,t^2,t\big].$$

Then, Evdokimov defined for a prime $p \nmid N$ Hecke operators on $M_k(N,\eta,\psi)$ by

$$T(1,1,p,p)F = T\big(\delta(p)\big)F = p^{k-3}\sum_j F|_k H_j,$$
$$T\big(1,p,p^2,p\big)F = T\big(\varepsilon(p)\big)F = p^{2k-6}\sum_j F|_k L_j,$$
$$T(p,p,p,p)F = p^{2k-6}\eta(p)F,$$

where the H_j, L_j satisfy $\Gamma\delta(p)\Gamma = \bigsqcup_j \Gamma H_j$, and $\Gamma\varepsilon(p)\Gamma = \bigsqcup_j \Gamma L_j$, $H_j \equiv \delta(p)$, $L_j \equiv \varepsilon(p) \pmod N$ with $\Gamma = \Gamma(N)$. Of course, these definitions are independent from the choice of H_j, L_j. For an eigenform $F \in M_k(N,\eta,\psi)$ at p with eigenvalues $\lambda(\delta(p))$, $\lambda(\varepsilon(p))$ for the above Hecke operators, Evdokimov defined the Andrianov L-function attached to F by

$$L^{ae}(s,F)_p = 1 - \lambda\big(\delta(p)\big)p^{-s} + \big(p\lambda\big(\varepsilon(p)\big) + p^{2k-5}\big(p^2+1\big)\eta(p)\big)p^{-2s} - \eta(p)\lambda\big(\delta(p)\big)p^{2k-3-3s} + \eta(p)^2 p^{4k-6-4s}.$$

Next, we recall the definition of the spinor L-function. For an automorphic form f on $GSp_2(\mathbb{A})$ which is right $GSp_2(\mathbb{Z}_p)$-invariant, the Hecke operators $T_p(\delta(p))$, $T_p(\varepsilon(p))$ are defined by

$$T_p\big(\delta(p)\big)f(g) = \sum_j f\big(g(H_j)_p^{-1}\big), \qquad T_p\big(\varepsilon(p)\big)f(g) = \sum_j f\big(g(L_j)_p^{-1}\big),$$

with $(H_j)_p, (L_j)_p$ being the images of H_j respectively L_j under the embedding $GSp_2(\mathbb{Q}) \to GSp_2(\mathbb{Q}_p)$. Using the eigenvalues $\lambda^\natural(\delta(p))$ and $\lambda^\natural(\varepsilon(p))$, local spinor L-function of f is defined by

$$L^{sp}(s,f)_p = 1 - \lambda^\natural(\delta(p))p^{-s} + \bigl(p\lambda^\natural(\varepsilon(p)) + p(p^2+1)\eta(p)\bigr)p^{-2s} - \eta(p)\lambda^\natural(\delta(p))p^{3-3s} + \eta(p)^2p^{6-4s}.$$

For a classical $F \in M_k(N,\eta,\psi)$, we extend F to a function $F^\natural$ on $GSp_2(\mathbb{A})$ as follows. By the strong approximation theorem for $Sp_2(\mathbb{A})$, any element $g \in GSp_2(\mathbb{A})$ can be decomposed as

$$g = \gamma g_\infty k t_\infty \times \prod_p \delta(t_p).$$

Here $\gamma \in Sp_2(\mathbb{Q})$, $g_\infty \in Sp_2(\mathbb{R})$, $k \in \prod_p \Gamma(N)_p$, and $t_\infty \in \mathbb{R}^\times$, $t_p \in \mathbb{Z}_p^\times$. We set

$$F^\natural(g) = F\bigl(g_\infty(\iota)\bigr)\det(c\iota+d)^{-k}, \quad g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \iota = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.$$

We call $F^\natural$ the 'canonical extension of F.'

Proposition 2.1. *Let $F \in M_k(\Gamma(N),\eta,\psi)$ be a classical form on $\mathfrak{H}_2$ and $F^\natural$ the canonical extension of F on $GSp_2(\mathbb{A})$. Suppose that F is an eigenform at p. Then, we have*

$$L^{ae}(s,F)_p = L^{sp}\bigl(s-k+3, F^\natural\bigr)_p.$$

Proof. It suffices to see $\lambda(\delta(p)) = p^{k-3}\lambda^\natural(\delta(p))$ and $\lambda(\varepsilon(p)) = p^{2k-6}\lambda^\natural(\varepsilon(p))$. This is clear by observing that $H_j\delta(p)^{-1} \equiv L_j\varepsilon(p)^{-1} \equiv 1 \pmod N$, and the way $F^\natural$ is defined. □

We now consider the case where $F \in S_3(\Gamma(4,8))$ has a character χ on $\Gamma(2)$, for our proof of the conjectures.

Proposition 2.2. *Suppose that $F \in S_3(\Gamma(4,8))$ is a Hecke eigenform with*

$$L^{ae}(s,F)_p = 1 - a_p p^{-s} + a_{p^2}p^{-2s} - a_p p^{3-3s} + p^{6-4s}$$

and has a character χ on $\Gamma(2)$. Then $F|\gamma$ is also a Hecke eigenform with

$$L^{ae}(s,F|\gamma) = 1 - \xi(p)a_p p^{-s} + a_{p^2}p^{-2s} - \xi(p)a_p p^{3-3s} + p^{6-4s},$$

for a certain function ξ on $\mathbb{Z}_2^\times$ defined modulo 8.

Proof. Put $\Gamma = \Gamma(8)$ and take an odd prime p. Then we compute the Hecke operator $T(\delta(p))$ for $F|_k\gamma$:

$$T\bigl(\delta(p)\bigr)(F|_k\gamma) = \sum_j F|_k\gamma H_j, \quad H_j \equiv H_1 \ (\mathrm{mod}\ 8), \ H_1 = \delta(p)$$

with $\Gamma\delta(p)\Gamma = \bigsqcup_j \Gamma H_j$. Instead of this computation for F, we consider that for $F^\natural$:

$$\sum_j F^\natural(\gamma_\infty H_{j,\infty} g_\infty) = \sum_j F^\natural\bigl(H_j^{-1}\gamma^{-1}\gamma_\infty H_{j,\infty} g_\infty\bigr)$$
$$= \sum_j F^\natural\bigl(g_\infty H_{j,2}^{-1}\gamma_2^{-1}H_{j,p}^{-1}\bigr), \tag{2.2}$$

where g_∞ is an element of $Sp_2(\mathbb{A})$ whose finite components are all 1 and $\gamma_v, H_{j,v}$ denote the images by the embedding $GSp_2(\mathbb{Q}) \to GSp_2(\mathbb{Q}_v)$. Here we use the left $GSp_2(\mathbb{Q})$-invariance and right $\prod_{v\neq 2} GSp_2(\mathbb{Z}_v)$-invariance of $F^\natural$. This computation is continued to

$$\begin{aligned}\sum_j F^\natural\bigl(g_\infty H_{j,2}^{-1}\gamma_2^{-1}H_{j,p}^{-1}\bigr) &= \sum_j F^\natural\bigl(g_\infty \gamma_2^{-1}\gamma_2 H_{j,2}^{-1}\gamma_2^{-1}H_{j,p}^{-1}\bigr)\\ &= \sum_j F^\natural\bigl(g_\infty \gamma_2^{-1}\gamma_2 H_{j,2}^{-1}\gamma_2^{-1}H_{j,2}H_{j,p}^{-1}\bigr)\\ &= \sum_j \chi_2\bigl(\bigl[\gamma_2, \delta(p)_2^{-1}\bigr]\bigr)F^\natural\bigl(g_\infty\gamma_2^{-1}H_{j,p}^{-1}\bigr)\\ &= \sum_j \lambda^\natural\bigl(\delta(p)\bigr)\chi_2\bigl(\bigl[\gamma_2, \delta(p)_2^{-1}\bigr]\bigr)F^\natural(\gamma_\infty g_\infty),\end{aligned}$$

where $[a,b] = aba^{-1}b^{-1}$ for $a, b \in GSp_2(\mathbb{Q}_2)$ and χ_2 denotes the 2-component of the extended χ, which is characterized by

$$\chi_2(k) = \chi(\alpha)^{-1}$$

for $k \in \Gamma(2)_2, \alpha \in \Gamma(2), \alpha \equiv k \pmod 8$. The computation for $T(\varepsilon(p))$ is also given by

$$T\bigl(\varepsilon(p)\bigr)(F|\gamma) = \chi_2\bigl(\bigl[\gamma_2, \varepsilon(p)_2^{-1}\bigr]\bigr)\lambda\bigl(\varepsilon(p)\bigr)(F|\gamma).$$

We observe that both of the maps

$$\mathbb{Z}_2^\times \ni t \to \chi_2\bigl(\bigl[\gamma_2, \delta(t)_2^{-1}\bigr]\bigr) \in \mathbb{C}^\times,$$
$$\mathbb{Z}_2^\times \ni t \to \chi_2\bigl(\bigl[\gamma_2, \varepsilon(t)_2^{-1}\bigr]\bigr) \in \mathbb{C}^\times$$

are defined modulo 8, and that the latter is always 1 since

$$\bigl[\varepsilon(p), Sp_2(\mathbb{Z}_2)\bigr] \subset \Gamma(4,8) \subset \ker(\chi_2),$$

reminding that the commutator subgroup of $\Gamma(2)$ is $\Gamma(4,8)$. This proves the assertion. □

Remark 2.3. Indeed, an example with a nontrivial ξ is given in [3].

In contrast, for a general automorphic form f on $GSp_2(\mathbb{A})$, the spinor L-function is stable under $Sp_2(\mathbb{Z})$-translations:

$$L^{sp}\big(s, f(\gamma_\infty g)\big) = L^{sp}\big(s, f(g)\big)$$

for every $g \in GSp_2(\mathbb{A})$ and $\gamma_\infty \in Sp_2(\mathbb{Z}) \subset Sp_2(\mathbb{R})$. This is clear from the definition. We note that $(F|\gamma)^\natural(g) = F^\natural(\gamma_\infty g)$ does not necessarily hold.

3. Review of the Yoshida lift

The Yoshida lift is a theta lift from a pair of automorphic forms on a definite quaternion algebra $D_\mathbb{Q}$ defined over $\mathbb{Q}$ to a Siegel modular form whose spinor L-function is the product of the L-functions of the pair. Jacquet–Langlands theory [7] associates cuspidal automorphic forms on $D_\mathbb{A}^\times$ to elliptic cuspforms. For every cuspidal automorphic form on $D_\mathbb{A}^\times$, there exists an elliptic cuspform having the same L-function. So, we can construct a Siegel modular form whose L-function is a product of that of a pair of elliptic modular forms.

We start with a short review of the Yoshida lift. Let $D_\mathbb{Q}$ be a definite quaternion algebra over $\mathbb{Q}$ attached to $a, b \in \mathbb{Q}_{>0}$:

$$D_\mathbb{Q} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, \quad I^2 = -a,\ J^2 = -b,\ IJ = -JI,$$

with the canonical involution $* : a + bI + cJ + dIJ \to a - bI - cJ - dIJ$. We denote by $\mathrm{N}(x) = x \cdot x^*$ and $\mathrm{Tr}(x) = x + x^*$ the reduced norm and trace of $x \in D_\mathbb{Q}$. We put $W_1 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}IJ \subset D_\infty$. Considering the action τ of $D_\infty^\times$ on W_1 such as $\tau(d)w = d^{-1}wd$, $d \in D_\infty^\times$, $w \in W_1$, we obtain a representation σ of $D_\infty^\times/\mathbb{R}^\times$. We denote by $\sigma_{2n} = \mathrm{Sym}^n(\sigma)$ the tensor n-tuple product representation on the space $W_n = \mathrm{Sym}^n(W_1)$.

Definition 3.1 *(Automorphic form of type* (σ_{2n}, R, χ)*).* Let R be an order in $D_\mathbb{Q}$ and $\chi = \bigotimes_p \chi_p$ be a product of character χ_p on $R_p^\times$ (χ_p is trivial at almost all p). We define an automorphic form on $D_\mathbb{A}^\times$ of type (σ_{2n}, R, χ) to be a W_n-valued function f on $D_\mathbb{A}^\times$ which satisfies the following conditions (1)–(3):

(1) For any $\gamma \in D_\mathbb{Q}^\times$ and $x \in D_\mathbb{A}^\times$, $f(\gamma x) = f(x)$.
(2) For any $h \in D_\infty^\times$, $f(xh_v) = \sigma_{2n}(h)f(x)$.
(3) For any $k_p \in R_\mathfrak{p}^\times$, $f(xk_p) = \chi_\mathfrak{p}(k_p)f(x)$.

We denote by $\mathcal{A}(\sigma_{2n}, R, \chi)$ the space of automorphic forms on $D_\mathbb{A}^\times$ of type (σ_{2n}, R, χ). If χ is trivial, we abbreviate it to $\mathcal{A}(\sigma_{2n}, R)$.

Remark 3.2. See [7] for the general definition of automorphic forms. Only the above types of automorphic forms are needed for our use in the Yoshida lift.

We only describe the Yoshida lift from a pair of eigenforms $f_1 \in \mathcal{A}(\sigma_0, R, \chi)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi)$ as follows. Associated to the pair, we take a certain W_1-valued Schwartz function (i.e., theta kernel or test function) $\Phi = \prod_v \Phi_v$ on $D_\mathbb{A}^2$ satisfying (i)–(iii):

(i) $\Phi_\infty(x_1, x_2) = P(x_1^* x_2)\exp(-2\pi(\mathrm{N}(x_1) + \mathrm{N}(x_2)))$ for $x_i \in D_\infty$, where $P(x) = P(a + bI + cJ + dIJ) = bI + cJ + dIJ$.
(ii) If χ_p on $R_p^\times$ is trivial, Φ_p is the characteristic function of R_p^2.
(iii) If χ_p is nontrivial, Φ_p has the property such as

$$\Phi_p\big(k_1^{-1}x_1k_2, k_1^{-1}x_2k_2\big) = \chi_p\big(k_1^{-1}k_2\big)\Phi_p(x_1, x_2), \quad k_i \in R_p^\times,\ x_j \in D_p. \tag{3.1}$$

Then, by the Weil representation of $Sp_2(\mathbb{A})$ in [14], we obtain a Siegel modular form on $Sp_2(\mathbb{A})$. The classical form of the Yoshida lift $\Theta_{\Phi, f_1\times f_2}(Z)$ from $f_1 \times f_2$ for a Schwartz function $\prod_{v\leqslant\infty}\Phi_v$ is

$$\sum_{i,j=1}^{h}(n_in_j)^{-1}\sum_{x_1,x_2\in D_\mathbb{Q}}\Phi_0\big(y_i^{-1}x_1y_j, y_i^{-1}x_2y_j\big)P_j\big(x_1^*x_2\big)f_1(y_i)\mathbf{e}[x_1, x_2, Z]. \tag{3.2}$$

The meanings of the symbols are as follows. We decompose

$$D_\mathbb{A}^\times = \bigsqcup_{1\leqslant i\leqslant h} D_\mathbb{Q}^\times y_i R_\mathbb{A}^\times \tag{3.3}$$

with $(y_i)_\infty = 1$ and denote $n_i = {}^\sharp(D_\mathbb{Q} \cap y_i R_\mathbb{A}^1 y_i^{-1})$. $\Phi_0 = \prod_{p<\infty}\Phi_p$. P_j means

$$P_j(a + bI + cJ + dIJ) = \mathrm{Tr}\big(f_2(y_j)(bI + cJ + dIJ)\big),$$

where we remark that P_j plays the role of the contribution of the Φ_∞. $\mathbf{e}[x_1, x_2, Z] = \mathbf{e}(\mathrm{N}(x_1)z_{11} + \mathrm{Tr}(x_1^*x_2)z_{12} + \mathrm{N}(x_2)z_{22})$, $Z = (z_{ij}) \in \mathfrak{H}_2$. Using this classical form, we can calculate the Fourier coefficients.

It is known that $\Theta_{\Phi, f_1\times f_2}$ is a cuspform of weight 3 and Hecke eigenform at almost all places. Its Andrianov L-function is described as follows. Suppose that χ_p is trivial and R_p is isomorphic to $M_2(\mathbb{Z}_p)$. By the computation in [9] which is a modification of Yoshida's original one, the Andrianov L-function of $\Phi_{f_1\times f_2}$ is given by

$$L^{ae}(s, \Theta_{\Phi, f_1\times f_2})_p = L(s-1, f_1)_p L(s, f_2)_p,$$

where the theta kernels are not fixed to be the characteristic functions of R^2. We note that, if the central character of f_1 is trivial, the same computation is used in [1] to describe the standard L-function as

$$Z(s, \Theta_{\Phi, f_1\times f_2})_p = \zeta(s)_p L(s-2, f_1 \otimes f_2)_p.$$

4. Proofs

In order to prove the conjectures, we need to check two things.

(1) To show the existence of eigenforms having the conjectured Andrianov L-functions in the irreducible $Sp_2(\mathbb{Z})$ module generated by F_i.
(2) To check eigenvalues of the eigenforms at 3, 5, and 7 (cf. Proposition 2.2).

For (1), we will construct eigenforms in $S_3(\Gamma(4,8))$ by the Yoshida lift and show the existence of such eigenforms in $Sp_2(\mathbb{Z}) \cdot F_i$. For (2), we will consult the table of [4]. We first fix some notations. In the remainder of this paper, we consider the definite quaternion algebra

$$D_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, \quad I^2 = J^2 = -1, IJ = -JI,$$

which is split at every odd prime. We will use the orders

$$\begin{aligned} \mathfrak{O} &= \mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}(1 + I + J + IJ)/2, \\ \mathfrak{O}(l) &= \mathbb{Z} + \varpi^l \mathfrak{O}, \quad N(\varpi) = 2, \quad l \in \mathbb{Z}_{\geqslant 0}, \\ R &= \mathbb{Z} + 2\mathbb{Z}I + 2\mathbb{Z}J + 2\mathbb{Z}IJ. \end{aligned}$$

Note that $\mathfrak{O}_p \simeq \mathfrak{O}(l)_p \simeq R_p \simeq M_2(\mathbb{Z}_p)$ at odd prime p and $\mathfrak{O}(l)_2^\times$ is a normal subgroup of $D_2^\times$. With respect to $\mathfrak{O}$ or R, we have decompositions of $D_{\mathbb{A}}^\times$ as

$$D_{\mathbb{A}}^\times = D_{\mathbb{Q}}^\times \mathfrak{O}_{\mathbb{A}}^\times = D_{\mathbb{Q}}^\times y_1 R_{\mathbb{A}}^\times \sqcup D_{\mathbb{Q}}^\times y_2 R_{\mathbb{A}}^\times,$$

for $y_1 = 1$ and $(y_2)_2 = I + J + IJ$, $(y_2)_v = 1$, $v \neq 2$. Here $\mathfrak{O}_{\mathbb{A}}^\times = D_\infty^\times \times \prod_{p<\infty} \mathfrak{O}_p^\times$ and so on.

4.1. Proof for F_2

Now, we start to prove the conjecture for F_2. We need first a pair of automorphic forms f_1, f_2 such that $L(s, f_1) = \zeta(s)\zeta(s-1)$, $L(s, f_2) = L(s, \rho_1)$. We can construct them in $\mathcal{A}(\sigma_0, R)$ and $\mathcal{A}(\sigma_2, R)$ as follows. By direct calculation, we have

$$\dim_{\mathbb{C}} \mathcal{A}(\sigma_0, R) = 2, \quad \text{and} \quad \dim_{\mathbb{C}} \mathcal{A}(\sigma_2, R) = 6.$$

Now define $f_1 \in \mathcal{A}(\sigma_0, R, 1)$ and $f_2 \in \mathcal{A}(\sigma_2, R, 1)$ by

$$\begin{aligned} &f_1(y_1) = f_1(y_2) = 1, \\ &f_2(y_1) = 2bI - cJ + 2dIJ, \qquad f_2(y_2) = -3cJ. \end{aligned}$$

Proposition 4.1. *The above f_1 and f_2 are Hecke eigenforms with*

$$L(s, f_1) = \zeta(s)\zeta(s-1), \qquad L(s, f_2) = L(s, \rho_1),$$

up to the Euler factor at 2.

Proof. The assertion for f_1 is clear. We give the proof for f_2. Since $\mathfrak{O}(3) \subset R$, Lemma 4.2 yields $\theta_{f_2} \in S_4(\Gamma_0(16))$ having the same L-function up to the Euler factor at 2.

The unique cuspform $\rho_1(z) \in S_4(\Gamma_0(8))$ yields two oldforms of level 16 (namely $\rho_1(z)$ and $\rho_1(2z)$). They give the same eigenvalue (namely, -4) of the Hecke operator T_3, by Stein's table in [13]. The newform of $S_4(\Gamma_0(16))$ has eigenvalue $+4$ for T_3. So f_2, corresponding to eigenvalue -4, comes from an oldform. □

Lemma 4.2. *For every Hecke eigenform $f \in \mathcal{A}(\sigma_2, \mathfrak{O}(l))$, there exists a Hecke eigenform $\theta_f \in S_4(\Gamma_0(2^{l+1}))$ having the same L-function, up to the Euler factor at 2.*

Proof. Let $V = \sum \mathbb{C} f_i$ be the subspace of $\mathcal{A}(\sigma_2, \mathfrak{O}(l))$ spanned by Hecke eigenforms f_i having the same L-function as f, outside of 2. We see that V is stable with respect to the right translation ρ of $D_2^\times$: $\rho(g) f'(x) = f'(xg)$, $f' \in V$, since

$$\rho(g) f'(xk) = f'(xkg) = f'\big(xgg^{-1}kg\big) = \rho(g) f'(x)$$

for every $k \in \mathfrak{O}(l)_2^\times$ and $g \in D_2^\times$ (note that $\mathfrak{O}(l)_2^\times$ is a normal subgroup of $D_2^\times$).

We take an irreducible component Ω taking values on $V_\Omega \subset V$. From a certain automorphic form in V_Ω, we take a function f_Ω, which is an automorphic form in the sense of [7, p. 330]. The right translation by $D_{\mathbb{A}}^\times$ of f_Ω determines the irreducible admissible representation $\pi' = \Omega \times \bigotimes_{v \neq 2} \pi'_v$ of $D_{\mathbb{A}}^\times$.

At ∞, the Weil representation in [7] associates σ_2 to a discrete series representation of $\mathcal{H}_{\mathbb{R}}$. By [5, p. 142], the discrete series are in the space of right $O_2(\mathbb{R})$-finite functions on $GL_2(\mathbb{R})$ such that

$$\phi\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g\right) = \mu_1(t_1)\mu_2(t_2)|t_1/t_2|^{1/2}\phi(g),$$

for the character $\mu_1(a) = |a|^{5/2}$, $\mu_2(a) = |a|^{1/2}$, $a \in \mathbb{R}^\times$.

At 2, Ω is associated to an irreducible admissible representation $\pi_2(\Omega)$ of $GL_2(\mathbb{Q}_2)$ by the Weil representation. We define a Schwartz function $\phi \in \mathcal{S}(D_2) \otimes V_\Omega$ by

$$\phi(k) := \Omega(k)v, \quad k \in \mathfrak{O}_2^\times,$$

for a nonzero $v \in V_\Omega$, and zero if $k \notin \mathfrak{O}_2^\times$. Noting $\Omega|\mathfrak{O}(l)_2^\times$ is trivial, we see ϕ is fixed by the action of $\Gamma_0(2^{l+1})$. Thus, the conductor of $\pi_2(\Omega)$ divides 2^{l+1}.

At the other places, by Theorem 4.4, π'_p are mapped to unramified π_p of $GL_2(\mathbb{Q}_p)$. π_p is related to a cuspform, so is infinite-dimensional, due to Deligne's theorem on Ramanujan's conjecture.

Summing up Theorem 14.4 of [7], Theorem 5.19 of [5] and the above discussions, we get the assertion. □

For the case of $\mathcal{A}(\sigma_0, \mathfrak{O}(l))$, a similar result to the previous lemma is obtained in almost the same way. So, we omit the proof.

Lemma 4.3. *Suppose that $f \in \mathcal{A}(\sigma_0, \mathfrak{O}(l))$ is an eigenform such that*

$$\int_{D_{\mathbb{Q}}^1 \backslash D_{\mathbb{A}}^1} f(h)\, dh = 0. \tag{4.1}$$

Then, there exists a Hecke eigenform $\theta_f \in S_2(\Gamma_0(2^{l+1}))$ having the same L-function, up to the Euler factor at 2.

Theorem 4.4. [10] *Suppose that a definite quaternion algebra $B_{\mathbb{Q}}$ ramifies at only one prime q and at ∞, and that an order $R' \subset B_{\mathbb{Q}}$ is isomorphic to $M_2(\mathbb{Z}_p)$ at every $p \neq q$.*

Then, the theta lifting from $\mathcal{A}(\sigma_{2n}, R')$ to elliptic modular forms is not vanishing. If $n > 0$, or if $n = 0$ and f satisfies (4.1)*, the image is in $S_{2n+2}(\Gamma(q^N))$ for some $N \in \mathbb{N}$.*

Remark 4.5. As mentioned after Theorem 14.4 of [7], we also think that every eigenform f is mapped to an eigen cuspform, except the case of $f(x) = \psi \circ N(x)$, $x \in D_{\mathbb{A}}$, for a certain character ψ on $\mathbb{Q}_{\mathbb{A}}^{\times}$. But, we do not know references showing it.

Next, we will compute the Yoshida lift from f_1 and f_2. We define a theta kernel $\Phi_2 \in \mathcal{S}(D_2^2)$ satisfying the condition (3.1) by

$$\Phi_2(x_1, x_2) = \begin{cases} \mathbf{e}((a_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1, \\ & \quad \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

We check the Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2}(Z)$ at $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not zero.

Theorem 4.6. *The Andrianov L-function of F_2 is equal to $\zeta(s-1)\zeta(s-2)L(s, \rho_1)$, up to the Euler factor at 2. The conjecture for F_2 is true.*

Proof. Put $\Theta(Z) = \Theta_{\Phi, f_1 \times f_2}(8^{-1}Z)$. We can see easily $\Theta \in S_3(\Gamma(4, 8))$ by the properties of the Weil representation at 2 in [14] and the definition of Φ_2. We observe that $\mathrm{N}(x_1), \mathrm{N}(x_2) \in \mathbb{Z}_2^{\times}$ whenever $\Phi_2(x_1, x_2) \neq 0$, and from the action of $\left(\begin{smallmatrix} 1 & S \\ 0 & 1 \end{smallmatrix}\right)$ on $\Theta_{\Phi, f_1 \times f_2}(Z)$ for $S = \left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$, we find $\Theta \notin S_3(\Gamma(4))$.

Hence, one of the seven irreducible $Sp_2(\mathbb{Z})$ modules (excluded that of $F_1 \in S_3(\Gamma(4))$) must contain a Hecke eigenform whose Andrianov L-function is equal to $\zeta(s-1)\zeta(s-2)L(s, \rho_1)$. Consulting the table of eigenvalues of F_i in [4], we see Θ is not orthogonal to the $Sp_2(\mathbb{Z})$ module of F_2.

Thus, observing the eigenvalues at $3, 5, 7$ of F_2, from Proposition 2.2, we find the precise Andrianov L-function of F_2 is equal to the conjectured one. □

4.2. Proof for F_3

We define $f_2^{(-1)} \in \mathcal{A}(\sigma_2, R)$ by

$$f_2^{(-1)}(y_1) = -2bI - 2cJ + dIJ, \qquad f_2^{(-1)}(y_2) = 3dIJ.$$

Similar to the proof of Proposition 4.1, we see $L(s, f_2^{(-1)}) = L(s, f_2 \otimes \omega_{-1})$, where ω_l denotes the quadratic character associated to $\mathbb{Q}(\sqrt{l})/\mathbb{Q}$.

We check $\Theta_{\Phi, f_1 \times f_2^{(-1)}}$ has nonzero Fourier coefficient at $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and thus get the next theorem analogous to Theorem 4.6.

Theorem 4.7. *The Andrianov L-function of F_3 is, up to the Euler factor at 2, equal to $\zeta(s-1)\zeta(s-2)L(s, \rho_1 \otimes \omega_{-1})$. The conjecture for F_3 is true.*

4.3. Proof for F_4

We define the character $\chi_4 = (\chi_4)_2 \times \prod_{v \neq 2} 1_v$ on $R_{\mathbb{A}}^{\times}$ with

$$(\chi_4)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^d,$$

for $k = 1 + 2a + 2bI + 2cJ + 2dIJ \in R_2^{\times}$ and calculate

$$\dim_{\mathbb{C}} \mathcal{A}(\sigma_0, R, \chi_4) = 2, \qquad \dim_{\mathbb{C}} \mathcal{A}(\sigma_2, R, \chi_4) = 6.$$

We define $f_1 \in \mathcal{A}(\sigma_0, R, \chi_4)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi_4)$ by

$$f_1(y_1) = 1, \qquad f_1(y_2) = 0,$$
$$f_2(y_1) = 2bI + cJ, \qquad f_2(y_2) = bI + 2dIJ.$$

Proposition 4.8. *The f_1 and f_2 are Hecke eigenforms and*

$$L(s, f_1) = L(s, \theta_\mu \otimes \omega_{-2}), \qquad L(s, f_2) = L(s, \rho_3 \otimes \omega_{-2}),$$

up to the Euler factor at 2.

Proof. We give only a proof for f_2, since that for f_1 is similar. Since $(\chi_4)_2$ is trivial on $\mathfrak{O}(5)^{\times}$, we have $\mathcal{A}(\sigma_2, R, \chi_4) \subset \mathcal{A}(\sigma_2, \mathfrak{O}(5))$. The same discussion as in the proof of Proposition 4.1 tells that the Jacquet–Langlands correspondence maps $\mathcal{A}(\sigma_2, R, \chi_4)$ ($\subset \mathcal{A}(\sigma_2, \mathfrak{O}(5))$) to $S_4(\Gamma_0(64))$. We calculate the Brandt matrices (representing matrix of the Hecke algebra on $\mathcal{A}(\sigma_2, R, \chi_4)$) and obtain

Space	Eigenvalues at 3	Eigenvalues at 5
$\mathcal{A}(\sigma_2, R, \chi_4)$	$\{\pm 8, 0\}$	$\{-22, 10\}$

We see f_2 has eigenvalues 10 at 5 and 8 at 3.

On the other hand, Stein's table tells that

Space	Eigenvalues at 3	Eigenvalues at 5
$\mathbb{C}\rho_3 \subset S_4(\Gamma_0(32))$	8	-10
$S_4(\Gamma_0(32))$	$\{\pm 8, \pm 4, 0\}$	$\{22, -10, 2\}$
$S_4(\Gamma_0(64))$	$\{\pm 8, \pm 4, 0\}$	$\{\pm 22, \pm 10, \pm 2\}$

Thus, by Proposition 3.64 of [12], we find that $\rho_3 \otimes \omega_{-2}$ belongs to $S_4(\Gamma_0(64))$. Stein's table tells that only $\rho_3 \otimes \omega_{-2}$ has eigenvalue 8 at 3 and 10 at 5.

Taking into account that $\mathcal{A}(\sigma_2, R, \chi_4)$ is spanned by eigenforms, we can easily conclude f_2 is an eigenform outside of 2 with L-function $L(s, \rho_3 \otimes \omega_{-2})$. $\square$

We define a theta kernel Φ_2 associated to the pair of f_1 and f_2 by

$$\Phi_2(x_1, x_2) = \begin{cases} \mathbf{e}((a_1+d_1+b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1, \\ & \quad \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then the Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2}(Z)$ at $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not zero, from which we obtain the next theorem analogous to Theorem 4.6.

Theorem 4.9. *The Andrianov L-function of F_4 is, up to the Euler factor at 2, equal to $L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \rho_3 \otimes \omega_{-2})$. The conjecture for F_4 is true.*

4.4. Proofs for F_5 and F_6

Suppose that the conjectures for F_5 and F_6 are true. By Proposition 2.2, we notice that there exist eigenforms with the same Andrianov L-function in the different modules $Sp_2(\mathbb{Z}) \cdot F_5$ and $Sp_2(\mathbb{Z}) \cdot F_6$. It is not sufficient to construct eigenforms in $S_3(\Gamma(4,8))$ having the Andrianov L-functions, different form the previous cases. The eigenforms obtained by the Yoshida lift may be in the same $Sp_2(\mathbb{Z})$ module. So, after the constructions, we will see that they are belonging to different $Sp_2(\mathbb{Z})$ modules.

We will first prove the conjecture for F_5. Define a character $\chi_5 = (\chi_5)_2 \times \prod_{v \neq 2} 1_v$ on $R_{\mathbb{A}}^\times$ with

$$(\chi_5)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^{b+c}.$$

We define $f_1 \in \mathcal{A}(\sigma_0, R, \chi_5)$ and $f_2 \in \mathcal{A}(\sigma_2, R, \chi_5)$ by

$$\begin{aligned} f_1(y_1) &= 1, & f_1(y_2) &= 0, \\ f_2(y_1) &= 0, & f_2(y_2) &= 2bI + 2cJ - dIJ. \end{aligned}$$

The next proposition is analogous to Proposition 4.8.

Proposition 4.10. *The above f_1 and f_2 are Hecke eigenforms outside of 2 with*

$$L(s, f_1) = L(s, \theta_\mu), \qquad L(s, f_2) = L(s, \theta_{\mu^3}),$$

up to the Euler factor at 2.

Associated to the pair f_1 and f_2, we define

$$\Phi_2(x_1, x_2) = \begin{cases} \mathbf{e}(d_2/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1 + J + IJ, \\ & \quad \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I + J \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

This theta kernel is the four-fold product of the Igusa theta constants (see Introduction and Main idea of [9])

$$\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}(Z),$$

which is the complement of F_5 of ten-fold product of all even theta constants. Using Proposition 6.2 of [4] and Lemma 2.2 of [11], we see the ten-fold product belongs to $S_5(\Gamma(2))$. Hence the four-fold product has the same character χ_{F_5} on $\Gamma(2)$ (note that χ_{F_5} is $\{\pm 1\}$-valued). Of course, $\Theta_{\Phi, f_1 \times f_2}$ has the same character χ_{F_5}. Thus, we conclude that $\Theta_{\Phi, f_1 \times f_2}$ is in the $Sp_2(\mathbb{Z})$-orbit of F_5, consulting the lengths of the orbits in Theorem 6.4 of [4] which is the classification of characters on $\Gamma(2)$. The Fourier coefficient of $\Theta_{\Phi, f_1 \times f_2}(Z)$ at $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is not zero. Hence, consulting eigenvalues at 3, 5, 7, we have

Theorem 4.11. *Up to the Euler factor at* 2*, the Andrianov* L*-function of* F_5 *is equal to* $L(s-1, \theta_\mu)L(s, \theta_{\mu^3})$*. The conjecture for* F_5 *is true.*

We are going to prove the conjecture for F_6. Define a character $\chi_6 = (\chi_6)_2 \times \prod_{v \neq 2} 1_v$ on $R_{\mathbb{A}}^\times$ with

$$(\chi_6)_2(1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^c.$$

We define $f_1' \in \mathcal{A}(\sigma_0, R, \chi_6)$ and $f_2' \in \mathcal{A}(\sigma_2, R, \chi_6)$ by

$$\begin{aligned} f_1'(y_1) &= 0, & f_1'(y_2) &= 1, \\ f_2'(y_1)(bI + cJ + dIJ) &= 0, & f_2'(y_2)(bI + cJ + dIJ) &= 2b - c + 2d. \end{aligned}$$

The next proposition is analogous to Proposition 4.8.

Proposition 4.12. *The above* f_1' *and* f_2' *are Hecke eigenforms outside of* 2 *and* $L(s, f_1') = L(s, \theta_\mu \otimes \omega_{-2})$*,* $L(s, f_2') = L(s, \theta_{\mu^3} \otimes \omega_{-2})$*, up to the Euler factor at* 2*.*

Associated to the pair f_1' and f_2', we define

$$\Phi_2'(x_1, x_2) = \begin{cases} \mathbf{e}((a_1 + c_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1, \\ & \quad \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier coefficient of $\Theta_{\Phi', f_1' \times f_2'}(Z)$ at $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not zero.

Theorem 4.13. *Up to the Euler factor at* 2*, the Andrianov* L*-function of* F_6 *is equal to* $L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \theta_{\mu^3} \otimes \omega_{-2})$*. The conjecture for* F_6 *is true.*

Proof. From the definitions we see, for $k = (1 + 2IJ)(1 + 2J)^{-1} \in R_2^1$,

$$(\chi_5)_2(k) = 1 \neq -1 = (\chi_6)_2(k).$$

Thus, using Lemma 4.14, we know that $\Theta_{\Phi', f_1' \times f_2'}$ cannot belong to $Sp_2(\mathbb{Z}) \cdot F_5$. Consulting the table in [4] for some Euler factors of Andrianov L-functions of F_i, we find that $\Theta_{\Phi', f_1' \times f_2'}$ belongs to $Sp_2(\mathbb{Z}) \cdot F_6$ (and not to the orbit $Sp_2(\mathbb{Z}) \cdot F_5$). Consulting the eigenvalues of F_6 at 3, 5, 7, we determine the precise Andrianov L-function of F_6. □

Lemma 4.14. *Let p be a bad prime and π_p be the Weil representation of $Sp_2(\mathbb{Q}_p)$. The property* (3.1) *of the theta kernel Φ_p is stable for translations by $Sp_2(\mathbb{Q}_p)$:*

$$\big(\pi_p(g)\Phi_p\big)\big(k_1^{-1}x_1k_2, k_1^{-1}x_2k_2\big) = \chi_p\big(k_1^{-1}k_2\big)\big(\pi_p(g)\Phi_p\big)(x_1, x_2)$$

for every $g \in Sp_2(\mathbb{Q}_p)$, $k_i \in R_p^1$ and $x_j \in D_p$.

Proof. Obvious from the fact that the action of $Sp_2(\mathbb{Q}_p)$ commutes with that of R_p^1 on Φ_p. □

Remark 4.15. van Geemen and Nygaard [3] showed that the L-function of the Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL\big(H^3(Y', \mathbb{Q}_l)\big) \simeq GL_4(\mathbb{Q}_l)$$

is equal to $L(s-1, \mu)L(s, \mu^3)$. Here Y' is a resolution of $\ker(\chi_{F_5}) \backslash \mathfrak{H}_2$. So, we have

$$L(s, \rho) = L(s, F_5).$$

4.5. Proof for F_1

In [3], the Andrianov L-function of F_1 was determined by using Oda lift [8] (Converse of Saito–Kurokawa lift).

Our method using Yoshida lift is also effective to F_1. We only write down the automorphic forms and theta kernel. We set $R' = \mathbb{Z} + 2\mathbb{Z}I + 2\mathbb{Z}J + \mathbb{Z}(I + J + IJ)$ and have $D_{\mathbb{A}}^\times = D^\times(R'_{\mathbb{A}})^\times$. Define the character χ on $(R'_2)^\times$ by $\chi(k) = \omega_{-1}(N(k))$. The automorphic forms are

$$f_1(1) = 1, \qquad f_2(1) = bI,$$

and we set the theta kernel Φ_2 by

$$\Phi_2(x_1, x_2) = \begin{cases} \mathbf{e}((a_1 + b_2)/4) & \text{if } x_1 = a_1 + b_1 I + c_1 J + d_1 IJ \equiv 1 + I, \\ & \quad \text{and } x_2 = a_2 + b_2 I + c_2 J + d_2 IJ \equiv I + J \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

One can show easily that $\Theta_{\Phi, f_1 \times f_2}$ belongs to $S_3(\Gamma(4))$. Its Fourier coefficient at $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is not zero.

Remark 4.16. In the constructions of Yoshida lifts above, we use odd Igusa theta constants. For example, for the above F_1, we use

$$\theta_{(1,0,1,0)}\theta_{(1,1,1,0)}\theta_{(0,1,0,0)}\theta_{(0,0,0,0)}(Z).$$

In the case of F_5, we use

$$\theta_{(0,0,0,1)}\theta_{(1,1,0,1)}\theta_{(0,1,0,1)}\theta_{(0,0,0,0)}(Z),$$

which is obtained from the four-fold product of even theta constants

$$\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}(Z)$$

by translating over $y_2 = I + J + IJ \in D_2^{\times}$. Using the polynomial P described in Section 3, one verifies that they do not vanish. In contrast, if Φ_2 is obtained from a four-fold product of even theta constants, then the theta kernel $\sum_{x_i \in D} P(x_1^* x_2) \times \Phi(x_1, x_2)\mathbf{e}[x_1, x_2, Z]$ vanishes.

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