A conjecture on a Shimura type correspondence for Siegel modular forms, and Harder's conjecture on congruences

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For modular forms of one variable there is the famous correspondence of Shimura between modular forms of integral weight and half integral weight (cf. [15]). In this paper, we propose a similar conjecture for vector valued Siegel modular forms of degree two and provide numerical evidence and conjectural dimensional equality (Section 1, Main Conjecture 1.1; a short announcement was made in [11].) We also propose a half-integral version of Harder's conjecture in [4]. Our version is deduced in a natural way from our Main Conjecture. While the original conjecture deals with congruences between eigenvalues of Siegel modular forms and modular forms of one variable our version is stated as a congruence between L-functions of a Siegel cusp form and a Klingen type Eisenstein series.

We give here a rough indication of the content of our Main Conjecture. This is restricted to the case of level one, but stated as a precise bijective correspondence as follows.

Conjecture *For any natural number $k \geq 3$ and any even integer $j \geq 0$, there is a linear isomorphism*

$$S^{+}_{\det^{k-1/2}\,\mathrm{Sym}(j)}\left(\Gamma_0(4), \left(\frac{-4}{*}\right)\right) \cong S_{\det^{j+3}\,\mathrm{Sym}(2k-6)}(\mathrm{Sp}(2,\mathbb{Z}))$$

which preserves L-functions.

Here the superscript $+$ means a certain subspace of new forms or a "level one" part. The details of the notation and our definitions of L-functions will be explained in section 1.

In section 1, after reviewing the definitions of Siegel modular forms of integral and half-integral weight and their L-functions, we give a precise statement of our Main Conjecture and a half-integral version of Harder's conjecture. In section 2 we compare dimensions and also give a supplementary conjecture on dimensions. In section 3 we give numerical examples which support our

https://doi.org/10.1017/CBO9780511543371.010

conjecture. In section 4 we explain how to calculate the numerical examples. In section 5 we review correspondence between Jacobi forms and Siegel modular forms of half-integral and define vector valued Klingen type Eisenstein series which is used in the half-integral version of Harder's conjecture. In the appendix we give tables of Fourier coefficients which we used.

The author would like to thank Professor Tsushima for showing him a conjectural dimension formula of vector valued Jacobi forms, which gave the author the motivation to start this research.

1 Main Conjecture

In this section, after reviewing the definitions quickly, we give our main conjecture.

1.1 Vector valued Siegel modular forms of integral weight

We denote by H_n the Siegel upper half space of degree n.

$$H_n = \left\{ Z = X + iY \in M_n(\mathbb{C}) \;\middle|\; X = {}^tX, Y = {}^tY \in M_n(\mathbb{R}), Y > 0 \right\},$$

where $Y > 0$ means that Y is positive definite. For any natural number N, we put

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{Z}) \;\middle|\; {}^tgJg = J, C \equiv 0 \bmod N \right\}$$

where $J = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ and 1_n or 0_n is the $n \times n$ unit or the zero matrix. When $n = 2$, we sometimes write $\Gamma_0^{(2)}(N) = \Gamma_0(N)$. We also write $\Gamma_n = \Gamma_0^{(n)}(1) = \mathrm{Sp}(n, \mathbb{Z})$.

Now we define vector valued Siegel modular forms of degree $n = 2$ of integral weight and their spinor L-functions. First we recall the irreducible representations of $\mathrm{GL}_2(\mathbb{C})$. For variables u_1, u_2 and $g \in \mathrm{GL}_2(\mathbb{C})$, we put $(v_1, v_2) = (u_1, u_2)g$. We define the $(j+1) \times (j+1)$ matrix $\mathrm{Sym}_j(g)$ by

$$(v_1^j, v_1^{j-1}v_2, \ldots, v_2^j) = (u_1^j, u_1^{j-1}u_2, \ldots, u_2^j)\mathrm{Sym}_j(g).$$

Then Sym_j gives the symmetric tensor representation of degree j of $\mathrm{GL}_2(\mathbb{C})$. We denote by $V_j \cong \mathbb{C}^{j+1}$ the representation space of Sym_j. The space V_j can be identified with the space of polynomials $P(u, v)$ in two variables u, v of homogeneous degree j, where the action is given by $P((u, v)g)$ for $g \in \mathrm{GL}_2$. If ρ is a rational irreducible representation of $\mathrm{GL}_2(\mathbb{C})$, then there exist an integer k and a positive integer j such that $\rho = \det^k \mathrm{Sym}_j$. We denote this representation by $\rho_{k,j}$. Any V_j-valued holomorphic function $F(Z)$ of H_2 is said to be a Siegel modular form of weight $\rho_{k,j}$ belonging to Γ_2 if we have

$$F(\gamma Z) = \rho_{k,j}(CZ + D)F(Z)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$. We denote by $A_{k,j}(\Gamma_2)$ the linear space over $\mathbb{C}$ of these functions. If j is odd, then -1_4 acts as multiplication by -1 and we have $A_{k,j}(\Gamma_2) = 0$. We define the Siegel Φ-operator by

$$(\Phi F)(\tau_1) = \lim_{\lambda \to \infty} F\begin{pmatrix} \tau_1 & 0 \\ 0 & i\lambda \end{pmatrix},$$

where $\tau_1 \in H_1$. It is well-known that all the components of the vector $\Phi(F)$ except for the first one always vanish and the first component is in $S_{k+j}(\Gamma_1)$ (e.g. [1]). If $\Phi(F) = 0$ for $F \in A_{k,j}(\Gamma_2)$, we say that F is a cusp form. We denote by $S_{k,j}(\Gamma_2)$ the space of cusp forms. If k is odd (and j is even), then since $S_{k+j}(\Gamma_1) = 0$, we have $A_{k,j}(\Gamma_2) = S_{k,j}(\Gamma_2)$.

Now we define Hecke operators and the spinor L functions. For any natural number m, we put

$$T(m) = \{\delta \in M_4(\mathbb{Z});\ {}^t\delta J \delta = mJ\}.$$

The action of $T(m)$ on $F \in M_{j.k}$ is defined by

$$F|_{(k,j)}T(m) = m^{2k+j-3} \sum_{M \in \Gamma_2 \backslash T(m)} F|_{(k,j)}[M]$$

where we put

$$F|_{(k,j)}M = \rho_{k,j}(CZ + D)^{-1}F(MZ)$$

for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in T(m)$. For a common Hecke eigenform $F \in S_{k,j}(\Gamma_2)$ we denote by $\lambda(p^\nu)$ the eigenvalue of $T(p^\nu)$, i.e., we put $T(p^\nu)F = \lambda(p^\mu)F$. The spinor L-function $L(s, F)$ of the common Hecke eigenform $F \in S_{k,j}(\Gamma_2)$ is defined to be

$$L(s, F, Sp) = \prod_{p:\,\text{prime}} L_p(s, F)$$

where $L_p(s, F)$ equals

$$\left(1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^{\mu-1})p^{-2s} - \lambda(p)p^{\mu-3s} + p^{2\mu-4s}\right)^{-1}$$

and $\mu = 2k + j - 3$ (cf. e.g. [1]).

1.2 Vector valued Siegel modular forms of half-integral weight

First we define Siegel modular forms of half integral weight with or without character. We denote by ψ the Dirichlet character modulo 4 defined by $\psi(a) = \left(\frac{-4}{a}\right)$ for any odd a. We define a character of $\Gamma_0(4)$ by $\psi(\det(D))$ for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(4)$ and denote this character also by ψ. To fix an automorphy factor of half-integral weight, we define a theta function on H_2 by

$$\theta(Z) = \sum_{p \in \mathbb{Z}^2} e({}^t pZp),$$

where $e(x) = \exp(2\pi i x)$. A vector valued Siegel modular form F of weight $\det^{k-1/2} \mathrm{Sym}_j$ belonging to $\Gamma_0(4)$ with character ψ^l ($l = 0$ or 1) is defined to be a V_j-valued holomorphic function $F(Z)$ of H_2 such that

$$F(\gamma Z) = \psi(\gamma)^l \left(\frac{\theta(\gamma Z)}{\theta(Z)}\right)^{2k-1} \mathrm{Sym}_j(CZ + D)F(Z)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(4)$. Let $A_{k-1/2,j}(\Gamma_0(4), \psi^l)$ denote the space of these functions. We note that if j is odd, then $A_{k-1/2,j}(\Gamma_0(4), \psi^l) = 0$ since -1_4 acts as multiplication by -1. We also see that any $F \in A_{k-1/2,j}(\Gamma_0(4), \psi)$ (here $l = 1$) is a cusp form since there are no half-integral modular forms of $\Gamma_0^{(1)}(4)$ with character ψ. For modular forms of half integral weight of one variable, Kohnen introduced the "plus" subspace to pick up a "level one" part and has shown that it is isomorphic to the space of modular forms of integral weight of level one (see [14]). Later it was shown that that this space is also isomorphic to the space of Jacobi forms of index one in Eichler-Zagier [3]. This notion of plus space was generalized to general degree and used in the comparison with holomorphic and skew holomorphic Jacobi forms of general degree (cf. [9], [5],[7]). We review this "plus" subspace for our case. We write the Fourier expansion of $F \in S_{k-1/2,j}(\Gamma_0(4), \psi)$ by

$$F(Z) = \sum_T a(T)e(\mathrm{Tr}(TZ))$$

where T runs over half-integral positive definite symmetric matrices. The subspace of $S_{k-1/2,j}(\Gamma_0(4), \psi^l)$ consisting of those F such that $a(T) = 0$ unless $T \equiv (-1)^{k+l-1}\mu^t\mu \bmod 4$ for some column vector $\mu \in \mathbb{Z}^2$ is called a plus subspace and denoted by $S^+_{k-1/2,j}(\Gamma_0(4), \psi^l)$. This is a higher dimensional analogue of the Kohnen plus space and should be regarded as the level one part of $S_{k-1/2,j}(\Gamma_0(4), \psi^l)$. In section 5, we review an isomorphism of this space to the space of holomorphic or skew holomorphic Jacobi forms.

The theory of Hecke operators on Siegel modular forms of half-integral weight was developped by Zhuravlev [21] [22]. (See also Ibukiyama [8] in case of vector valued forms of degree two.) We review it here. (Our normalization is slighly different from his original definition.) We define $\widetilde{\mathrm{GSp}}^{+}(2,\mathbb{R})$ as the set of elements $(g, \phi(Z))$ where

$$g \in \mathrm{GSp}^{+}(2,\mathbb{R}) = \left\{ g \in \mathrm{GL}_4(\mathbb{R}) \;\middle|\; {}^t g J g = n(g) J, n(g) \in \mathbb{R}_{+}^{\times} \right\}$$

and $\phi(Z)$ is a holomorphic function such that $|\phi(Z)| = |\det(CZ + D)|^{1/2}$. $\widetilde{\mathrm{GSp}}^{+}(2,\mathbb{R})$ becomes a group via the product

$$(g_1, \phi_1(Z))(g_2, \phi_2(Z)) = (g_1 g_2, \phi_1(g_2 Z)\phi_2(Z)).$$

We can identify $\Gamma_0(4)$ with a subgroup $\widetilde{\Gamma}_0(4)$ of $\widetilde{\mathrm{GSp}}^{+}(2,\mathbb{R})$ by embedding $\gamma \mapsto (\gamma, \theta(\gamma Z)/\gamma(Z))$. For any element $(g, \phi(Z)) \in \widetilde{\mathrm{GSp}}^{+}(2,\mathbb{R})$ with ${}^t g J g = m^2 J$, we put

$$g' = m^{-1} g = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

We define an action of $\mathrm{GSp}^{+}(2,\mathbb{R})$ on V_j-valued functions F on H_2 by

$$F|_{k-1/2,j}[(g, \phi(Z))] = \mathrm{Sym}_j(C_1 Z + D_1)^{-1} \phi(Z)^{-2k+1} F(gZ).$$

For any prime number p, we put

$$K_1(p^2) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, p^{1/2} \right) \qquad K_2(p^2) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}, p \right)$$

For the $\widetilde{\Gamma}_0(4)$ double cosets

$$T_i(p) = \tilde{\Gamma}_0(4) K_i(p^2) \tilde{\Gamma}_0(4) = \cup_\nu \tilde{\Gamma}_0(4) \tilde{g_\nu}$$

we define

$$F|_{k-1/2,j} T_i(p) = p^{i(k+j-7/2)} \sum_{\nu} F|_{k-1/2,\rho}[\tilde{g_\nu}] \psi(\det(D))$$

where D is right lower 2×2-matrix of $n(g_\nu)^{-1/2} g_\nu \in \mathrm{Sp}(2,\mathbb{R})$ and g_ν is the projection of $\tilde{g_\nu}$ on its first argument. For any odd prime p and any $F \in A_{k-1/2,j}(\Gamma_0(4), \psi^l)$, assume that $F|T_1(p) = \lambda(p) F$ and $F|T_2(p) = \omega(p) F$. We put $\lambda^*(p) = \psi(p)^l \lambda(p)$, where, as before, $\psi(p) = \left(\frac{-1}{p}\right)$. Then the Euler p-factor of the L-function of F is defined to be

$$\left(1 - \lambda^*(p) p^{-s} + (p\omega(p) + p^{\nu-2}(1+p^2)) p^{-2s} - \lambda^*(p) p^{\nu-3s} + p^{2\nu-4s}\right)^{-1},$$

where $\nu = 2k + 2j - 3$. We remark that when $p = 2$, we can also define an Euler 2-factor for $F \in S^{+}_{k-1/2,j}(\Gamma_0(4), \psi)$ in the same way as in [7]. Indeed, we can similarly define $T_i^*(2)$ as in [9] and [5] by the pull back of the Hecke operators on holomorphic or skew holomorphic Jacobi forms. Denoting by $\lambda^*(2)$ and $\omega^*(2) = \omega(2)$ the eigenvalues of these operators, we can then define an Euler 2-factor as above. For details, see [7] or section 4.2 and 5 of this paper.

1.3 Main Conjecture

We propose the following conjecture.

Conjecture 1.1. *For any integer $k \geq 3$ and even integer $j \geq 0$, there exists a linear isomorphism ϕ of $S^+_{k-1/2,j}(\Gamma_0(4), \psi)$ onto $S_{j+3,2k-6}(\Gamma_2)$ such that*

$$L(s, F) = L(s, \phi(F), \mathrm{Sp}).$$

In the above, the scalar valued case occurs only when $j = 0$ on the left or $k = 3$ on the right, respectively and when both sides are scalar valued, they are zero. So it is essential to treat the vector valued forms. The above conjecture is false when j is odd, since the left hand side is zero and the right hand side is not zero in general in this case. It is not clear which kind of modification is necessary for odd j.

This conjecture can be proved in principle by Selberg trace formula, but no concrete trace formula is known at the moment except for dimension formulas. To use trace formulas, we need a conjecture comparing not only $T(p^\delta)$ or $T_i(p)$ but all the Hecke operators. So we would like to describe this. Let a, b, c and d be natural numbers such that $a \leq b \leq d \leq c$ with $a+c = b+d = 2\delta$ for some natural integer δ. We denote by $T(p^a, p^b, p^c, p^d)$ the Hecke operator obtained by the Γ_2 double coset determined by the diagonal matrix whose diagonal components are (p^a, p^b, p^c, p^d). If we take sums of two of a, b, c, d, respectively, we have six sums but only two of them equal δ and there are four other terms. Our guess for comparing the Hecke operators is

$$T(p^a, p^b, p^c, p^d) \to \psi(p)^\delta T_{\mathrm{half}}(p^{a+b}, p^{a+d}, p^{c+d}, p^{b+c})$$

up to normalizing factors, where T_{half} is the Hecke operator defined by the $\widetilde{\Gamma}_0(4)$ double coset containing the diagonal matrix in the parenthesis.

We explain several reasons why we believe our Main Conjecture.

(1) The above weight correspondence is explained as follows. By the Langlands conjectures, Siegel modular forms of $\mathrm{Sp}(2, \mathbb{Q})$ should correspond to automorphic forms belonging to the compact twist whose real form is $\mathrm{Sp}(2) = \{g \in M_2(\mathbb{H}); g^t g = 1_2\}$, where $\mathbb{H}$ denotes the Hamilton quaternions. It was observed by Y. Ihara (cf. [12]) that the holomorphic discrete series representation of $\mathrm{Sp}(2, \mathbb{R})$ corresponding to the weight $\det^k \mathrm{Sym}_j$ should correspond

to the irreducible representation of $\mathrm{Sp}(2)$ corresponding to the Young diagram $(k+j-3, k-3)$ by comparing the character of the representations. On the other hand, $\mathrm{Sp}(2)$ is isogenous to $SO(5)$ and starting from automorphic forms belonging to $SO(5)$, by the theta correspondence we can construct Siegel modular forms of the double cover of $\mathrm{Sp}(2,\mathbb{R})$ of weight $\det^{(j+5)/2}\mathrm{Sym}_{k-3}$ (cf. [8]). In other words, the weight $\det^{k-1/2}\mathrm{Sym}(j)$ should correspond the weight $\det^{j+3}\mathrm{Sym}_{2k-6}$ as stated as above and we have no other choice. By the way, we cannot prove our conjecture by this theta correspondence, since in our case the level equals one, while the compact $\mathrm{Sp}(2)$ has always a level greater than one.

(2) The dimension of $S_{k,j}(\Gamma_2)$ is known by Igusa for $j=0$ and by Tsushima for $j>0$ under the condition that $k \geq 5$. On the other hand, the dimension of half-integral Siegel modular forms are known by Tsushima. Furthermore, the dimension of holomorphic Jacobi forms and skew holomorphic Jacobi forms are known also by Tsushima as far as we assume a standard vanishing theorem of cohomology which has not been proved in the non-scalar valued case. So, we have a proven dimension formula for $S_{k,j}(\Gamma_2)$ and conjectural dimensions for $S^{+}_{k-1/2,j}(\Gamma_0(4),\psi)$ for $k>4$. We can compare these two and we can show that they coincide.

These proven and conjectural dimensions are given in the table in section 2.

(3) Several numerical examples of Euler factors which support the conjecture will be given for spaces of small dimensions in section 3.

1.4 A half-integral version of Harder's conjecture

In his paper [4], Harder proposed a conjecture on certain congruences between eigenvalues of Siegel modular forms and modular forms of one variables. I understand that it can be stated as follows. Let $f \in S_{2k+j-2}(SL_2(\mathbb{Z}))$ be a common Hecke eigenform of weight $2k+j-2$ of one variable with p-th eigenvalue $c(p)$. Then there exists a Siegel modular form $F \in S_{k,j}(\Gamma_2)$ which is a Hecke common eigenform with eigenvalues $\lambda(p^\delta)$ with respect to $T(p^\delta)$ such that the following condition is satisfied.

$$
\begin{aligned}
1-\lambda(p)T &+ \left(\lambda(p)^2-\lambda(p^2)-p^{2k+j-4}\right)T^2 \\
&\qquad -\lambda(p)p^{2k+j-3}T^3+p^{4k+2j-6}T^4 \\
&\equiv \left(1-p^{k-2}T\right)\left(1-p^{k+j-1}T\right)\left(1-c(p)T+p^{2k+j-3}T^2\right) \bmod \mathfrak{l},
\end{aligned}
$$

where T is a variable and $\mathfrak{l}$ is a certain prime ideal which divides a certain critical value of $L(s,f)$. (For a deeper explanation, see [4].) The left hand side is the Euler p-factor of F if we put $T=p^{-s}$. But, as far as we can see from examples, there are no Siegel modular forms having the right hand side as the Euler p-factor.

Now since we have a conjectural correspondence between $S_{k,j}(\Gamma_2)$ and $S^+_{(j+5)/2,k-3}(\Gamma_0(4),\psi)$ for odd k, we can give a half-integral version of Harder's conjectures, and in this case we can say more. We assume that $k \geq 3$ and $j \geq 0$ is even. We take $f \in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ as above. Then there exists $g \in S^+_{k+j/2-1/2}(\Gamma_0^{(1)}(4))$ which corresponds to f by Shimura correspondence (cf. Kohnen [14]). As we shall see in section 5, when $j+3>5$, associates to g there exists a Klingen type Eisenstein series $E_{((j+5)/2,k-3)}(Z,g) \in S^+_{(j+5)/2,k-3}(\Gamma_0(4))$ (*without character*) such that

$$L(s, E_{(j+5)/2,k-3}(Z,g)) = \zeta(s-k+2)\zeta(s-k-j+1)L(s,f)$$

Hence we propose

Conjecture 1.2. *Assume that $k>5$. For any Klingen type Eisenstein series $E(Z,g) \in S^+_{k-1/2,j}(\Gamma_0(4))$ as above (with $g \in S^+_{k+j-1/2}(\Gamma_0^{(1)}(4)) \cong S_{2k+2j-2}(\mathrm{SL}_2(\mathbb{Z}))$), there exists a Hecke eigen cusp form*

$$F \in S^+_{k-1/2,j}(\Gamma_0(4),\psi)$$

such that the Hecke eigenvalues are congruent to those of $E(Z,g)$ modulo the above ideal $\mathfrak{l}$.

This type of congruence between cusp forms and Eisenstein series are well-known for the one variable case, so it seems interesting to state Harder's conjecture in this way.

2 Dimension formulas

We review here Tsushima's formula for $\dim S_{k,j}(\Gamma_2)$ for $k \geq 5$ in [18]. Tsushima also gave a conjectural dimension formulas for vector valued holomorphic or skew holomorphic Jacobi forms of any index under the assumption that $k \geq 4$ and assuming a standard conjecture on the vanishing of obstruction cohomology, which is satisfied when $j=0$. By the isomorphism we shall define in section 5 this implies also conjectural dimension formulas for the plus space $S^+_{k-1/2,j}(\Gamma_0(4),\psi^l)$. He stated his results in the form of polynomials in k and j defined accordingly to the residue classes of k, j modulo certain natural numbers. Here we restate these results using generating functions and find:

Theorem 2.1. *For $k \geq 4$ and even $j \geq 2$,* $\dim S_{j+3,2k-6}(\Gamma_2)$ *is equal to the conjectural formula of* $\dim S^+_{k-1/2,j}(\Gamma_0(4),\psi)$.

For small k or j, examples for the dimensions in question are given as follows.

$$\sum_{j=0}^{\infty} \dim S_{5,j}(\Gamma_2)s^j = \frac{s^{18}+s^{20}+s^{24}}{(1-s^6)(1-s^8)(1-s^{10})(1-s^{12})}$$

$$\sum_{j=0}^{\infty} \dim S_{7,j}(\Gamma_2)s^j = \frac{s^{12}+s^{14}+s^{16}+s^{18}+s^{20}}{(1-s^6)(1-s^8)(1-s^{10})(1-s^{12})}$$

$$\sum_{k=1}^{\infty} \dim S^+_{k-1/2,0}(\Gamma_0(4),\psi)t^k = \frac{t^{21}}{(1-t^3)(1-t^4)(1-t^5)(1-t^6)}$$

$$\sum_{k=1}^{\infty} \mathrm{cdim} S^+_{k-1/2,2}(\Gamma_0(4),\psi)t^k = \frac{t^{12}(1+t+t^3)}{(1-t^3)(1-t^4)(1-t^5)(1-t^6)}$$

$$\sum_{k=1}^{\infty} \mathrm{cdim} S^+_{k-1/2,4}(\Gamma_0(4),\psi)t^k = \frac{t^9(1+t+t^2+t^3+t^4)}{(1-t^3)(1-t^4)(1-t^5)(1-t^6)}$$

where cdim means the conjectured dimension.

For the reader's convenience, we quote here Tsushima's formula for $\dim S_{k,j}(\Gamma_2)$ for any odd $k \geq 5$ and even j using generating functions. The values $\dim S_{3,j}(\Gamma_2)$ are not known, but in Table 1 below, we give them as our conjecture. (See below.)

We have

$$\sum_{j=0}^{\infty}\sum_{k=3}^{\infty} \dim S_{2j+3,2k-6}(\Gamma_2)t^k s^j$$
$$= \frac{f(t,s)}{(1-s^2)(1-s^3)(1-s^5)(1-s^6)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}$$

where $f(t,s)$ is given below and $\dim S_{3,2k-6}(\Gamma_2)$ (the case $j = 0$) are conjectural values.

The coefficients of $t^k s^j$ of $f(t,s)$ are given in Table 1. A part of our main conjecture says that we should have

$$S^+_{5/2,j}(\Gamma_0(4),\psi) \cong S_{j+3,0}(\Gamma_2)$$

and

$$S_{3,2k-6}(\Gamma_2) \cong S^+_{k-1/2,0}(\Gamma_0(4),\psi).$$

In each case, while the dimension of the right hand side is known for all k or j, the one for the left hand side is not known. So, assuming these isomorphisms, we are naturally led to the following conjecture on dimension.

Table 1.

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	–1	0
5	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0
6	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	–1
7	0	0	0	1	1	1	2	2	1	–1	0	0	–1	–1	0	1	–1
8	0	0	0	1	1	1	2	2	1	–1	–1	0	–2	–2	0	1	–1
9	0	0	1	2	2	1	0	0	–1	–3	–3	0	–1	–1	0	1	–1
10	0	0	1	1	1	1	0	–1	–2	–3	–4	–1	–2	–1	0	1	1
11	0	0	1	1	1	1	–2	–3	–4	–3	–4	–2	0	1	1	0	1
12	0	1	1	0	–1	–1	–3	–4	–3	–2	–2	–1	1	1	0	0	2
13	0	1	1	–1	–1	–1	–3	–5	–4	0	0	–1	2	3	1	–1	1
14	0	0	0	0	–1	–2	–2	–2	–2	–1	1	0	2	2	1	0	1
15	0	1	0	–1	–2	–2	–1	–2	–1	2	4	1	2	2	1	–1	–1
16	0	0	0	0	–1	–1	0	0	0	1	3	1	1	1	1	0	–1
17	0	0	0	0	–1	–1	1	1	1	1	3	2	0	0	0	0	–1
18	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	–1
19	0	0	0	0	0	0	0	1	2	1	0	1	0	–1	–1	0	0
20	0	0	0	0	1	1	0	0	0	0	–1	0	0	0	0	0	0
21	1	0	–1	–1	0	0	–1	1	2	1	–1	0	0	–1	–1	0	1

Conjecture 2.2. *We have*

$$\sum_{j=0}^{\infty} \dim S^{+}_{5/2,j}(\Gamma_0(4),\psi)t^j = \frac{t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

$$\sum_{j=0}^{\infty} \dim S_{3,j}(\Gamma_2)s^j = \frac{s^{36}}{(1-s^6)(1-s^8)(1-s^{10})(1-s^{12})}.$$

Actually, if $j > 0$, these conjectured dimensions are equal to those obtained by putting $k = 3$ in the general formula of $\dim S^{+}_{k-1/2,j}(\Gamma_0(4),\psi)$ or by putting $k = 3$ in the general formula of $\dim S_{k,j}(\Gamma_2)$.

3 Numerical examples

We have the following table of dimensions of $S_{k,j}(\Gamma_2)$ due to Tsushima (cf. [18]).

(k,j)	(5,18)	(5,20)	(5,22)	(5,24)	(5,26)
dim $S_{k,j}(\Gamma_2)$	1	1	0	2	2
(k,j)	(7,10)	(7,12)	(7,14)	(7,16)	(7,18)
dim $S_{k,j}(\Gamma_2)$	0	1	1	1	2

We also have the following table of dimensions of $S_{k-1/2,j}(\Gamma_2)$ and conjectural dimensions of $S^+_{k-1/2,j}(\Gamma_0(4),\psi)$ due to Tsushima (cf. [19], [20]).

(k,j)	(12,2)	(13,2)	(14,2)	(15,2)	(16,2)
dim $S_{k-1/2,j}(\Gamma_0(4),\psi)$	32	45	58	77	96
dim $S^+_{k-1/2,j}(\Gamma_0(4),\psi)$	1	1	0	2	2
(k,j)	(8,4)	(9,4)	(10,4)	(11,4)	(12,4)
dim $S_{k-1/2,j}(\Gamma_0(4),\psi)$	20	32	45	65	86
dim $S^+_{k-1/2,j}(\Gamma_0(4),\psi)$	0	1	1	1	2

We give below the basis of the above spaces and their Euler 2 and 3 factors excluding the case $S_{7,18}(\Gamma_2)$ and $S_{23/2,4}(\Gamma_0(4))$. The Fourier coefficients we used will be given in the Appendix.

3.1 Eigenforms of integral weight

We construct elements in $S_{k,j}(\Gamma_2)$ by theta functions with harmonic polynomials. For any $x=(x_i)$, $y=(y_i)\in\mathbb{C}^8$, we put $(x,y)={}^t xy$. Let $a,b\in\mathbb{C}^8$ be vectors such that $(a,a)=(a,b)=(b,b)=0$. We use the lattice E_8 defined by

$$E_8=\Big\{x=(x_i)\in\mathbb{Q}^8\;\Big|\;2x_i\in\mathbb{Z},\,x_i-x_j\in\mathbb{Z},\,\sum_{i=1}^{8}x_i\in 2\mathbb{Z}\Big\}.$$

This is the unique unimodular lattice of rank 8 up to isomorphism. For a variable $Z\in H_2$, we write

$$Z=\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}$$

and for integers $k \geq 4$ and $j \in \mathbb{Z}_{\geq 0}$, we define $\vartheta_{k,j,a,b}(Z)$ to be the sum

$$\sum_{x,y \in E_8} \begin{vmatrix} (x,a) & (x,b) \\ (y,a) & (y,b) \end{vmatrix}^{k-4} (xu+yv,a)^j e\left(\frac{1}{2}((x,x)\tau + 2(x,y)z + (y,y)\omega)\right),$$

where we write $e(x) = \exp(2\pi i x)$. Then identifying V_j with homogeneous polynomials in u and v, we have $\theta_{k,j,a,b} \in A_{k,j}(\Gamma_2)$. (This is more or less forklore and we omit the proof.) Now a nuisance here is that this theta function often vanishes identically and we must choose a and b carefully to get non-zero forms. Here we put

$$a_1 = (2,1,i,i,i,i,i,0) \qquad b_1 = (1,-1,i,i,1,-1,-i,i)$$

or

$$a_2 = (3,2i,i,i,i,i,i,0) \qquad b_2 = (1,i,-1,i,1,i,-i,1).$$

We define

$$\begin{aligned}
F_{5,18} &= \theta_{5,18,a_1,b_1} \\
F_{5,20} &= \theta_{5,20,a_1,b_1} \\
f_{5,24a} &= \theta_{5,24,a_1,b_1}/(2^{25}\cdot 3\cdot 5^3\cdot 13) \\
f_{5,24b} &= \theta_{5,24,a_2,b_2}/(2^{25}\cdot 3^7\cdot 5^4\cdot 13) \\
f_{5,26a} &= \theta_{5,26,a_1,b_1}/(2^{28}\cdot 3^2\cdot 5^4\cdot 13) \\
f_{5,26b} &= \theta_{5,26,a_2,b_2}/(2^{28}\cdot 3^6\cdot 5^4\cdot 13) \\
F_{7,12} &= \theta_{7,12,a_1,b_1}/(2^{22}\cdot 3^2\cdot 5^5\cdot 7^2) \\
F_{7,14} &= \theta_{7,14,a_1,b_1}/(2^{23}\cdot 3^4\cdot 5^2\cdot 11\cdot 181) \\
F_{7,16} &= \theta_{7,16,a_1,b_1}/(2^{27}\cdot 3^3\cdot 5^3\cdot 11^2\cdot 19)
\end{aligned}$$

Then these forms are non-zero, $F_{k,j} \in S_{k,j}(\Gamma_2)$ and $f_{k,ja}, f_{k,jb} \in S_{k,j}(\Gamma_2)$ for all the above forms. Moreover, the $F_{k,j}$ are common Hecke eigenforms. We also put

$$\begin{aligned}
F_{5,24a} &= -28741829 f_{5,24a} + (-966968929 + 821420\sqrt{4657}) f_{5,24b} \\
F_{5,24b} &= 28741829 f_{5,24b} + (966968929 + 821420\sqrt{4657}) f_{5,24b} \\
F_{5,26a} &= (-171241458523 + 631327288\sqrt{99661}) f_{5,26a} \\
&\quad - 5095416151 f_{5,26b} \\
F_{5,26b} &= (171241458523 + 631327288\sqrt{99661}) f_{5,26a} \\
&\quad + 5095416151 f_{5,26b}.
\end{aligned}$$

Then $F_{5,24a}$, $F_{5,24b}$, $F_{5,26a}$, $F_{5,26b}$ are also common Hecke eigenforms.

3.2 Structure of half-integral weight

Since it seems to be difficult to compute the plus space directly, we first give basis of $S_{k-1/2,j}(\Gamma_0(4),\psi)$ and then, calculating Fourier coefficients we find elements in the plus space. We consider the graded ring $A = \sum_{k=0}^{\infty} A_{2k}(\Gamma_0(4),\psi^k)$ of scalar valued Siegel modular forms of even weight belonging to $\Gamma_0(4)$ with character ψ^k for weight k. For each j, the module $\oplus_{k=1}^{\infty} S_{k-1/2,j}(\Gamma_0(4),\psi)$ is an A-module. The explicit structure of A was given in [7]. It is a weighted polynomial ring $A = \mathbb{C}[f_1, g_2, x_2, f_3]$ generated by the following algebraically independent four forms

$$\begin{aligned} f_1 &= (\theta_{0000}(2Z))^2 = \theta^2, \\ x_2 &= (\theta_{0000}(2Z)^4 + \theta_{0001}(2Z)^4 + \theta_{0010}(2Z)^4 + \theta_{0011}(2Z)^4)/4, \\ g_2 &= (\theta_{0000}(2Z)^4 + \theta_{0100}(2Z)^4 + \theta_{1000}(2Z)^4 + \theta_{1100}(2Z)^4), \\ f_3 &= (\theta_{0001}(2Z)\theta_{0010}(2Z)\theta_{0011}(2Z))^2 \end{aligned}$$

where, for any $m = (m', m'') \in \mathbb{Z}^4$, we define theta constants $\theta_m(Z)$ as usual by

$$\theta_m(Z) = \sum_{p\in\mathbb{Z}^2} e\left(\frac{1}{2}{}^t(p+\frac{m'}{2})Z(p+\frac{m'}{2}) + {}^t(p+\frac{m'}{2})\frac{m''}{2}\right).$$

For $j=2$ or $j=4$, the explicit structure of $\oplus_{k=1}^{\infty} S_{k-1/2,j}(\Gamma_0(4),\psi)$ as A-module is given in [10]. When $j=2$, this is a free A-module of rank 3 generated by $F_{11/2,3} \in S_{11/2,3}(\Gamma_0(4),\psi)$ and $G_{13/2,2}$, $H_{13/2,2} \in S_{13/2,2}(\Gamma_0(4),\psi)$. When $j = 4$, the module $\oplus_{k=1}^{\infty} S_{k-1/2,4}(\Gamma_0(4),\psi)$ is a non-free A-module generated $F_{9/2,4a}$, $F_{9/2,4b}$, $F_{9/2,4c} \in S_{9/2,4}(\Gamma_0(4),\psi)$ and $F_{11/2,4a}$, $F_{11/2,4b}$, $F_{11/2,4c} \in S_{11/2,4}(\Gamma_0(4),\psi)$. The fundamental relation of the generators of the A-module is given by

$$\begin{aligned} &(18f_1^2 - 6g_2 - 12x_2)F_{11/2,4a} + (576f_1^2 - 96g_2 - 768x_2)F_{11/2,4c} \\ &\quad + (-30f_1^3 + 27f_3 + 13f_1g_2 + 8f_1x_2)F_{9/2,4a} \\ &\quad + (-24f_1^3 + 4f_1g_2 + f_1x_2)F_{9/2,4b} + (-6f_1^3 + 9f_3 + 3f_1g_2)F_{9/2,4c} = 0 \end{aligned}$$

in the space $M_{15/2,4}(\Gamma_0(4),\psi)$. All these modular forms are constructed by Rankin-Cohen type differential operators starting from θ, g_2, x_2, f_3. For details, we refer to [10] and we omit them here.

3.3 Eigenforms of half-integral weight in the plus space

By calculating enough Fourier coefficients, we compute basis for various plus spaces (assuming the conjectured dimension formulas hold true). Then by calculating the action of Hecke operators $T_i(3)$, we can give common eigenforms. We define $F_{k-1/2,j}$ or $F_{k-1/2,j\epsilon}$ ($\epsilon = a$ or b) as below. For each k, j, this is a common eigenforms of all the Hecke operators $T_i(p)$ ($i = 1, 2$) belonging to $S^+_{k-1/2,j}(\Gamma_0(4), \psi)$.

The form $F_{23/2,2}$ is defined to be $1/2717908992$ times

$$\begin{aligned}
&(1134 f_1^5 - 1728 f_1^3 x_2 - 378 f_1^3 g_2 - 162 f_1^2 f_3 + 648 f_1 x_2^2 + 324 f_1 g_2 x_2 + 108 x_2 f_3 \\
&+ 54 g_2 f_3) G_{13/2,2} + (-4680 f_1^5 + 8640 f_1^3 x_2 - 1188 f_1^3 g_2 + 12636 f_1^2 f_3 \\
&- 13968 f_1 x_2^2 + 72 f_1 g_2^2 + 936 f_1 g_2 x_2 - 2376 x_2 f_3 + 108 g_2 f_3) H_{13/2,2} + (297 f_1^6 \\
&+ 54 f_1^4 x_2 - 81 f_1^4 g_2 - 540 f_1^3 f_3 - 180 f_1^2 x_2^2 - 108 f_1^2 g_2 x_2 - 9 f_1^2 g_2^2 + 360 f_1 x_2 f_3 \\
&+ 180 f_1 g_2 f_3 + 8 x_2^3 + 12 g_2 x_2^2 + 6 g_2^2 x_2 + g_2^3) F_{11/2,2}.
\end{aligned}$$

The form $F_{25/2,2}$ is defined to be $1/3623878656$ times

$$\begin{aligned}
&(1053 f_1^6 - 2826 f_1^4 x_2 - 333 f_1^4 g_2 - 648 f_1^3 f_3 + 828 f_1^2 x_2^2 + 684 f_1 g_2 x_2 - 9 f_1^2 g_2^2 \\
&+ 432 f_1 x_2 f_3 + 216 f_1 g_2 f_3 + 392 x_2^3 + 204 g_2 x_2^2 + 6 g_2^2 x_2 + g_2^3) G_{13/2,2} \\
&+ (-4374 f_1^6 + 828 f_1^4 x_2 + 3654 f_1^4 g_2 - 1296 f_1^3 f_3 + 1656 f_1^2 x_2^2 - 72 f_1^2 g_2 x_2 \\
&- 738 f_1^2 g_2^2 + 864 f_1 x_2 f_3 + 432 f_1 g_2 f_3 - 176 x_2^3 - 552 g_2 x_2^2 - 228 g_2^2 x_2 \\
&+ 2 g_2^3) H_{13/2,2} + (54 f_1^7 - 54 f_1^5 g_2 + 612 f_1^5 x_2 + 3456 f_1^4 f_3 + 18 f_1^3 g_2^2 \\
&- 168 f_1^3 g_2 x_2 - 696 f_1^3 x_2^2 - 1584 f_1^2 x_2 f_3 - 1152 f_1^2 g_2 f_3 + 176 f_1 x_2^3 + 72 f_1 g_2 x_2^2 \\
&- 12 f_1 g_2^2 x_2 - 2 f_1 g_2^3 - 240 g_2 x_2 f_3 - 480 x_2^2 f_3) F_{11/2,2}.
\end{aligned}$$

The form $f_{29/2,2a}$ is defined to be $1/14495514624$ times

$$\begin{aligned}
&G_{13/2,2}(2430 f_1^8 - 49086 f_1^6 x_2 - 1134 f_1^6 g_2 + 3888 f_1^5 f_3 + 117612 f_1^4 x_2^2 \\
&+ 16362 f_1^4 g_2 x_2 + 162 f_1^4 g_2^2 + 23328 f_1^3 x_2 f_3 - 1296 f_1^3 g_2 f_3 - 65448 f_1^2 x_2^3 \\
&- 28944 f_1^2 g_2 x_2^2 - 162 f_1^2 g_2^2 x_2 - 18 f_1^2 g_2^3 + 1944 f_1^2 f_3^2 - 8640 f_1 g_2 x_2 f_3 \\
&- 17280 f_1 x_2^2 f_3 + 6 g_2^3 x_2 + 36 g_2^2 x_2^2 + 2760 g_2 x_2^3 + 5424 x_2^4 - 648 g_2 f_3^2 \\
&- 1296 x_2 f_3^2) + H_{13/2,2}(-3645 f_1^8 + 73548 f_1^6 x_2 + 28512 f_1^6 g_2 + 153576 f_1^5 f_3 \\
&- 46656 f_1^4 x_2^2 - 154116 f_1^4 g_2 x_2 - 4266 f_1^4 g_2^2 - 173664 f_1^3 x_2 f_3 - 47952 f_1^3 g_2 f_3 \\
&+ 72144 f_1^2 x_2^3 + 72432 f_1^2 g_2 x_2^2 + 26676 f_1^2 g_2^2 x_2 - 216 f_1^2 g_2^3 + 42768 f_1^2 f_3^2 \\
&+ 7560 f_1 g_2^2 f_3 + 12960 f_1 g_2 x_2 f_3 - 56160 f_1 x_2^2 f_3 + 15 g_2^4 + 132 g_2^3 x_2 - 2928 g_2^2 x_2^2 \\
&- 7440 g_2 x_2^3 - 2352 x_2^4 - 1296 g_2 f_3^2 + 10368 x_2 f_3^2) + F_{11/2,2}(-1620 f_1^9 \\
&- 972 f_1^7 g_2 + 15012 f_1^7 x_2 - 20736 f_1^6 f_3 + 756 f_1^5 g_2^2 - 3564 f_1^5 g_2 x_2 - 15624 f_1^5 x_2^2
\end{aligned}$$

$$
\begin{aligned}
&- 89856 f_1^4 x_2 f_3 + 6912 f_1^4 g_2 f_3 + 816 f_1^3 x_2^3 + 2688 f_1^3 g_2 x_2^2 - 36 f_1^3 g_2^2 x_2 - 84 f_1^3 g_2^3 \\
&- 16848 f_1^3 f_3^2 + 34560 f_1^2 g_2 x_2 f_3 + 79200 f_1^2 x_2^2 f_3 + 5616 f_1 g_2 f_3^2 + 11232 f_1 x_2 f_3^2 \\
&- 52 f_1 g_2^3 x_2 - 312 f_1 g_2^2 x_2^2 + 720 f_1 g_2 x_2^3 + 2272 f_1 x_2^4 - 3360 g_2 x_2^2 f_3 - 6720 x_2^3 f_3).
\end{aligned}
$$

Next, the form $f_{29/2,2b}$ is defined as 1/28991029248 times

$$
\begin{aligned}
&G_{13/2,2}(+486 f_1^8 - 12636 f_1^6 x_2 - 162 f_1^6 g_2 + 972 f_1^5 f_3 + 30888 f_1^4 x_2 x_2 \\
&+ 4104 f_1^4 g_2 x_2 + 7776 f_1^3 x_2 f_3 - 324 f_1^3 g_2 f_3 - 17424 f_1^2 x_2^3 - 7560 f_1^2 g_2 x_2^2 \\
&+ 486 f_1^2 f_3^2 - 2808 f 1 g_2 x_2 f_3 - 5616 f 1 x_2^2 f_3 + 768 g_2 x_2^3 + 1536 x_2^4 - 162 g_2 f_3^2 \\
&- 324 x_2 f_3^2) + H_{13/2,2}(-486 f_1^8 + 23976 f_1^6 x_2 + 5508 f_1^6 g_2 + 38880 f_1^5 f_3 \\
&- 17280 f_1^4 x_2^2 - 38880 f_1^4 g_2 x_2 - 756 f_1^4 g_2^2 - 38880 f_1^3 x_2 f_3 - 14904 f_1^3 g_2 f_3 \\
&+ 15840 f_1^2 x_2^3 + 19152 f_1^2 g_2 x_2^2 + 6912 f_1^2 g_2 g_2 x_2 - 72 f_1^2 g_2^3 + 8748 f_1^2 f_3^2 \\
&+ 2376 f 1 g_2^2 f_3 + 3888 f 1 g_2 x_2 f_3 - 12096 f 1 x_2^2 f_3 + 6 g_2^4 + 48 g_2^3 x_2 - 816 g_2^2 x_2^2 \\
&- 2112 g_2 x_2^3 - 672 x_2^4 - 324 g_2 f_3^2 + 1944 x_2 f_3^2) + F_{11/2,2}(-405 f_1^9 - 243 f_1^7 g_2 \\
&+ 3780 f_1^7 x_2 - 7209 f_1^6 f_3 + 189 f_1^5 g_2^2 - 918 f_1^5 g_2 x_2 - 3600 f_1^5 x_2^2 - 22302 f_1^4 x_2 f_3 \\
&+ 2457 f_1^4 g_2 f_3 - 144 f_1^3 x_2^3 + 588 f_1^3 g_2 x_2^2 - 21 f_1^3 g_2^3 - 6156 f_1^3 f_3^3 - 27 f_1^2 g^2 f_3 \\
&+ 8964 f_1^2 g_2 x_2 f_3 + 20916 f_1^2 x_2^2 f_3 + 2052 f_1 g_2 f_3^2 + 4104 f_1 x_2 f_3^2 - 14 f_1 g_2^3 x_2 \\
&- 84 f_1 g_2^2 x_2^2 + 216 f_1 g_2 x_2^3 + 656 f_1 x_2^4 + 3 g_2^3 f_3 + 18 g_2^2 x_2 f_3 - 924 g_2 x_2^2 f_3 \\
&- 1896 x_2^3 f_3).
\end{aligned}
$$

We go on defining

$$
\begin{aligned}
F_{29/2,2a} &= 442 f_{29/2,2a} + (-4207 + 15\sqrt{4657}) f_{29/2,2b} \\
F_{29/2,2b} &= 442 f_{29/2,2a} + (-4207 - 15\sqrt{4657}) f_{29/2,2b}.
\end{aligned}
$$

We define $f_{31/2,2a}$ as 1/1391569403904 times

$$
\begin{aligned}
&F_{11/2}(+162 f_1^{10} - 216 f_1^8 x_2 - 162 f_1^8 g_2 + 162 f_1^7 f_3 1152 f_1^6 x_2^2 - 180 f_1^6 g_2 x_2 \\
&+ 54 f_1^6 g_2^2 - 1188 f_1^5 x_2 f_3 - 162 f_1^5 g_2 f_3 - 2208 f_1^4 x_2^3 + 72 f_1^4 g_2 x_2^2 + 144 f_1^4 g_2^2 x_2 \\
&- 6 f_1^4 g_2^3 + 54 f_1^3 g_2^2 f_3 + 504 f_1^3 g_2 x_2 f_3 + 2520 f_1^3 x_2^2 f_3 - 20 f_1^2 g_2^3 x_2 - 120 f_1^2 g_2^2 x_2^2 \\
&+ 336 f_1^2 g_2 x_2^3 + 992 f_1^2 x_2^4 - 432 f_1^2 x_2 f_3^2 - 6 f_1 g_2^3 f_3 - 36 f_1 g_2^2 x_2 f_3 - 648 f_1 g_2 x_2^2 f_3 \\
&- 1200 f_1 x_2^3 f_3 + 144 g_2 x_2 f_3^2 + 288 x_2^2 f_3^2) + G_{13/2}(567 f_1^9 - 999 f_1^7 g_2 + 756 f_1^7 x_2 \\
&- 1377 f_1^6 f_3 + 405 f_1^5 g_2^2 + 594 f_1^5 g_2 x_2 - 3024 f_1^5 x_2^2 + 3618 f_1^4 x_2 f_3 + 513 f_1^4 g_2 f_3 \\
&+ 1584 f_1^3 x_2^3 + 540 f_1^3 g_2 x_2^2 - 216 f_1^3 g_2^2 x_2 - 45 f_1^3 g_2^3 - 648 f_1^3 f_3^2 - 27 f_1^2 g_2^2 f_3 \\
&- 972 f_1^2 g_2 x_2 f_3 - 1836 f_1^2 x_2^2 f_3 + 216 f_1 g_2 f_3^2 + 432 f_1 x_2 f_3^2 - 6 f_1 g_2^3 x_2 \\
&- 36 f_1 g_2^2 x_2^2 - 72 f_1 g_2 x_2^3 - 48 f_1 x_2^4 + 3 g_2^3 f_3 + 18 g_2^2 x_2 f_3 + 36 g_2 x_2^2 f_3 + 24 x_2^3 f_3) \\
&+ H_{13/2}(-2754 f_1^9 + 594 f_1^7 g_2 + 13176 f_1^7 x_2 - 6642 f_1^6 f_3 + 378 f_1^5 g_2^2
\end{aligned}
$$

$$- 4860 f_1^5 g_2 x_2 - 13824 f_1^5 x_2^2 + 11124 f_1^4 x_2 f_3 + 3618 f_1^4 g_2 f_3 + 2592 f_1^3 x_2^3$$
$$+ 2232 f_1^3 g_2 x_2^2 + 288 f_1^3 g_2^2 x_2 - 90 f_1^3 g_2^3 - 1296 f_1^3 f_3^2 - 486 f_1^2 g_2^2 f_3 - 2808 f_1^2 g_2 x_2 f_3$$
$$- 3672 f_1^2 x_2^2 f_3 + 432 f1 g_2 f_3^2 + 864 f1 x_2 f_3^2 - 12 f1 g_2^3 x_2 + 216 f1 g_2^2 x_2^2$$
$$+ 1008 f1 g_2 x_2^3 + 1056 f1 x_2^4 + 6 g_2^3 f_3 - 108 g_2^2 x_2 f_3 - 504 g_2 x_2^2 f_3 - 528 x_2^3 f_3).$$

Next we define $f_{31/2,2b}$ as $1/521838526464$ times

$$F_{11/2}(2673 f_1^{10} - 3078 f_1^8 x_2 - 2511 f_1^8 g_2 - 47628 f_1^7 f_3 - 15336 f_1^6 x_2^2 + 864 f_1^6 g_2 x_2$$
$$+ 702 f_1^6 g_2^2 + 1944 f_1^5 x_2 f_3 1 + 16524 f_1^5 g_2 f_3 - 144 f_1^4 x_2^3 + 6264 f_1^4 g_2 x_2^2$$
$$+ 540 f_1^4 g_2^2 x_2 - 18 f_1^4 g_2^3 + 77760 f_1^4 f_3^2 - 324 f_1^3 g_2^2 f_3 + 9072 f_1^3 g_2 x_2 f_3$$
$$+ 45360 f_1^3 x_2^2 f_3 - 15 f_1^2 g_2^4 - 192 f_1^2 g_2^3 x_2 - 360 f_1^2 g_2^2 x_2^2 + 3840 f_1^2 g_2 x_2^3$$
$$+ 7824 f_1^2 x_2^2 - 25920 f_1^2 g_2 f_3^2 - 51840 f_1^2 x_2 f_3^2 + 36 f_1 g_2^3 f_3 + 216 f_1 g_2^2 x_2 f_3$$
$$- 8208 f_1 g_2 x_2^2 f_3 - 16992 f_1 x_2^3 f_3 + 1 g_2^5 + 10 g_2^4 x_2 - 8 g_2^3 x_2^2 - 208 g_2^2 x_2^3 - 496 g_2 x_2^2$$
$$- 352 x_2^5) + G_{13/2}(10206 f_1^9 - 10206 f_1^7 g_2 - 29160 f_1^7 x_2 - 164754 f_1^6 f_3$$
$$+ 3402 f_1^5 g_2^2 + 22356 f_1^5 g_2 x_2 - 23328 f_1^5 x_2^2 + 251748 f_1^4 x_2 f_3 + 55890 f_1^4 g_2 f_3$$
$$+ 68256 f_1^3 x_2^3 + 1944 f_1^3 g_2 x_2^2 - 5184 f_1^3 g_2^2 x_2 - 378 f_1^3 g_2^3 + 23328 f_1^3 f_3^2$$
$$- 486 f_1^2 g_2^2 f_3 - 48600 f_1^2 g_2 x_2 f_3 - 87480 f_1^2 x_2^2 f_3 - 7776 f_1 g_2 f_3^2 - 15552 f_1 x_2 f_3^2$$
$$+ 324 f_1 g_2^3 x_2 + 1944 f_1 g_2^2 x_2^2 - 11664 f_1 g_2 x_2^3 - 28512 f_1 x_2^4 + 54 g_2^3 f_3 + 324 g_2^2 x_2 f_3$$
$$- 1944 g_2 x_2^2 f_3 - 4752 x_2^3 f_3) + H_{13/2}(-43740 f_1^9 + 18468 f_1^7 g_2 + 136080 f_1^7 x_2$$
$$+ 813564 f_1^6 f_3 + 2916 f_1^5 g_2^2 - 48600 f_1^5 g_2 x_2 - 15552 f_1^5 x_2^2 - 1417176 f_1^4 x_2 f_3$$
$$+ 96228 f_1^4 g_2 f_3 - 212544 f_1^3 x_2^3 + 159408 f_1^3 g_2 x_2^2 - 2592 f_1^3 g_2^2 x_2 - 1620 f_1^3 g_2^3$$
$$- 1819584 f_1^3 f_3^2 + 1620 f_1^2 g_2^2 f_3 - 71280 f_1^2 g_2 x_2 f_3 1483920 f_1^2 x_2^2 f_3 - 15552 f_1 g_2 f_3^2$$
$$+ 342144 f_1 x_2 f_3^2 + 72 f_1 g_2^4 + 1224 f_1 g_2^3 x_2 - 13392 f_1 g_2^2 x_2^2 - 97056 f_1 g_2 x_2^3$$
$$+ 614592 f_1 x_2^4 + 108 g_2^3 f_3 - 1944 g_2^2 x_2 f_3 - 14256 g_2 x_2^2 f_3 + 104544 x_2^3 f_3).$$

We define

$$F_{31/2,2a} = (-144 + 48\sqrt{99661}) f_{31/2,2a} + f_{31/2,2b}$$
$$F_{31/2,2b} = (-144 - 48\sqrt{99661}) f_{31/2,2a} + f_{31/2,2b}$$

We define $F_{17/2,4}$ to be $1/884736$ times

$$(F_{11/2,4a}(36 f_1^3 - 12 f_1 g_2 - 24 f_1 x_2) + F_{11/2,4b}(-144 f_1^3 + 48 f_1 g_2 + 96 f_1 x_2)$$
$$+ F_{11/2,4c}(1728 f_1^3 - 576 f_1 g_2 - 1152 f_1 x_2 + 1728 f_3) + F_{9/2,4a}(27 f_1^4 - 270 f_1 f_3$$
$$- 72 f_1^2 g_2 + 84 f_1^2 x_2 + 3 g_2^2 + 44 g_2 x_2 + 76 x_2^2) + F_{9/2,4b}(-342 f_1^4 - 72 f_1 f_3$$
$$+ 156 f_1^2 g_2 + 216 f_1^2 x_2 - 14 g_2^2 - 24 g_2 x_2 + 8 x_2^2) + F_{9/2,4c}(-63 f_1^4 - 90 f_1 f_3$$
$$+ 108 f_1^2 x_2 + g_2^2 + 4 g_2 x_2 + 4 x_2^2))$$

We set $F_{19/2,4} := f_{19/2,4}/589824$, where $f_{19/2,4}$ is

$$F_{11/2,4c}(-1296f_1^4 + 96f_1^2g_2 - 2112f_1^2x_2 - 16g_2^2 - 64g_2x_2 - 64x_2^2 + 3456f_1f_3)$$
$$+ F_{9/2,4a}(-15f_1^5 - 70f_1^3g_2364f_1^3x_2 + 9f_1g_2^2 + 36f_1g_2x_2 - 396f_1x_2^2 + 96f_1^2f_3$$
$$+ 16g_2f_3 - 40x_2f_3) + F_{9/2,4b}(+150f_1^5 + 44f_1^3g_2 - 440f_1^3x_2 - 10f_1g_2^2$$
$$- 40f_1g_2x_2 + 248f_1x_2^2 + 336f_1^2f_3 + 16g_2f_3 - 304x_2f_3) + F_{9/2.4c}(-21f_1^5$$
$$- 18f_1^3g_2 + 132f_1^3x_2 + 3f_1g_2^2 + 12f_1g_2x_2 - 132f_1x_2^2 + 48f_1^2f_3 - 24x_2f_3)$$

and finally $F_{21/2,4} := f_{21/2,4}/10616832$, with $f_{21/2,4}$ given by

$$F_{11/2,4b}(-162f_1^5 + 432f_1^3x_2 + 54f_1^3g_2 - 162f_1^2f_3 - 216f_1x_2^2 - 108f_1g_2x_2$$
$$+ 108x_2f_3 + 54g_2f_3) + F_{11/2,4c}(+7776f_1^5 - 22464f_1^3x_2 + 432f_1^3g_2 + 9072f_1^2f_3$$
$$+ 11520f_1x_2^2 + 288f_1g_2x_2 - 144f_1g_2^2 - 6048x_2f_3 - 432g_2f_3)F_{9/2,4a}(+459f_1^6$$
$$+ 486f_1^4x_2 - 729f_1^4g_2 - 810f_1^3f_3 - 972f_1^2x_2^2 + 540f_1^2g_2x_2 + 108f_1^2g_2^2$$
$$+ 1188f_1x_2f_3 - 54f_1g_2f_3 + 8x_2^3 + 12g_2x_2^2 + 6g_2^2x_2 + 1g_2^3 - 243f_3^2)$$
$$+ F_{9/2,4b}(-1269f_1^6 - 882f_1^4x_2 + 1395f_1^4g_2 + 1944f_1^3f_3 + 3444f_1^2x_2^2$$
$$- 1092f_1^2g_2x_2 - 219f_1^2g_2^2 - 4752f_1x_2f_3 + 108f_1g_2f_3 + 8x_2^3 + 12g_2x_2^2 + 6g_2^2x_2$$
$$+ g_2^3 + 1296f_3^2) + F_{9/2,4c}(+108f_1^6 + 288f_1^4x_2 - 234f_1^4g_2 - 324f_1^3f_3 - 384f_1^2x_2^2$$
$$+ 156f_1^2g_2x_2 + 39f_1^2g_2^2 + 432f_1x_2f_3 - 81f_3^2)$$

3.4 Numerical Examples of L-functions

In this section, we give our results on the Euler 2-factors and 3-factors of the common eigenforms listed above. We verified that they all support our conjecture. We shall explain how to calculate these examples in section 4.

$\mathbf{S_{5,18}(\Gamma_2)}$ and $\mathbf{S^+_{23/2,2}(\Gamma_0(4)}, \psi)$.

$$\begin{aligned} H_2(s, F_{23/2,2}) &= H_2(s, F_{5,18}) \\ &= 1 + 2880T - 26378240T^2 + 2880 \cdot 2^{25}T^3 + 2^{50}T^4 \\ H_3(s, F_{23/2,2}) &= H_3(s, F_{5,18}) \\ &= 1 + 538970T + 204622302870T^2 + 539870 \cdot 3^{25}T^3 + 3^{50}T^4 \end{aligned}$$

$\mathbf{S_{5,20}(\Gamma_2)}$ and $\mathbf{S^+_{25/2,2}(\Gamma_0(4)}, \psi)$.

$$\begin{aligned} H_2(s, F_{25/2,2}) &= H_2(s, F_{5,20}) \\ &= 1 + 240T - 29204480T^2 + 240 \cdot 2^{27}T^3 + 2^{54}T^4 \\ H_3(s, F_{25/2,2}) &= H_3(s, F_{5,20}) \\ &= 1 - 1645560T - 2281745279610T^2 - 1645560 \cdot 3^{27}T^3 + 3^{54}T^4 \end{aligned}$$

https://doi.org/10.1017/CBO9780511543371.010

$S_{5,24}(\Gamma_2)$ and $S^+_{29/2,2}(\Gamma_0(4),\psi)$.

$$H_2(s,F_{29/2,a}) = 1-(-8040+600\sqrt{4657})T+(742973440-1843200\sqrt{4657})T^2 - (-8040+600\sqrt{4657})2^{31}T^3+2^{62}T^4$$

$$H_2(s,F_{29/2,b}) = 1-(-8040-600\sqrt{4657})T+(742973440+1843200\sqrt{4657})T^2 - (-8040-600\sqrt{4657})2^{31}T^3+2^{62}T^4$$

$$H_3(s,F_{29/2,a}) = 1-(4187160-194400\sqrt{4657})T+196830(65242301 + 4016320\sqrt{4657})T^2-(4187160-194400\sqrt{4657})3^{31}T^3 + 3^{62}T^4$$

$$H_3(s,F_{29/2,b}) = 1-(4187160+194400\sqrt{4657})T+196830(65242301 - 4016320\sqrt{4657})T^2-(4187160+194400\sqrt{4657})3^{31}T^3 + 3^{62}T^4$$

$S_{5,26}(\Gamma_2)$ and $S^+_{31/2,2}(\Gamma_0(4),\psi)$.

$$H_2(s,F_{31/2,a}) = 1-(27072+192\sqrt{99661})T+(4836327424 - 9732096\sqrt{99661})T^2-(27072+192\sqrt{99661})\cdot 2^{33}T^3 + 2^{66}T^4$$

$$H_2(s,F_{31/2,b}) = 1-(27072-192\sqrt{99661})T+(4836327424 + 9732096\sqrt{99661})T^2-(27072-192\sqrt{99661})\cdot 2^{33}T^3 + 2^{66}T^4$$

$$\begin{aligned} H_3(s,F_{31/2,a}) &= H_3(s,F_{5,26a}) \\ &= 1-(-9567144-59904\sqrt{99661})T+(-268954900275114 \\ &\quad + 16134754093056\sqrt{99661})T^2-(-9567144 \\ &\quad - 59904\sqrt{99661})\cdot 3^{33}T^3+3^{66}T^4 \end{aligned}$$

$$\begin{aligned} H_3(s,F_{31/2,b}) &= H_3(s,F_{5,26b}) \\ &= 1-(-9567144+59904\sqrt{99661})T+(-268954900275114 \\ &\quad - 16134754093056\sqrt{99661})T^2-(-9567144 \\ &\quad + 59904\sqrt{99661})\cdot 3^{33}T^3+3^{66}T^4 \end{aligned}$$

$S_{7,12}(\Gamma_2)$ and $S^+_{17/2,4}(\Gamma_0(4),\psi)$.

$$\begin{aligned} H_2(s,F_{17/2,4}) &= H_2(s,f_{7,12}) \\ &= 1+480T+5754880T^2+480\cdot 3^{26}T^3+3^{52}T^4 \end{aligned}$$

$$\begin{aligned} H_3(s,F_{17/2,4}) &= H_3(s,f_{7,12}) \\ &= 1+73080T-97880212890T^2+73080\cdot 3^{23}T^3+3^{46}T^4 \end{aligned}$$

$\mathbf{S}_{7,14}(\Gamma_2)$ and $\mathbf{S}^+_{19/2,4}(\Gamma_0(4),\psi)$.

$H_2(s, F_{19/2,4}) = 1 + 3696T + 18116608T^2 + 3696 \cdot 2^{25}T^3 + 2^{50}T^4$
$H_3(s, F_{19/2,4}) = 1 - 511272T + 377292286422T^2 - 511272 \cdot 3^{25}T^3 \cdot 3^{50}T^4$

$\mathbf{S}_{7,16}(\Gamma_2)$ and $\mathbf{S}^+_{21/2,4}(\Gamma_0(4),\psi)$.

$H_2(s, F_{21/2,4}) = 1 - 13440T + 166912000T^2 - 13440 \cdot 2^{27}T^3 + 2^{54}T^4$
$H_3(s, F_{21/2,4}) = 1 + 1487160T - 2487701893050T^2 + 1487160 \cdot 3^{27}T^3 + 3^{54}T^4$

4 How to calculate eigenvalues

In this section, we would like to show how to calculate the Euler factors of section 3 from the Fourier coefficients of the corresponding Siegel modular forms.

4.1 Integral weight

For the theory of vector valued forms of integral weight, we refer to Arakawa [1]. For a common eigenform $F \in A_{k,j}(\Gamma_2)$, we write the Fourier expansion as

$$F(Z) = \sum_T A(T)e(\mathrm{Tr}(TZ))$$

where T runs over 2×2 half-integral positive semi-definite symmetric matrices and $A(T) \in \mathbb{C}^{j+1}$. By automorphy, we have $A(UT\,{}^tU) = \rho_{k,j}(U)A(T)$ for any $U \in \mathrm{GL}_2(\mathbb{Z})$. For the Hecke operator $T(p^\delta)$, denote by $A(p^\delta;T)$ the Fourier coefficient at T of $T(p^\delta)F$. For a fixed $T = \begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}$, and for any $U \in \mathrm{GL}_2(\mathbb{Z})$, we define a_U, b_U, c_U by the relation ${}^tUTU = \begin{pmatrix} a_U & b_U/2 \\ b_U/2 & c_U\end{pmatrix}$, and for any non-negative integers α,β, we put

$$d_{\alpha,\beta} = \begin{pmatrix} p^\alpha & 0 \\ 0 & p^{\alpha+\beta}\end{pmatrix}.$$

Let $R(p^\beta)$ denote a complete set of representatives of $\mathrm{SL}_2(\mathbb{Z})/{}^t\Gamma_0^{(1)}(p^\beta)$. For example, we can take

$$R(p) = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\ x \in \mathbb{Z}/p\mathbb{Z}\right\},$$

$$R(p^2) = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}py & 1\\ -1 & 0\end{pmatrix};\ x \in \mathbb{Z}/p^2\mathbb{Z},\ y \in \mathbb{Z}/p\mathbb{Z}\right\}$$

Here, for simplicity, we write $\rho_j = \mathrm{Sym}_j$. Then we have

$$A(p^\delta; T) = \sum_{\alpha+\beta+\gamma=\delta} p^{\beta(k+j-2)+\gamma(2k+j-3)} \sum_{\substack{U\in R(p^\beta),\\ a_U\equiv 0 p^{\beta+\gamma},\\ b_U\equiv c_U\equiv 0 \bmod p^\gamma}} \rho_j(d_{0,\beta}U)^{-1} A\left(p^\alpha \begin{pmatrix} a_U p^{-\beta-\gamma} & b_U p^{-\gamma}/2 \\ b_U p^{-\gamma}/2 & c_U p^{\beta-\gamma}\end{pmatrix}\right)$$

For practical use, we need more explicit examples which are given below. Here we write $A(p^\delta, T) = A(p^\delta, (a, c, b))$ for $T = \begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}$ for simplicity.

$$\begin{aligned}
A(2, (1, 1, 1)) &= A(2, 2, 2),\\
A(4, (1, 1, 1)) &= A(4, 4, 4),\\
A(2, (2, 2, 2)) &= A(4, 4, 4) + 2^{k+j-2}\Big(\rho_j \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2^{-1}\end{smallmatrix}\right) A(1, 4, 2) + \rho_j \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2\end{smallmatrix}\right)^{-1} A(3, 4, 6)\\
&\quad + \rho_j \left(\begin{smallmatrix} 0 & 1 \\ -2 & 0\end{smallmatrix}\right)^{-1} A(1, 4, -2)\Big) + 2^{2k+j-3} A(1, 1, 1)\\
&= A(4, 4, 4) + 2^{k-2}\Big(\rho_j \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1\end{smallmatrix}\right) A(1, 3, 0) + \rho_j \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1\end{smallmatrix}\right) A(1, 3, 0)\\
&\quad + \rho_j \left(\begin{smallmatrix} 1 & -1 \\ 2 & 0\end{smallmatrix}\right) A(1, 3, 0)\Big) + 2^{2k+j-3} A(1, 1, 1),\\
A(2, (1, 1, 0)) &= A(2, 2, 0) + 2^{k+j-2}\rho_j \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2\end{smallmatrix}\right)^{-1} A(1, 2, 2)\\
&= A(2, 2, 0) + 2^{k-2}\rho_j \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1\end{smallmatrix}\right) A(1, 1, 0),\\
A(4, (1, 1, 0)) &= A(4, 4, 0) + 2^{k-2}\rho_j \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1\end{smallmatrix}\right) A(2, 2, 0),\\
A(2, (2, 2, 0)) &= A(4, 4, 0) + 2^{k-2}\rho_j \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1\end{smallmatrix}\right) A(2, 2, 0) + 2^{k-2}\Big(\rho_j \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1\end{smallmatrix}\right) A(1, 4, 0)\\
&\quad + \rho_j \left(\begin{smallmatrix} 0 & -1 \\ 2 & 0\end{smallmatrix}\right) A(1, 4, 0)\Big) + 2^{2k+j-3} A(1, 1, 0),\\
A(3, (1, 1, 0)) &= A(3, 3, 0),\\
A(9, (1, 1, 0)) &= A(9, 9, 0),\\
A(3, (3, 3, 0)) &= A(9, 9, 0) + 3^{k-2}\Big(\rho_j \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1\end{smallmatrix}\right) A(1, 9, 0) + \rho_j \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1\end{smallmatrix}\right) A(2, 5, 2)\\
&\quad + (-1)^k \rho_j \left(\begin{smallmatrix} -2 & 1 \\ 1 & 1\end{smallmatrix}\right) A(2, 5, 2) + \rho_j \left(\begin{smallmatrix} 0 & -1 \\ 3 & 0\end{smallmatrix}\right) A(1, 9, 0)\Big)\\
&\quad + 3^{2k+j-3} A(1, 1, 0),\\
A(3, (1, 1, 1)) &= A(3, 3, 3) + 3^{k+j-2}\rho_j \left(\begin{smallmatrix} 1 & -1/3 \\ 0 & 1/3\end{smallmatrix}\right) A(1, 3, 3)\\
&= A(3, 3, 3) + 3^{k-2}\rho_j \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1\end{smallmatrix}\right) A(1, 1, 1),\\
A(9, (1, 1, 1)) &= A(9, 9, 9) + 3^{k-2}\rho_j \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1\end{smallmatrix}\right) A(3, 3, 3),
\end{aligned}$$

$$A(3,(3,3,3)) = A(9,9,9) + 3^{k-2}\rho_j\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} A(3,3,3) + 3^{k-2}\Big(\rho_j\left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right) A(1,7,1) + (-1)^k\rho_j\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} A(1,7,1) + \rho_j\begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} A(1,7,-1)\Big) + 3^{2k+j-3}A(1,1,1).$$

4.2 Half-integral weight

This section is essentially due to Zhuravlev [21], [22]. The scalar valued case is explained in detail in [7]. For $h \in S^+_{k-1/2,j}(\Gamma_0(4),\psi)$, we write the Fourier expansion as

$$h(Z) = \sum_T A(T)e(\mathrm{Tr}(TZ)).$$

When p is odd, we use the notation $A(T_i(p);(a,c,b)) = A(T_i(p),T)$ for the Fourier coefficients of $T_i(p)h$ for $i = 1, 2$. Then we have

$$A(T_1(p);T) = \alpha_{1,1,0}(T) + \alpha_{1,1,1}(T) + \alpha_{1,2,0}(T),$$

where

$$\alpha_{1,1,0}(T) = p^{2k+j-4}\psi(p)\sum_{U\in R(p)}\rho_j(U)^{-1}\rho_j\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)A\Big(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)UT\,{}^tU\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big),$$

and

$$\alpha_{1,1,1}(T) = \psi(p)p^j\rho_j(U^{-1})\rho_j\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)\sum_{U\in R(p)}A\Big(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}UT\,{}^tU\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\Big)$$

and

$$\alpha_{1,2,0}(T) = \begin{cases} \left(\frac{(-1)^{k+1}a}{p}\right)p^{k+j-2}A(T) & \text{if } p\nmid a \text{ and } p|\det(2T) \\ \left(\frac{(-1)^{k+1}c}{p}\right)p^{k+j-2}A(T) & \text{if } p|a \text{ and } p|\det(2T) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have

$$A(T_2(p);T) = \sum_{0\le i+j\le 2}\alpha_{2,i,j}(T),$$

where

$$\alpha_{2,0,0}(T) = p^{4k+3j-8}A(p^{-2}T)$$

$$\alpha_{2,0,1}(T) = p^{2k+2j-5}\sum_{U\in R(p^2)}\rho_j(U^{-1})\rho_j\left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)A\Big(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p \end{smallmatrix}\right)UT\,{}^tU\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & p \end{smallmatrix}\right)\Big)$$

$$\alpha_{2,0,2}(T) = p^jA(p^2T)$$

$$\alpha_{2,1,0}(T) = p^{3k+2j-7}\psi(p)^k \sum_{U\in R(p)} \left(\frac{m}{p}\right) \rho_j(U^{-1})\left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \times A\left(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) UT{}^tU\left(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right),$$

where $UT{}^tU = \left(\begin{smallmatrix} * & * \\ * & m \end{smallmatrix}\right)$. Further,

$$\alpha_{2,1,1}(T) = p^{k+2j-3}\psi(p)^k \sum_{U\in R(p)} \left(\frac{m}{p}\right) \rho_j(U^{-1})\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)\right) \times A\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\right) UT{}^tU\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\right),$$

where $UT{}^tU = \left(\begin{smallmatrix} m & * \\ * & * \end{smallmatrix}\right)$, and

$$\alpha_{2,2,0}(T) = \begin{cases} -p^{2k+2j-6}A(T) & \text{if } p \nmid \det(2T) \\ (p-1)p^{2k+2j-6}A(T) & \text{if } p \mid \det(2T). \end{cases}$$

When $p = 2$, we denote by $T_i^*(2)$ $(i = 1, 2)$ the Hecke operator obtained as the pullback of the Hecke operators on Jacobi forms at 2 (see section 5). This amounts to take $\psi(p)T_i(p)$ for odd p. We must modify the above formula of Fourier coefficients $A(T_i^*(2), T)$ of $T_i^*(p)F$, but this can be done in the same way as in [7] p.516–517. To be more precise, we omit $\psi(p)$ in $\alpha_{1,1,0}(T)$ and $\alpha_{1,1,1}(T)$, replace $k+1$ by k in $\left(\frac{(-1)^{k+1}a}{p}\right)$ and $\left(\frac{(-1)^{k+1}c}{p}\right)$ in $\alpha_{1,2,0}(T)$, and replace everywhere the condition $p \mid \det(2T)$ or $p \nmid \det(2T)$ by $8 \mid \det(T)$ or $8 \nmid \det(T)$, respectively, and interpret the symbol $\left(\frac{x}{p}\right)$ as 0, 1 or -1 for $x \equiv 0$ mod 4, 1 mod 8, or 5 mod 8, respectively.

We have the following formulas.

$$\begin{aligned} &A(T_1(3), (3,3,2)) \\ &= \psi(3)\Big(\rho_j\left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right) A(3,27,-6) + \rho_j\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) A(3,27,6) \\ &\quad +3^j\rho_j\left(\begin{smallmatrix} 1 & -1/3 \\ 0 & 1/3 \end{smallmatrix}\right) A(8,27,24) + 3^j\rho_j\left(\begin{smallmatrix} 1 & -2/3 \\ 0 & 1/3 \end{smallmatrix}\right) A(19,27,42)\Big) \\ &= \psi(3)\Big(\rho_j\left(\begin{smallmatrix} 2 & -1 \\ 1 & 1 \end{smallmatrix}\right) A(8,11,8) + \rho_j\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) A(3,27,6) \\ &\quad +\rho_j\left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right) A(3,27,-6) + \rho_j\left(\begin{smallmatrix} 1 & -2 \\ 1 & 1 \end{smallmatrix}\right) A(19,4,4)\Big) \end{aligned}$$

$$\begin{aligned} &A(T_2(3), (3,3,2)) \\ &= 3^j A(27,27,18) + 3^{2k+j-5}\big(\rho_j\left(\begin{smallmatrix} 0 & -1 \\ 9 & 3 \end{smallmatrix}\right) A(4,27,-20) \\ &\quad +\rho_j\left(\begin{smallmatrix} 9 & -3 \\ 0 & 1 \end{smallmatrix}\right) A(4,27,20)\big) + 3^{k+j-3}\psi(3)^k\big(-\rho_j\left(\begin{smallmatrix} 3 & -1 \\ 0 & 1 \end{smallmatrix}\right) A(8,27,24) \\ &\quad +\rho_j\left(\begin{smallmatrix} 3 & -2 \\ 0 & 1 \end{smallmatrix}\right) A(19,27,21)\big) - 3^{2k+2j-6}A(3,3,2) \end{aligned}$$

$$\begin{aligned}
&= 3^j A(27,27,18) + 3^{2k+j-5}\big(\rho_j\left(\begin{smallmatrix}3&-3\\2&1\end{smallmatrix}\right)A(4,3,4)\\
&\quad +\rho_j\left(\begin{smallmatrix}2&-1\\3&3\end{smallmatrix}\right)A(4,3,-4)\big) + 3^{k+j-3}\psi(3)^k\big(-\rho_j\left(\begin{smallmatrix}2&-1\\1&1\end{smallmatrix}\right)A(8,11,8)\\
&\quad +\rho_j\left(\begin{smallmatrix}1&-2\\1&1\end{smallmatrix}\right)A(19,4,4)\big) - 3^{2k+2j-6}A(3,3,2)
\end{aligned}$$

$A(T_1(3),(3,4,0))$

$$\begin{aligned}
&= \psi(3)\Big(\rho\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)A(3,36,0) + \rho_j\left(\begin{smallmatrix}3&-1\\0&1\end{smallmatrix}\right)A(7,36,24)\\
&\quad +\rho_j\left(\begin{smallmatrix}3&-2\\0&1\end{smallmatrix}\right)A(19,36,48) + \rho_j\left(\begin{smallmatrix}0&-1\\3&0\end{smallmatrix}\right)A(4,27,0)\Big)\\
&\quad +3^{k+j-2}A(3,4,0)
\end{aligned}$$

$A(T_2(3),(3,4,0))$

$$\begin{aligned}
&= 3^j A(27,36,0) + 3^{k+j-3}\psi(3)^k\Big(\rho_j\left(\begin{smallmatrix}3&-1\\0&1\end{smallmatrix}\right)A(7,36,24)\\
&\quad +\rho_j\left(\begin{smallmatrix}3&-2\\0&1\end{smallmatrix}\right)A(19,36,48) + \rho_j\left(\begin{smallmatrix}0&-1\\0&3\end{smallmatrix}\right)A(4,27,0)\Big)\\
&\quad +2\cdot 3^{2k+2j-6}A(3,4,0)
\end{aligned}$$

$A(T_1(3),(1,4,0))$

$$\begin{aligned}
&= \rho_j\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)A(1,36,0) + 3^j\rho_j\begin{pmatrix}1&-1/3\\0&1/3\end{pmatrix}A(5,36,24)\\
&\quad +3^j\rho_j\begin{pmatrix}1&-2/3\\0&1/3\end{pmatrix}A(17,36,48) + 3^j\rho_j\begin{pmatrix}0&-1/3\\1&0\end{pmatrix}A(4,9,0)\\
&= \rho_j\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)A(1,36,0) + \rho_j\left(\begin{smallmatrix}2&-1\\1&1\end{smallmatrix}\right)A(5,17,14)\\
&\quad +\rho_j\left(\begin{smallmatrix}1&-2\\1&1\end{smallmatrix}\right)A(17,5,14) + \rho_j\left(\begin{smallmatrix}0&-1\\3&0\end{smallmatrix}\right)A(4,9,0)
\end{aligned}$$

$A(T_2(3),(1,4,0))$

$$\begin{aligned}
&= 3^j A(9,36,0) + 3^{k+j-3}\psi(3)^k\Big(\rho_j\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)A(1,36,0)\\
&\quad -\rho_j\left(\begin{smallmatrix}3&-1\\0&1\end{smallmatrix}\right)A(5,36,24) - \rho_j\left(\begin{smallmatrix}3&-2\\0&1\end{smallmatrix}\right)A(17,36,48)\\
&\quad +\rho_j\left(\begin{smallmatrix}0&-1\\3&0\end{smallmatrix}\right)A(4,9,0)\Big) - 3^{2k+2j-6}A(1,4,0)
\end{aligned}$$

$A(T_1^*(2),(1,4,0))$

$$\begin{aligned}
&= \rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(1,16,0) + \rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(5,16,16)\\
&\quad +\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(4,4,0) + 2^{2k+j-4}\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(1,1,0)
\end{aligned}$$

$A(T_1^*(2),(4,5,0))$

$$\begin{aligned}
&= \rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(4,20,0) + \rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(9,20,20)\\
&\quad +\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(5,16,0) + 2^{2k+j-4}\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(1,5,0)
\end{aligned}$$

$A(T_2^*(2),(1,4,0))$

$$= 2^{2k+j-5}\Big(\rho_j\left(\begin{smallmatrix}0&-1\\4&0\end{smallmatrix}\right)A(1,4,0) + \rho_j\left(\begin{smallmatrix}1&-1\\2&2\end{smallmatrix}\right)A(2,2,0)\Big) + 2^j A(4,16,0)$$

$$
\begin{aligned}
&+2^{3k+2j-7}e\Big(((-1)^{k+1}+1)/8\Big)\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(1,1,0)\\
&+2^{k+j-3}\Big(e\Big(((-1)^{k+1}+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(1,16,0)\\
&+e\Big((5(-1)^{k+1}+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(5,16,16)\Big)-2^{2k+2j-6}A(1,4,0)
\end{aligned}
$$

$$
\begin{aligned}
&A(T_2^*(2),(4,5,0))\\
&=\ 2^{2k+j-5}\Big(\rho_j\left(\begin{smallmatrix}4&0\\0&1\end{smallmatrix}\right)A(1,20,0)+\rho_j\left(\begin{smallmatrix}4&-2\\0&1\end{smallmatrix}\right)A(6,20,20)\Big)\\
&\quad+2^jA(16,20,0)+2^{3k+2j-7}e\Big(((-1)^{k+1}5+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(1,5,0)\\
&\quad+2^{k+j-3}(e(((-1)^{k+1}9+1)/8)\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(9,20,20)\\
&\quad+\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(5,16,0))-2^{2k+2j-6}A(4,5,0)
\end{aligned}
$$

In the above, for $F\in S^+_{k-1/2}(\Gamma_0(4).\psi)$, by definition of the plus space, we have $A(1,4,0)=0$ if k is odd. So we can always assume that k is even.

$$
\begin{aligned}
&A(T_1^*(2),(3,4,0))\\
&=\ 2^{2k+j-4}\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(1,3,0)+\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(3,16,0)\\
&\quad+\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(7,16,16)+\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(4,12,0)
\end{aligned}
$$

$$
\begin{aligned}
&A(T_2^*(2),(3,4,0))\\
&=\ 2^{2k+j-5}\Big(\rho_j\left(\begin{smallmatrix}0&-1\\4&0\end{smallmatrix}\right)A(1,12,0)+(-1)\rho_j\left(\begin{smallmatrix}0&1\\4&-2\end{smallmatrix}\right)A(4,12,12)\Big)\\
&\quad+2^jA(12,16,0)+2^{3k+2j-7}e\Big(((-1)^{k+1}3+1)/8\Big)\rho_j\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)A(1,3,0)\\
&\quad+2^{k+j-3}\Big(e\Big(((-1)^{k+1}3+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(3,16,0)\\
&\quad+e\Big(((-1)^{k+1}7+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(7,16,16)\Big)-2^{2k+2j-6}A(3,4,0)
\end{aligned}
$$

$$
\begin{aligned}
&A(T_1^*(2),(3,3,2))\\
&=\ 2^{2k+j-4}\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(2,3,4)+\rho_j\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)A(3,12,4)\\
&\quad+\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(8,12,16)+(-1)\rho_j\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)A(3,12,4)\\
&\quad-2^{k+j-2}A(3,3,2)
\end{aligned}
$$

$$
\begin{aligned}
&A(T_2^*(2),(3,3,2))\\
&=\ 2^{2k+j-5}\Big(\rho_j\left(\begin{smallmatrix}4&-1\\0&1\end{smallmatrix}\right)A(2,12,8)+\rho_j\left(\begin{smallmatrix}4&-3\\0&1\end{smallmatrix}\right)A(9,12,20)\Big)\\
&\quad+2^jA(12,12,8)+2^{3k+2j-7}e\Big(((-1)^{k+1}3+1)/8\Big)\rho_j\left(\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right)A(2,3,4)\\
&\quad+2^{k+j-3}e\Big(((-1)^{k+1}3+1)/8\Big)
\end{aligned}
$$

$$\times\Big(\rho_j \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) A(3,12,4) + (-1)\rho_j \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right) A(3,12,4)\Big)$$
$$+2^{2k+2j-6} e\big(((-1)^{k+1}3+1)/4\big) A(3,3,2)$$

In the above formula, if k is even, then by the very definition of the plus space, we have $A(3,3,2)=0$, so we can always assume that k is odd.

5 Jacobi forms and Klingen type Eisenstein series

Here, for the reader's convenience, we explain the isomorphism between vector valued holomorphic or skew holomorphic Jacobi forms of index one and the plus space of vector valued Siegel modular forms of half integral weight of general degree. We also define Klingen type Jacobi Eisenstein series, and hence Klingen type Eisenstein series of half integral weight in the plus space. For simplicity, we assume here that $n=2$ though the results can be easily generalized. Most of the materials in this section are in [3], [17], [2], [9], [7], [5], [13], and [23], [6]. A definition of vector valued Klingen Eisenstein series has not been treated in the above papers and we sketch it here but it is almost the same as in the known cases.

5.1 Definition of holomorphic and skew holomorphic Jacobi forms

The symplectic group $\mathrm{Sp}(n,\mathbb{R})$ acts on $H_n \times \mathbb{C}^n$ by

$$M(\tau,z) = ((a\tau+b)(c\tau+d)^{-1}, {}^t(c\tau+d)^{-1}\tau)$$

for

$$(\tau,z)\in H_n\times\mathbb{C}^n, \quad M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{Sp}(n,\mathbb{R}) \quad (a,b,c,d\in M_n(\mathbb{R}))\,.$$

The Jacobi group $\mathrm{Sp}(2,\mathbb{R})^J$ is defined by

$$\{(M,([\mu,\nu],\kappa));\, M\in \mathrm{Sp}(2,\mathbb{R}),\ \mu,\nu\in\mathbb{R}^2, \kappa\in\mathbb{R}\}$$

as a set, with product given by

$$(M,([0,0],0))(1_{2n},([\lambda,\mu],\kappa)) = (M,([\lambda,\mu],\kappa))$$
$$(1_{2n},([\lambda,\mu],\kappa))(M,([0,0],0)) = (M,\ ([\,{}^ta\lambda + {}^tc\mu,\ {}^tb\lambda + {}^tc\mu],\kappa))$$
$$(M,([0,0],0))(M',[(0,0),0)) = (MM',([0,0],0))$$

and

$$(1_{2n},([\lambda,\mu],\kappa))(1_{2n},([\lambda',\mu'],\kappa'))$$
$$= (1_{2n},([\lambda+\lambda',\mu+\mu'],\kappa+\kappa'+{}^t\lambda\mu'-{}^t\mu\lambda'))\,.$$

We identify $\mathrm{Sp}(n,\mathbb{R})$ and the Heisenberg group $H(\mathbb{Z}) = \{([\lambda,\mu],\kappa)\}$ with the corresponding subgroup of $\mathrm{Sp}(n,\mathbb{R})^J$, respectively. We define Γ_n^J the subgroup of $Sp(n,\mathbb{R})^J$ such that $M \in \Gamma_n$, λ, $\mu \in \mathbb{Z}^n$, $\kappa \in \mathbb{Z}$. For any irreducible polynomial representation ρ of $\mathrm{GL}_n(\mathbb{C})$ and for any function $F(\tau,z)$ on $H_n \times \mathbb{C}^n$, we define two kinds of group actions $|\gamma$ or $|^{sk}\gamma$ of $\mathrm{Sp}(n,\mathbb{R})^J$ (of index 1). One is

$$\begin{aligned} F|_1([\lambda,\mu],\kappa) &= e({}^t\lambda\tau\lambda + 2{}^t\lambda z + {}^t\lambda\mu + \kappa)F(\tau, z+\tau\lambda+\mu) \\ F|_{\rho,1}M &= e(-{}^tz(c\tau+d)^{-1}cz)\rho(c\tau+d)^{-1}F(M\tau,\ {}^t(c\tau+d)z) \end{aligned}$$

($M \in \mathrm{Sp}(n,\mathbb{R})$, λ, $\mu \in \mathbb{R}^n$, $\kappa \in \mathbb{R}$) and the other one is given by

$$F|_1^{sk}([\lambda,\mu],\kappa) = F|_1([\lambda,\mu],\kappa)$$

and

$$\begin{aligned} F|_{\rho,1}^{sk}M = {} & e(-{}^tz(c\tau+d)^{-1}cz) \\ & \times \left(\frac{|\det(c\tau+d)|}{\det(c\tau+d)}\right)\overline{\rho(c\tau+d)^{-1}}F(M\tau,\ {}^t(c\tau+d)z), \end{aligned}$$

where $\overline{*}$ means complex conjugation. We say that F is a holomorphic Jacobi form of weight ρ of index 1 if
(0) F is holomorphic on $H_n \times \mathbb{C}^n$.
(1) $F|_{\rho,1}\gamma = F$ for any $\gamma \in \Gamma_n^J$ and
(2) $F(\tau,z)$ has the Fourier expansion of the following form,

$$F(\tau,z) = \sum_{N \in L_n^*, r \in \mathbb{Z}^n} A(N,r)e(\mathrm{tr}(N\tau) + {}^trz),$$

where we denote by L_n^* the set of all half integral symmetric matrices and (N,r) runs over all elements in $L_n^* \times \mathbb{Z}^n$ such that $4N - r{}^tr \geq 0$ (positive semi-definite). The space of such functions is denoted by $J_{\rho,1}$. For $\mu \in \mathbb{Z}^n$, we define

$$\vartheta_\mu(\tau,z) = \sum_{p \in \mathbb{Z}^n} e\left({}^t(p+\mu/2))\tau(p+\mu/2) + 2{}^t(p+\mu/2)z\right).$$

Then for any $F \in J_{\rho,1}$, we can write

$$F(\tau,z) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} h_\mu(\tau)\vartheta_\mu(\tau,z),$$

where the $h_\mu(\tau)$ are holomorphic and uniquely determined by F. We define

$$\sigma(F) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} h_\mu(4\tau).$$

The definition of skew holomorphic Jacobi forms was introduced by Skoruppa for $n = 1$ and by Arakawa for general n. We say that $F(\tau, z)$ is a skew holomorphic Jacobi form of weight ρ of index 1 if
(0) F is holomorphic with respect to z and real analytic with respect to the real and the imaginary part of τ.
(1) $F|^{sk}_{\rho,1}\gamma = F$ for any $\gamma \in \Gamma^J_n$. and
(2) F has a Fourier expansion of the following form.

$$F(\tau, z) = \sum_{N \in L_n^*, r \in \mathbb{Z}^n} A(N, r) e(tr(N\tau - \frac{1}{2} i(4N - r^t r)y)) e({}^t r z) \ ,$$

where y is the imaginary part of τ and (N, r) runs over $L_n^* \times \mathbb{Z}^n$ such that $r^t r - 4N \geq 0$. The space of such functions is denoted by $J^{\text{skew}}_{\rho,1}$. For any $F \in J_{\rho,1}$, we have

$$F(\tau, z) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} h_\mu(\tau) \vartheta_\mu(\tau, z)$$

where the $h_\mu(\tau)$ are uniquely determined functions. We define

$$\sigma(F) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} h_\mu(-4\overline{\tau}).$$

The definition of the Hecke operators $T_i(p)$ $(0 \leq i \leq n)$ associated with the diagonal matrix $K_i(p^2) = (1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)$ for holomorphic or skew holomorphic Jacobi forms is given in [9] or [5] for the scalar valued case. The vector valued case is obtained by replacing the automorphy factor $\det^k$ by ρ. For any ρ, we write $\rho = \det^k \rho_0$ where k is the largest integer such that ρ_0 is a polynomial representation. Then Siegel modular forms of half-integral weight $\det^{k-1/2} \rho_0$ are defined similarly by taking as automorphy factor $(\theta(\gamma\tau)/\theta(\tau))^{2k-1} \rho_0(c\tau + d)$. The definition of the plus space is the same as in the scalar valued case, parity depending only on k and the character. The definition of Hecke operators $T_{\text{half},i}(p)$ on half-integral weight associated with the $\widetilde{\Gamma}_0(4)$ double coset containing $(K_i(p^2), p^{(n-i)/2})$ is similar.

Theorem 5.1 (cf. [9],[7], [5],[13]). *For any irreducible representation $\rho = \det^k \rho_0$ as above, the linear map σ gives an isomorphism*

$$J_{\rho,1}(\Gamma_2) \cong A^+_{\det^{k-1/2} \rho_0}(\Gamma_0(4), \psi^k).$$

and

$$J^{\text{skew}}_{\rho,1}(\Gamma_2) \cong A^+_{\det^{k-1/2} \rho_0}(\Gamma_0(4), \psi^{k-1}).$$

For odd primes p, this isomorphism σ commutes with Hecke operators, i.e.,

$$T_i(p^2)F = p^{(3n+i)/2}\left(\frac{-1}{p}\right)^{(k+\delta)i} T_{\mathrm{half},i}(p)(\sigma(F))$$

where $\delta = 0$ or 1 for holomorphic or skew holomorphic Jacobi forms, respectively.

For simplicity we assume now that $n = 2$ and denote by $\rho_{k,j}$ the representation $\det^k \mathrm{Sym}_j$ as before. The Φ operator on Siegel modular forms of half-integral weight is defined as usual. As for Jacobi forms, for any function $F(\tau, z)$ on $H_2 \times \mathbb{C}^2$, the Siegel Φ-operator is defined by

$$(\Phi F)(\tau_1, z_1) = \lim_{\lambda\to\infty} F\left(\begin{pmatrix}\tau_1 & 0\\ 0 & i\lambda\end{pmatrix}, \begin{pmatrix} z_1 \\ 0\end{pmatrix}\right),$$

where $(\tau_1, z_1) \in H_1 \times \mathbb{C}$ (cf. [23]). If $F \in J_{\rho_{k,j},1}$ or $J^{\mathrm{skew}}_{\rho_{k,j},1}$, then we have $\Phi(F) = \phi e_1$ where $e_1 = {}^t(1, 0, \ldots, 0)$ and ϕ is a Jacobi form of degree one belonging to $J_{k+j,1}$ or $J^{\mathrm{skew}}_{k+j,1}$, respectively. It is well-known and easy to see that $J_{k+j,1} = 0$ or $J^{\mathrm{skew}}_{k+j,1} = 0$ if $k + j$ is odd or even respectively. We sometimes identify ϕe_1 with ϕ. We would like to define F such that $\Phi(F) = \phi$ for a given ϕ. This can be done using Klingen type Jacobi Eisenstein series similarly as in [23] and [6]. To define these we need some notation. We put

$$P_1(\mathbb{Z}) = \left\{\begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 0 & \mathbb{Z}\end{pmatrix}\right\} \cap \Gamma_2$$

and

$$P_1(\mathbb{Z})^J = \left\{(M, ([\lambda, \mu], \kappa)) \;\middle|\; M \in P_1(\mathbb{Z}); \lambda = \begin{pmatrix}\lambda_1 \\ 0\end{pmatrix} \in \mathbb{Z}^2, \mu \in \mathbb{Z}^2, \kappa \in \mathbb{Z}\right\}.$$

We denote by j an even natural number. First of all, we assume that k is even. We take a holomorphic Jacobi form $\phi(\tau_1, z_1) \in J_{k+j,1}$ of weight $k + j$ and of index 1 where $(\tau_1, z_1) \in H \times \mathbb{C}$. For $\tau = \left(\begin{smallmatrix}\tau_1 & z_0 \\ z_0 & \tau_2\end{smallmatrix}\right) \in H_2$ and $z = \left(\begin{smallmatrix} z_1 \\ z_2\end{smallmatrix}\right) \in \mathbb{C}^2$, we define f_ϕ by $f_\phi(\tau, z) = \phi(\tau, z)$. We put

$$E_{(k,j)}(\tau, z, \phi) = \sum_{\gamma \in P_1(\mathbb{Z})^J \backslash \Gamma_2^J} (f_\phi|_{\rho_{k,j},1}\gamma)(\tau, z).$$

This converges when $k > 5$. We have $E_{(k,j)} \in J_{\rho_{k,j},1}$ and $\Phi(F) = \phi e_1$. Next we assume that k is odd. For $\phi \in J^{\mathrm{skew}}_{k+j,1}$, we put

$$E^{\mathrm{skew}}_{(k,j)}(\tau, z, \phi) = \sum_{\gamma \in P_1(\mathbb{Z})^J \backslash \Gamma_2^J} (f_\phi|^{sk}_{\rho_{k,j},1}\gamma)(\tau, z).$$

This converges also when $k > 5$. We have $E^{\mathrm{skew}}_{(k,j)} \in J^{\mathrm{skew}}_{\rho_{k,j},1}$ and $\Phi(F) = \phi e_1$. We can show that $E_{(k,j)}(\tau, z, \phi)$ or $E^{\mathrm{skew}}_{(k,j)}(\tau, z, \phi)$ is a Hecke eigenform if and only if ϕ is so. For any $F \in J_{\rho_{k,j},1}$ or $J^{\mathrm{skew}}_{\rho_{k,j},1}$, it is easy to see that we have $\Phi(\sigma(F)) = \sigma(\Phi(F))$. We have $\sigma(E_{(k,j)}) \in A^{+}_{k-1/2,j}(\Gamma_0(4))$ and $\sigma(E_{(k,j)}) \in A^{+}_{k-1/2,j}(\Gamma_0(4))$. We call these functions also the Klingen type Eisenstein series of half-integral weight. Now by Eichler-Zagier [3] or Skoruppa [16], the above ϕ corresponds to a modular form g of half-integral weight $(k + j) - 1/2$ of one variable in the plus space. By Kohnen [14], this g corresponds to a modular form f of integral weight $2k+2j-2$. If ϕ is a Hecke eigen form, then so is f. It is proved in the same way as in [7] pp.517–518 that $L(s, \sigma(E_{(k,j)}))$ or $L(s, \sigma(E^{\mathrm{skew}}_{(k,j)}))$ is equal to

$$\zeta(s - j - 1)\zeta(s - 2k - j + 4)L(s, f).$$

Appendix: Table of Fourier coefficients

We give here some Fourier coefficients which are needed to calculate Euler 3-factors.

A.1 Integral weight

For the sake of simplicity, in the tables below, we give $\binom{j}{\nu}^{-1}$ times the ν-th component of the Fourier coefficients.

Fourier coefficients of $F_{5,18}$

(1, 1, 1)	(0, 0, 0, 0, 0, −13, −39, −56, −42, 0, 42, 56, 39, 13, 0, 0, 0, 0, 0)
(3, 3, 3)	(0, −72176832, −72176832, −35823060, 530712, 22121775, 28950129, 24964254, 14112630, 0, −14112630, −24964254, −28950129, −22121775, −530712, 35823060, 72176832, 72176832, 0)
(1, 7, 1)	(0, 0, 0, 0, 0, 55692, 167076, −312984, −2031624, −752976, 12641832, 29628144, 1511244, −186309396, −743285088, −545257440, 7212022272, 66466842624, 363771233280)

Fourier coefficients of $F_{5,20}$

(1, 1, 1)	(0, 0, 0, 0, 0, −10, −30, −47, −48, −30, 0, 30, 48, 47, 30, 10, 0, 0, 0, 0, 0)
(3, 3, 3)	(0, −271257984, −271257984, −152285184, −33312384, 10991250, −19374282, −71372997, −91968912, −62764362, 0, 62764362, 91968912, 71372997, 19374282, −10991250, 33312384, 152285184, 271257984, 271257984, 0)
(1, 7, 1)	(0, 0, 0, 0, 0, 42840, 128520, 264708, 459072, −1507896, −10082880, −8142120, −8142120, 66806208, 194800572, −6014376, −1196526600, −3357573120, −17302150656, −115867874304, −1848572928, 3544437657600)

Fourier coefficients of $f_{5,24a}$ and $f_{5,24b}$. We denote by (a, c, b, ν) the ν-th component of the Fourier coefficients at (a, c, b). If $a = c$, then the ν-th component is easily obtained from the $(j - \nu)$-th component by the relation

$$(-1)^k \rho_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & a \end{pmatrix} = \begin{pmatrix} a & b/2 \\ b/2 & a \end{pmatrix}.$$

So we omit half of the components.

Fourier coefficients of $f_{5,24a}$ and $f_{5,24b}$

Matrix	$f_{5,24a}$	$f_{5,24b}$
(1, 1, 1, 0)	0	0
(1, 1, 1, 1)	0	0
(1, 1, 1, 2)	0	0
(1, 1, 1, 3)	0	0
(1, 1, 1, 4)	0	0
(1, 1, 1, 5)	−1065725580	32022220
(1, 1, 1, 6)	−3197176740	96066660
(1, 1, 1, 7)	−4633759802	139577242
(1, 1, 1, 8)	−3614881088	109997888
(1, 1, 1, 9)	−490246470	17470950
(1, 1, 1, 10)	2280138630	−65162790
(1, 1, 1, 11)	2398195767	−69714887
(1, 1, 1, 12)	0	0

(3, 3, 3, 0)	0	0
(3, 3, 3, 1)	−57680814147751872	1965334455387072
(3, 3, 3, 2)	−57680814147751872	1965334455387072
(3, 3, 3, 3)	−79892317549985472	2530636395440832
(3, 3, 3, 4)	−102103820952219072	3095938335494592
(3, 3, 3, 5)	−88644356379770868	2616377386560948
(3, 3, 3, 6)	−39513923832640860	1091953548639900
(3, 3, 3, 7)	13841245593223650	−520288602797250
(3, 3, 3, 8)	3997492080187536	−1263304492279200
(3, 3, 3, 9)	32753921485717278	−972251330000958
(3, 3, 3, 10)	11358118125502722	−274488111823842
(3, 3, 3, 11)	−1274040532609515	98637117316155
(3, 3, 3, 12)	0	0
(1, 7, 1, 0)	0	0
(1, 7, 1, 1)	0	0
(1, 7, 1, 2)	0	0
(1, 7, 1, 3)	0	0
(1, 7, 1, 4)	0	0
(1, 7, 1, 5)	4565568384720	−137183190480
(1, 7, 1, 6)	13696705154160	−411549571440
(1, 7, 1, 7)	−20125272546792	560249344872
(1, 7, 1, 8)	−144419047573248	4161562046208
(1, 7, 1, 9)	−40199266068696	1269696712536
(1, 7, 1, 10)	978023462535000	−27744078783960
(1, 7, 1, 11)	1416905286935292	−40456367945532
(1, 7, 1, 12)	−4941068164623360	140166523284480
(1, 7, 1, 13)	−14069719692603036	396569445933276
(1, 7, 1, 14)	20781880935902856	−614348042111496
(1, 7, 1, 15)	115846315862908680	−3315904201378440
(1, 7, 1, 16)	−45366806527725312	1588017523099392
(1, 7, 1, 17)	−905887869644583000	26592097111617240
(1, 7, 1, 18)	−722781032781399984	18020442954662064
(1, 7, 1, 19)	5659154519219412624	−176346931613794704
(1, 7, 1, 20)	13313426015445373440	−382512025094976000
(1, 7, 1, 21)	−31897628338076621568	1006879261703793408
(1, 7, 1, 22)	−274312629286921032192	7649076209441812992
(1, 7, 1, 23)	775542991272035728896	−27722461484811050496
(1, 7, 1, 24)	18282369435943326658560	−563378797820507074560

Fourier coefficients of $f_{5,26a}$ and $f_{5,26b}$

Matrix	$f_{5,26a}$	$f_{5,26b}$
(1, 1, 1, 0)	0	0
(1, 1, 1, 1)	0	0
(1, 1, 1, 2)	0	0
(1, 1, 1, 3)	0	0
(1, 1, 1, 4)	0	0
(1, 1, 1, 5)	−65362127	−698150645
(1, 1, 1, 6)	−196086381	−2094451935
(1, 1, 1, 7)	−293270485	−3083495479
(1, 1, 1, 8)	−258012162	−2559872886
(1, 1, 1, 9)	−92619264	−581765568
(1, 1, 1, 10)	99390228	1629055260
(1, 1, 1, 11)	197894213	2672858375
(1, 1, 1, 12)	150772479	1957582341
(1, 1, 1, 13)	0	0
(3, 3, 3, 0)	0	0
(3, 3, 3, 1)	8828682282969120	267842148592629600
(3, 3, 3, 2)	8828682282969120	267842148592629600
(3, 3, 3, 3)	−5270746587314076	47392227548316684
(3, 3, 3, 4)	−19370175457597272	−173057693495996232
(3, 3, 3, 5)	−22713697964516067	−234166624356665745
(3, 3, 3, 6)	−15301314108070461	−135934565033691855
(3, 3, 3, 7)	−4293834304435281	7922247106615029
(3, 3, 3, 8)	3147931030214646	83687574697944498
(3, 3, 3, 9)	4514603375393028	59187990031424172
(3, 3, 3, 10)	1948236106302108	−16207124944380300
(3, 3, 3, 11)	−793744931583747	−69036654729809745
(3, 3, 3, 12)	−1374600847934457	−59198826380455971
(3, 3, 3, 13)	0	0
(1, 7, 1, 0)	0	0
(1, 7, 1, 1)	0	0
(1, 7, 1, 2)	0	0
(1, 7, 1, 3)	0	0
(1, 7, 1, 4)	0	0
(1, 7, 1, 5)	280011352068	2990877363180
(1, 7, 1, 6)	840034056204	8972632089540
(1, 7, 1, 7)	−64803927852	−7006354420644

(1, 7, 1, 8)	−4179374640360	−69897700767096
(1, 7, 1, 9)	−5957294015376	−58447436865840
(1, 7, 1, 10)	7439078119248	294803996050800
(1, 7, 1, 11)	87684101757252	770654941762860
(1, 7, 1, 12)	356834320429356	30375725391396
(1, 7, 1, 13)	100273026367800	−3441731807036952
(1, 7, 1, 14)	−4078946072879964	−8820348405789492
(1, 7, 1, 15)	−9910281374768412	−3591595424266740
(1, 7, 1, 16)	20223662622618096	57878193051642960
(1, 7, 1, 17)	108510171590526048	109772358846004512
(1, 7, 1, 18)	−84232703324738808	−697171710871429416
(1, 7, 1, 19)	−1093102324323664140	−2574974409947325252
(1, 7, 1, 20)	−10267763962648572	7181473549651696620
(1, 7, 1, 21)	11129160276266097060	5590177686127992780
(1, 7, 1, 22)	8613076555184317920	49190764056017351328
(1, 7, 1, 23)	−12115302660676205208	−472827529196854612704
(1, 7, 1, 24)	−282688453093682845440	−1514111683998966877440
(1, 7, 1, 25)	1122572073240871925760	4629136865780229580800
(1, 7, 1, 26)	8479594535903665574400	6404196822113033152000

Fourier coefficients of $F_{7,12}$, $F_{7,14}$ and $F_{7,16}$

Matrix	$F_{7,12}$	$F_{7,14}$	$F_{7,16}$
(1, 1, 1, 0)	0	0	0
(1, 1, 1, 1)	0	0	0
(1, 1, 1, 2)	0	0	0
(1, 1, 1, 3)	−3	−11	13
(1, 1, 1, 4)	−6	−22	26
(1, 1, 1, 5)	−5	−23	30
(1, 1, 1, 6)	0	−14	25
(1, 1, 1, 7)	5	0	14
(1, 1, 1, 8)	6	14	0
(1, 7, 1, 0)	0	0	0
(1, 7, 1, 1)	0	0	0
(1, 7, 1, 2)	0	0	0
(1, 7, 1, 3)	12852	47124	−55692
(1, 7, 1, 4)	25704	94248	−111384

Fourier coefficients of $F_{7,12}$, $F_{7,14}$ and $F_{7,16}$ (cont.)

Matrix	$F_{7,12}$	$F_{7,14}$	$F_{7,16}$
(1, 7, 1, 5)	−183060	−69948	−280440
(1, 7, 1, 6)	−613440	−445464	−562860
(1, 7, 1, 7)	139860	−617904	839664
(1, 7, 1, 8)	3482136	−172872	5725440
(1, 7, 1, 9)	−9955764	7715484	12379752
(1, 7, 1, 10)	−80317440	36284472	15574860
(1, 7, 1, 11)	−97843680	90293148	−20239920
(1, 7, 1, 12)	367804800	151132608	−187748136
(1, 7, 1, 13)	*	−237714048	−298479636
(1, 7, 1, 14)	*	−2784862080	820572480
(1, 7, 1, 15)	*	*	−21346022880
(1, 7, 1, 16)	*	*	−198792921600
(3, 3, 3, 0)	0	0	0
(3, 3, 3, 1)	−1443420	−22517352	107469180
(3, 3, 3, 2)	−1443420	−22517352	107469180
(3, 3, 3, 3)	−312201	−17099181	44380791
(3, 3, 3, 4)	819018	−11681010	−18707598
(3, 3, 3, 5)	923085	−7094385	−50519700
(3, 3, 3, 6)	0	−3339306	−51055515
(3, 3, 3, 7)	−923085	0	−30740472
(3, 3, 3, 8)	−819018	3339306	0

A.2 half-integral weight

The coefficients in the next tables are given as row vectors, though they are regarded as column vectors in the main text.

Fourier coefficients of $F_{23/2,2}$

T	coefficient
(4, 9, 0)	(0, 47115, 0)
(1, 4, 0)	(0, 1, 0)
(1, 36, 0)	(−12729, −62577, −79704)
(9, 36, 0)	(0, −15424819875, 0)
(5, 17, 14)	(−12729, −37119, −29856)

Fourier coefficients of $F_{25/2,2}$

T	coefficient
(3, 3, 2)	(−1, 0, 1)
(3, 4, 4)	(1, 2, 0)
(11, 8, 8)	(−88659, −177318, 0)
(27, 27, 18)	(186439477563, 0, −186439477563)
(19, 4, 4)	(291057, 582114, 0)
(3, 27, 6)	(0, 506412, 506412)

Fourier coefficients of $f_{29/2,a}$

T	coefficient
(3, 4, 0)	(0, 148, 0)
(3, 36, 0)	(0, −639309348, 0)
(4, 27, 0)	(0, −313727592, 0)
(7, 36, 24)	(182700984, 954103848, 565559136)
(19, 36, 48)	(205843728, −177014424, −565559136)
(27, 36, 0)	(0, −2064912998383584, 0))

Fourier coefficients of $f_{29/2,b}$

T	coefficient
(3, 4, 0)	(0, 16, 0)
(3, 36, 0)	(0, −85729104, 0)
(4, 27, 0)	(0, −33480480, 0)
(7, 36, 24)	(23092896, 117705696, 57084768)
(19, 36, 48)	(37528032, 3536160, −57084768)
(27, 36, 0)	(0, −176777487932544, 0)

Fourier coefficients of $f_{31/2,2a}$ and $f_{31/2,2b}$

T	$f_{31/2,2a}$	$f_{31/2,2b}$
(1, 4, 0)	(0, 0, 0)	(0, 1, 0)
(1, 36, 0)	(0, 0, 0)	(0, −3139803, 0)
(4, 9, 0)	(0, −408, 0)	(0, 114939, 0)

Fourier coefficients of $f_{31/2,2a}$ and $f_{31/2,2b}$ (cont.)

T	$f_{31/2,2a}$	$f_{31/2,2b}$
(5, 17, 14)	(−232, 68, 188)	(−5433, −42063, −36240)
(5, 36, 24)	(−232, −396, 24)	(−5433, −52929, −83736)
(9, 36, 0)	(0, 6557038128, 0)	(0, −198464650763139, 0)
(17, 5, 14)	(−188, −68, 232)	(36240, 42063, 5433)
(17, 36, 48)	(−188, −444, −24)	(36240, 114543, 83736)

Fourier coefficients of $F_{17/2,4}$

T	coefficient
(3, 4, 0)	(0, −2, 0, 16, 0)
(7, 36, 24)	(22596, 249036, 738720, 569952, -211104)
(3, 36, 0)	(0, −3078, 0, −867024, 0)
(19, 36, 48)	(268776, 1325868, 2237760, 1414368, 211104)
(4, 27, 0)	(0, 31200, 0, −357660, 0)
(27, 36, 0)	(0, 1747574352, 0, −11599739136, 0)

Fourier coefficients of $F_{19/2,4}$

T	coefficient
(1, 4, 0)	(0, 0, 0, −1, 0)
(5, 7, 14)	(−6279, −35121, −74448, −68520, −23640)
(1, 36, 0)	(0, 0, 0, 113643, 0)
(17, 36, 48)	(23640, 163080, 421848, 484137, 208008)
(4, 9, 0)	(0, 15885, 0, −7128, 0)
(9, 36, 0)	(0, −93533616, 0, 1200752667, 0)

Fourier coefficients of $F_{21/2,4}$

T	coefficient
(3, 3, 2)	(0, 1, 0, −1, 0)
(8, 11, 8)	(0, −39798, −59697, −44307, −12204)
(4, 3, 4)	(0, −2, −3, −1, 0)

(3, 27, 6)	$(0, -115182, -345546, -634230, -403866)$
(19, 4, 4)	$(141102, 304965, 68283, 45522, 0)$
(27, 27, 18)	$(-8571080448, -53384619699, 0, 53384619699, 8571080448)$

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