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Siegel Modular Forms of Genus 2 and Level 2: Cohomological Computations and Conjectures

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1 Introduction

In this paper, we study the cohomology of certain local systems on moduli spaces of principally polarized abelian surfaces with a level 2 structure that corresponds to prescribing a number of Weierstrass points in case the abelian surface is the Jacobian of a curve of genus 2. These moduli spaces are defined over $\mathbb{Z}[1/2]$, and we can calculate the trace of Frobenius on the alternating sum of the étale cohomology groups of these local systems by counting the number of pointed curves of genus 2 with a prescribed number of Weierstrass points that separately or taken together are defined over the given finite field. This cohomology is intimately related to vector-valued Siegel modular forms. Two of the present authors carried out this scheme for local systems on the moduli space $\mathcal{A}_2$ of level 1 in [11]. This provided new information on Siegel modular forms and led for example to a precise formulation of a conjecture of Harder about congruences between genus 1 and genus 2 modular forms and also to experimental evidence supporting it, cf. [13, 16].

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Here, we extend this scheme to level 2, where new phenomena appear. In order to be able to extract information on Siegel modular forms, we must subtract the contributions to the cohomology from the boundary, that is, the Eisenstein cohomology, and the endoscopic contributions. We determine the contribution of the Eisenstein cohomology together with its $\mathbb{S}_6$-action for the full-level two-structure and on the basis of our computations, we make precise conjectures on the endoscopic contribution. We also make a prediction about the existence of a vector-valued analogue of the Saito–Kurokawa lift. Assuming these conjectures that are based on ample numerical evidence, we obtain the traces of the Hecke-operators $T(p)$ for $p \leq 37$ on the remaining spaces of "genuine" Siegel modular forms. We present a number of examples of one-dimensional spaces of eigenforms where these traces coincide with the Hecke eigenvalues to illustrate this. We hope that the experts on lifting and endoscopy will be able to prove our conjectures.

2 The Moduli Spaces $\mathcal{A}_2(w^n)$

Let $\mathcal{M}_2$ be the moduli space of curves of genus 2 and $\mathcal{A}_2$ the moduli space of principally polarized abelian surfaces. These are Deligne–Mumford stacks defined over $\mathrm{Spec}(\mathbb{Z})$, and by the Torelli map, we can view $\mathcal{M}_2$ as an open substack of $\mathcal{A}_2$. A curve of genus 2 admits a unique morphism of degree 2 to $\mathbb{P}^1$ and its ramification points are called the Weierstrass points. If the element 2 is invertible on the base, there are six Weierstrass points. We now can look at covers of $\mathcal{M}_2$, namely, for $0 \leq n \leq 6$, we consider the stack $\mathcal{M}_2(w^n)$ that is the moduli space of pairs $(C, (r_1, \ldots, r_n))$ of curves of genus 2 together with n ordered Weierstrass points. These are Deligne–Mumford stacks defined over $\mathrm{Spec}(\mathbb{Z}[1/2])$.

Let $(C, r_1, \ldots, r_6)$ be a genus 2 curve with its six numbered Weierstrass points, which together form the set W. A Weierstrass point r_1 defines an embedding of C into its Jacobian $\mathrm{Jac}(C)$ given by $q \mapsto q - r_1$. This provides us with a set of five points of order 2 of $\mathrm{Jac}(C)$, namely, $\{r_i - r_1 : i = 2, \ldots, 6\}$. The nonzero points of order 2 on $\mathrm{Jac}(C)$ correspond bijectively to the pairs $\{r_i, r_j\}$ with $i \neq j$. For the Weil pairing, we have $\langle r_j - r_i, r_k - r_i \rangle = 1$ for $j \neq k$. The map $(\mathbb{Z}/2)^W \to \mathrm{Jac}(C)[2]$ defined by $a \mapsto \sum a_i(r_i - r_1)$ makes it possible to identify $\mathrm{Jac}(C)[2]$ with the kernel of the summation map $\sum : (\mathbb{Z}/2)^W \to (\mathbb{Z}/2)$ modulo the diagonally embedded $\mathbb{Z}/2$. The symplectic form can be identified with $(a, b) \mapsto \sum a_i b_i$. Hence, we obtain an embedding of the symmetric group $\mathbb{S}_6$ into $\mathrm{GSp}(4, \mathbb{Z}/2)$ (and $\mathrm{Sp}(4, \mathbb{Z}/2)$) and this is an isomorphism.

By associating to a (decorated) curve its (decorated) Jacobian, we have an embedding

$$\mathcal{M}_2(w^6) \hookrightarrow \mathcal{A}_2[2],$$

where $\mathcal{A}_2[2]$ is the moduli space of principally polarized abelian surfaces X with a full level two-structure, that is, an isomorphism of the kernel $X[2]$ of multiplication $2_X \times 2$ on X with a fixed symplectic module $((\mathbb{Z}/2)^4, \langle\ ,\ \rangle)$. The symmetric group $\mathbb{S}_6$ acts on $\mathcal{M}_2(w^6)$ and the group $\mathrm{GSp}(4, \mathbb{Z}/2)$ acts on $\mathcal{A}_2[2]$ and the embedding defines the isomorphism of $\mathbb{S}_6$ with $\mathrm{GSp}(4, \mathbb{Z}/2)$ given above. Let us identify $\mathrm{GSp}(4, \mathbb{Z}/2)$ with $\mathbb{S}_6$ under this isomorphism and define the quotient stacks

$$\mathcal{A}_2(w^n) := \frac{\mathcal{A}_2[2]}{\mathbb{S}_{6-n}},$$

where $\mathbb{S}_{6-n}$ is the subgroup of $\mathbb{S}_6$ fixing $\{1, \ldots, n\}$ pointwise. Note that we have inclusions $\mathcal{M}_2(w^n) \hookrightarrow \mathcal{A}_2(w^n)$, and equalities $\mathcal{A}_2(w^0) = \mathcal{A}_2$ and $\mathcal{A}_2(w^6) = \mathcal{A}_2[2]$.

Starting instead on the side of the abelian varieties, we let U be a symplectic space of dimension 4 over $\mathbb{Z}/2$ with basis e_1, e_2, f_1, f_2, and with symplectic form $\langle\ ,\ \rangle$ such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle e_i, f_j\rangle = \delta_{i\,j}$ (Kronecker delta). One observes that U contains six (maximal) sets V_i $(i = 1, \ldots, 6)$ of five vectors $u_j \in U - \{0\}$ with $\langle u_j, u_k\rangle = 1$ for $j \neq k$. For instance, one such set is $\{e_1, f_1, e_1 + f_1 + f_2, e_1 + e_2 + f_1, e_1 + e_2 + f_1 + f_2\}$. We have $\#(V_i \cap V_j) = 1$ if $i \neq j$. The action of $\mathrm{GSp}(4, \mathbb{Z}/2)$ on these six sets V_i defines an isomorphism of $\mathrm{GSp}(4, \mathbb{Z}/2)$ with $\mathbb{S}_6$ but this is compatible with the one above as a Weierstrass point r_i on a curve of genus 2 determines a set of five points $r_j - r_i$ $(j \neq i)$ of order 2 and they satisfy $\langle r_j - r_i, r_k - r_i\rangle = 1$ for $j \neq k$.

Consider the inverse image $\Gamma_2(w^n)$ under $\mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/2) \to \mathbb{S}_6$ of a subgroup $\mathbb{S}_{6-n}$ of $\mathbb{S}_6$ fixing the set V_i for each i between 1 and n. Then, the orbifold $\Gamma_2(w^n)\backslash\mathcal{H}_2$, with $\mathcal{H}_2$ the Siegel upper half space of degree 2, is the complex fibre of the moduli stack $\mathcal{A}_2(w^n)$. By $\Gamma_2[2] = \Gamma_2(w^6)$, we shall mean the kernel of $\mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/2)$.

3 Local Systems

Let $\pi : \mathcal{X} \to \mathcal{A}_2$ denote the universal abelian surface. The moduli space $\mathcal{A}_2$ carries natural local systems $\mathbb{V}_{l,m}$ indexed by the pairs (l, m) of integers with $l \geq m \geq 0$ that correspond to irreducible representations of $\mathrm{GSp}(4)$, cf. [11]. The local system $\mathbb{V}_{1,0}$ is the one defined by $\mathbb{V} := R^1\pi_*\mathbb{Q}_\ell$ and $\mathbb{V}_{l,m}$ is of weight $l+m$ and occurs "for the first time" in $\mathrm{Sym}^{l-m}(\mathbb{V}) \otimes \mathrm{Sym}^m(\wedge^2\mathbb{V})$. When $l > m > 0$, the local system is called *regular*. By pullback to $\mathcal{A}_2(w^n)$, we obtain local systems that we will denote by the same symbol $\mathbb{V}_{l,m}$. We are interested in the cohomology (with compact support) of these local systems, more precisely in the motivic Euler characteristic

$$e_c(\mathcal{A}_2(w^n), \mathbb{V}_{l,m}) = \sum_{i=0}^{6}(-1)^i\big[H_c^i(\mathcal{A}_2(w^n), \mathbb{V}_{l,m})\big],$$

where this expression is taken in the Grothendieck group of an appropriate category (for instance, the category of mixed Hodge structures). Note that this cohomology is zero if $l+m$ is odd and thus we will from now on only consider the case when $l+m$ is even.

For any $\mathcal{A}$, among the moduli spaces considered, there is a natural map $H_c^*(\mathcal{A}, \mathbb{V}_{l,m}) \to H^*(\mathcal{A}, \mathbb{V}_{l,m})$ whose image is called the inner cohomology and denoted by $H_!^*(\mathcal{A}, \mathbb{V}_{l,m})$. By work of Faltings and Chai, one knows that for regular λ, the cohomology groups $H_!^i(\mathcal{A}, \mathbb{V}_{l,m})$ vanish for $i \neq 3$ (see [9, Corollary to Theorem 7, p. 84] and [10, pp. 233–7]). Moreover, one knows that $H^3(\mathcal{A}, \mathbb{V}_{l,m})$ (resp. $H_c^3(\mathcal{A}, \mathbb{V}_{l,m})$) carry mixed Hodge structures of weights $\geq l+m+3$ (resp. $\leq l+m+3$) and that $H_!^3(\mathcal{A}, \mathbb{V}_{l,m})$ carries a pure Hodge structure with Hodge filtration

$$(0) \subseteq F^{l+m+3} \subseteq F^{l+2} \subseteq F^{m+1} \subseteq F^0 = H_!^3(\mathcal{A}, \mathbb{V}_{l,m}).$$

The first step in this Hodge filtration is connected to Siegel modular forms by the isomorphism

$$F^{l+m+3} \cong S_{l-m,m+3}(\Gamma_2(w^n)),$$

where $S_{l-m,m+3}(\Gamma_2(w^n))$ is the complex vector space of vector-valued cusp forms of weight $(l-m, m+3)$ on the group $\Gamma_2(w^n)$, cf. [10, Theorem 5.5], see also [15, Theorem 17]. By a Siegel modular form of weight (j,k), we mean a vector-valued function on the Siegel upper half space that transforms with the factor of automorphy:

$$\mathrm{Sym}^j(c\tau + d)\det(c\tau + d)^k,$$

for $(a, b; c, d) \in \mathrm{Sp}(4, \mathbb{Z})$ or in a subgroup $\Gamma_2(w^n)$. We can equivalently define a Siegel modular form of weight (j,k) as a section of $\mathrm{Sym}^j(\Omega_{\mathcal{X}/\mathcal{A}_2}) \otimes (\wedge^2\Omega_{\mathcal{X}/\mathcal{A}_2})^{\otimes k}$ with $\Omega_{\mathcal{X}/\mathcal{A}_2}$ the cotangent bundle of $\mathcal{X}$ along the zero section. Thus, the cohomology of the moduli spaces considered is closely connected to Siegel modular forms of the corresponding groups. In [11], two of the three present authors studied the cohomology in the case of level 1 (i.e. on $\mathcal{A}_2$) using counts of curves of genus 2 of compact type defined over finite fields (cf. also [15]).

Recall that the Eichler–Shimura theorem says (cf. [8]) that for the local system $\mathbb{V}_k := \mathrm{Sym}^k(\mathbb{V})$ (with $\mathbb{V} := R^1\pi_*\mathbb{Q}_l$), on the moduli space $\mathcal{A}_1$ of elliptic curves with universal family $\pi : \mathcal{X}_1 \to \mathcal{A}_1$, one has for even $k \geq 2$,

$$-e_c(\mathcal{A}_1, \mathbb{V}_k) = S[\mathrm{SL}(2,\mathbb{Z}), k+2] + 1,$$
$$-e(\mathcal{A}_1, \mathbb{V}_k) = S[\mathrm{SL}(2,\mathbb{Z}), k+2] + L^{k+1},$$

where we from now on denote by $L := h^2(\mathbb{P}^1)$ the Tate motive of weight 2 and by $S[\mathrm{SL}(2,\mathbb{Z}), k+2]$ the motive for cusp forms of weight $k+2$ of $\mathrm{SL}(2,\mathbb{Z})$ as constructed by Scholl, cf. [20] (see also [7] for an alternative construction). For $k=0$, one can use the same formulas if one puts $S[\mathrm{SL}(2,\mathbb{Z}), 2] := -L - 1$.

4 The Eisenstein Cohomology

The compactly supported cohomology has a natural map to the usual cohomology and the kernel is called the *Eisenstein cohomology*. The corresponding motivic Euler characteristic is denoted by $e_{\mathrm{Eis}}(\mathcal{A}, \mathbb{V}_{l,m})$. By the *full Eisenstein cohomology*, we mean the difference between the compactly supported and the usual cohomology, with corresponding Euler characteristic,

$$e_{\mathrm{Eis}^{\mathrm{f}}}(\mathcal{A}, \mathbb{V}_{l,m}) := e_c(\mathcal{A}, \mathbb{V}_{l,m}) - e(\mathcal{A}, \mathbb{V}_{l,m}).$$

For example, for genus 1 we have by Eichler–Shimura $e_{\mathrm{Eis}}(\mathcal{A}_1, \mathbb{V}_k) = -1$, and for the full Eisenstein cohomology $e_{\mathrm{Eis}^{\mathrm{f}}}(\mathcal{A}_1, \mathbb{V}_k) = L^{k+1} - 1$.

Remark 4.1. The full Eisenstein cohomology is anti-invariant under Poincaré duality and $e_{\mathrm{Eis}}(\mathcal{A}, \mathbb{V}_{l,m})$ determines the full Eisenstein cohomology by antisymmetrizing. The converse also holds by considerations of weights if λ is regular, cf. [19]. □

We shall write $\Gamma(2)$ for the full-level two congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and $\Gamma_0(N)$ for the congruence subgroup of matrices $(a, b; c, d)$ with $N|c$. For any of these groups Γ, we will write $S_k(\Gamma)$ for the space of cusp forms of Γ of weight k, and $S[\Gamma, k]$ for the motive associated to this space. The motive $S[\Gamma, k]$ has a Hodge realization that decomposes as $S_k(\Gamma) \oplus \bar{S}_k(\Gamma)$. For $\Gamma = \Gamma_0(N)$, let $S_k(\Gamma)^{\mathrm{new}}$ denote the subspace of newforms in $S_k(\Gamma)$, and $S[\Gamma, k]^{\mathrm{new}}$ the corresponding motive, constructed in [20].

Theorem 4.2. For regular pairs (l, m), the Eisenstein cohomology of the local system $\mathbb{V}_{l,m}$ on the moduli space $\mathcal{A}_2[2]$ is given by

$$15 \dim S_{l-m+2}(\Gamma(2)) - 15 \dim S_{l+m+4}(\Gamma(2))\, L^{m+1} + 15 \begin{cases} S[\Gamma(2), m+2] + 3 & \text{if } m \text{ even} \\ -S[\Gamma(2), l+3] & \text{if } m \text{ odd.} \end{cases}$$

□

This can be proved as in [14] using the BGG-complex of Faltings–Chai (see [10]), by first computing $e_{\mathrm{Eis^f}}(\mathcal{A}_2[2], \mathbb{V}_{l,m})$ and then deducing $e_{\mathrm{Eis}}(\mathcal{A}_2[2], \mathbb{V}_{l,m})$, see Remark 4.1. If the pair (l, m) is not regular, we still expect the formula to hold as long as we put $S[\Gamma(2), 2] := -L - 1$ in case $m = 0$ and $\dim S_2(\Gamma(2)) := -1$ in case $l = m$.

The factors 15 in the formula come from the fact that the Satake compactification of $\mathcal{A}_2[2]$ has 15 one-dimensional and 15 zero-dimensional boundary components.

The group $\mathbb{S}_6$ acts on $\mathcal{A}_2[2]$ and this induces an action on the Eisenstein cohomology of a local system. We can decompose this piece of the cohomology into irreducible representations for $\mathbb{S}_6$. Note that we can identify $S_k(\Gamma(2))$ with $S_k(\Gamma_0(4))$ via the map $f(z) \mapsto f(2z)$ and the corresponding motive can be split as

$$S[\Gamma_0(4), k] = S[\Gamma_0(4), k]^{\mathrm{new}} + 2\, S[\Gamma_0(2), k]^{\mathrm{new}} + 3\, S[\mathrm{SL}(2, \mathbb{Z}), k].$$

Notation 4.3. Define $\tau_{N,k} := \dim S_k(\Gamma_0(N))^{\mathrm{new}}$. □

We also need notation concerning representations of $\mathrm{Sp}(4, \mathbb{Z}/2) \cong \mathbb{S}_6$. Let Q (resp. P) be the subgroup of $\mathrm{Sp}(4, \mathbb{Z}/2)$ that fixes a nonzero vector v (resp. a two-dimensional totally isotropic subspace V of U). It acts on $v^{\perp}/\langle v \rangle$ (resp. V), and this defines a map onto $\mathrm{SL}(2, \mathbb{Z}/2) \cong \mathbb{S}_3$. Starting from the trivial, the two-dimensional and the alternating representations of $\mathbb{S}_3$, inflating to Q (resp. P) and inducing to $\mathrm{Sp}(4, \mathbb{Z}/2)$, one obtains representations A, B, C (resp. A', B', C') of $\mathrm{Sp}(4, \mathbb{Z}/2)$. As representations of

$\mathbb{S}_6$, they are the sums of the following irreducible representations $s[p]$, p a partition of 6:

$$\begin{aligned} A &= s[6]+s[5,1]+s[4,2], & A' &= s[6]+s[4,2]+s[2^3],\\ B &= s[4,2]+s[3,2,1]+s[2^3], & B' &= s[5,1]+s[4,2]+s[3,2,1],\\ C &= s[3,1^3]+s[2,1^4], & C' &= s[4,1^2]+s[3^2]. \end{aligned}$$

We note that $\dim A = \dim A' = \dim C = \dim C' = 15$ while $\dim B = \dim B' = 30$.

Theorem 4.4. For regular pairs (l,m), the contributions in Theorem 4.2 to the Eisenstein cohomology of the local system $\mathbb{V}_{l,m}$ can be decomposed under the action of $\mathbb{S}_6$ as follows. The term $15\dim S_k(\Gamma_0(4))$ with $k = l-m+2$ or $k = l+m+4$ decomposes as

$$\tau_{1,k}\cdot A' + (\tau_{1,k}+\tau_{2,k})\cdot B' + \tau_{4,k}\cdot C'$$

while the term $15S[\Gamma_0(4),k]$ with $k = m+2$ or $k = l+3$ can be written as

$$(A+B)\otimes S[\Gamma_0(1),k] + B\otimes S[\Gamma_0(2),k]^{\mathrm{new}} + C\otimes S[\Gamma_0(4),k]^{\mathrm{new}}$$

and finally $15\cdot 3\,L^0 = (A+B)\,L^0$. □

For $l = m$, we conjecture that $15\dim S_2(\Gamma_0(4))$ decomposes as $-A'$, and for $m = 0$ that the term $15S[\Gamma_0(4),2]$ decomposes as $A\cdot(-L-1)$. The theorem can be proved by the method of [14] taking into account the action of $\mathbb{S}_6$ on the boundary components.

Remark 4.5. The coefficient $\tau_{1,k}$ (resp. $\tau_{1,k}+\tau_{2,k}$, resp. $\tau_{4,k}$) equals the multiplicity of the trivial (resp. the two-dimensional, resp. the alternating) representation in $S_k(\Gamma(2))$, viewed as a representation of $\mathrm{SL}(2,\mathbb{Z}/2)\cong\mathbb{S}_3$. Correspondingly, the term $15S[\Gamma(2),k]$ can be written as

$$A\otimes S[\Gamma(2),k]_3 + B\otimes S[\Gamma(2),k]_{2,1} + C\otimes S[\Gamma(2),k]_{1^3},$$

with $S[\Gamma(2),k]_\mu = \mathrm{Hom}_{\mathbb{S}_3}(s[\mu], S[\Gamma(2),k])$ for μ a partition of 3. □

The $\mathbb{S}_6$-decomposition of the Eisenstein cohomology on $\mathcal{A}_2[2]$ allows one to deduce the formulas for $\mathcal{A}_2(w^n)$ for $0\le n\le 6$. For example, for $\mathcal{A}_2(w^1)$ and $\mathcal{A}_2(w^3)$, we find the following.

Corollary 4.6. For regular (l, m), the Eisenstein cohomology of $\mathbb{V}_{l,m}$ on $\mathcal{A}_2(w^1)$ is given by

$$\dim S_{l-m+2}(\Gamma_0(2)) - \dim S_{l+m+4}(\Gamma_0(2))\, L^{m+1} + \begin{cases} 2(S[m+2]+1) & \text{if } m \text{ even} \\ -2\, S[l+3] & \text{if } m \text{ odd.} \end{cases}$$

□

Corollary 4.7. For regular (l, m), the Eisenstein cohomology of $\mathbb{V}_{l,m}$ on $\mathcal{A}_2(w^3)$ is given by

$$4 \dim S_{l-m+2}(\Gamma_0(4)) - 4 \dim S_{l+m+4}(\Gamma_0(4))\, L^{m+1} + \begin{cases} 3S[\Gamma_0(1), m+2] + 3S[\Gamma_0(2), m+2] + S[\Gamma_0(4), m+2] + 12 & \text{if } m \text{ even} \\ -3S[\Gamma_0(1), l+3] - 3S[\Gamma_0(2), l+3] - S[\Gamma_0(4), l+3] & \text{if } m \text{ odd.} \end{cases}$$

□

5 Counting Points Over Finite Fields

Recall that the moduli space $\mathcal{A}_2[2]$ of principally polarized abelian surfaces with level 2 structure can be identified with the moduli space of tuples $(C, r_1, \ldots, r_6)$, where C is either an irreducible genus 2 curve or a pair of genus 1 curves intersecting in one point, and where $(r_1, \ldots, r_6)$ is a six-tuple of marked Weierstrass points. In the case of two intersecting elliptic curves, these are the points of order 2 on the two elliptic curves taking the intersection point as origin on both.

For an odd prime number p, we consider this moduli space over the field $\overline{\mathbb{F}}_q$, where q is a power of p. Let $H^i_{ét}$ denote the compactly supported ℓ-adic étale cohomology. The natural action of $\mathbb{S}_6$ on $\mathcal{A}_2[2]$ induces a decomposition of $H^i_{ét}(\mathcal{A}_2[2] \otimes \overline{\mathbb{F}}_q, \mathbb{V}_{l,m})$ into pieces denoted $H^i_{ét,\mu}(\mathcal{A}_2[2] \otimes \overline{\mathbb{F}}_q, \mathbb{V}_{l,m})$ (with $H^i_{ét,\mu} = R_\mu \otimes \mathrm{Hom}_{\mathbb{S}_6}(R_\mu, H^i_{ét})$ for an irreducible representation R_μ of $\mathbb{S}_6$ indexed by the partition μ of 6). We wish to compute the trace of Frobenius F_q on the Euler characteristic

$$e_{ét,\mu}(\mathcal{A}_2[2] \otimes \overline{\mathbb{F}}_q, \mathbb{V}_{l,m}) := \sum_i (-1)^i H^i_{ét,\mu}(\mathcal{A}_2[2] \otimes \overline{\mathbb{F}}_q, \mathbb{V}_{l,m}).$$

The necessary information to compute this, for any partition μ and pair (l, m), was found for all odd $q \leq 37$ with the aid of the computer. We indicate below how this was done.

We will denote by k a finite field and by k_2 a degree 2 extension of k.

5.1 Irreducible curves of genus 2

Let $P_2(k) \subset k[x]$ be the set of all square-free polynomials of degree 5 or 6. Each element $f \in P_2(k)$ defines a curve C_f of genus 2 defined by $y^2 = f(x)$. For each $f \in P_2(k)$ and $k \in \mathcal{K} := \{\mathbb{F}_q : 2 \nmid q, q \leq 37\}$, we computed the following: (1) the number of points of C_f defined over k, (2) the number of points of C_f defined over k_2, (3) the fields of definition of all six ramification points of the canonical map $C_f \to \mathbb{P}^1$ given by $(x, y) \to x$.

For a partition ν of 6, let $P_2(\nu, k) \subset P_2(k)$ be the subset of polynomials f defining curves C_f, which have fields of definition of their ramification points given by ν. Using the Lefschetz trace formula, we can now, for each pair of numbers n_1, n_2, and $k \in \mathcal{K}$, compute

$$a(\mathcal{M}_2, \nu, n_1, n_2) := \frac{\sum_{f \in P_2(\nu,k)} a_1(C_f)^{n_1} \cdot a_2(C_f)^{n_2}}{|\mathrm{GL}_2(k)|}, \tag{5.1}$$

where $a_1(C_f) := \mathrm{Tr}(F_q, H^1_{\acute{e}t}(C_f))$ and $a_2(C_f) := \mathrm{Tr}(F_q^2, H^1_{\acute{e}t}(C_f))$. Note that $|\mathrm{GL}_2(k)|$ is the number of k-isomorphisms between the curves of $P_2(\nu, k)$ (for the actual group, see [2, Section 3]).

5.2 Pairs of elliptic curves

Similarly, let $P_1(k) \subset k[x]$ consist of all square-free polynomials $f(x) \in k[x]$ of degree 3 and let $\mathcal{K}' := \{\mathbb{F}_q, \mathbb{F}_{q^2} : 2 \nmid q, q \leq 37\}$ be a collection of finite fields. Each element of $P_1(k)$ defines an elliptic curve C_f given by $y^2 = f(x)$ with $x = \infty$ as origin. For each element $k \in \mathcal{K}'$ and $f \in P_1(k)$ with corresponding curve C_f, we computed the following: (1) the number of points of C_f defined over k, (2) the fields of definition of the three affine ramification points of the map $C_f \to \mathbb{P}^1$ given by $(x, y) \to x$.

To get the analogue of equation (5.1) for the pairs of elliptic curves joined at the origin, we should sum over all possibilities of distributing the ramification points and the marked points (which correspond to the monomials $a_1(C_f)^{n_1} a_2(C_f)^{n_2}$) on the two elliptic curves. Let us define $a(\mathcal{A}_{1,1}, \nu, n_1, n_2)$ to be the sum, over all ordered choices of partitions ρ and σ of 3 such that $\nu = \rho + \sigma$ and integers $m_1 \leq n_1$ and $m_2 \leq n_2$, of the following. We put $I'_k := |\mathrm{GL}_2(k)|/(|k| + 1)$, which is the number of k-isomorphisms between the curves of $P_1(k)$, and we divide into two cases according to if there is an automorphism that interchanges the two elliptic curves or not.

Case (i) If $\rho \neq \sigma$ or $(m_1, m_2) \neq (n_1 - m_1, n_2 - m_2)$, we add $1/2$ times

$$\left(\sum_{f \in P_1(\rho,k)} a_1(C_f)^{m_1} \cdot a_2(C_f)^{m_2}/I'_k\right) \cdot \left(\sum_{f \in P_1(\sigma,k)} a_1(C_f)^{n_1-m_1} \cdot a_2(C_f)^{n_2-m_2}/I'_k\right).$$

Case (ii) If $\rho = \sigma$ and $(m_1, m_2) = (n_1 - m_1, n_2 - m_2)$, we have the contribution from pairs of elliptic curves that are defined over k,

$$1/2 \cdot \left(\sum_{f \in P_1(\rho,k)} a_1(C_f)^{m_1} \cdot a_2(C_f)^{m_2}/I'_k\right)^2.$$

Moreover, if in addition $n_1 = 0$ and $\nu_i = 0$ for all odd i, then the two elliptic curves together with marked ramification and ordinary points may also form a conjugate pair. We construct these by taking an elliptic curve defined over k_2, and join it at the origin with its Frobenius conjugate. Define the partition $\nu^{1/2} := [1^{\nu_2} 2^{\nu_4} 3^{\nu_6}]$. We then add

$$1/2 \cdot \left(\sum_{f \in P_1(\nu^{1/2},k_2)} a_1(C_f)^{n_2}/I'_{k_2}\right).$$

In both these formulas, the factor $1/2$ is due to the extra automorphism.

5.3 Adding the contributions from the two strata

To the irreducible representation of $\mathbb{S}_6$ indexed by the partition μ of 6, we can associate s_μ, the ordinary Schur polynomial, and to an irreducible representation of the symplectic group $\mathrm{Sp}(4, \mathbb{Q})$ indexed by the partition $\lambda = [l, m]$, the Schur polynomial $s_{<\lambda>}$, see [12, Appendix A]. Written in terms of the power sums p_i, we have $s_\mu = \sum_\nu \alpha^\mu_\nu \cdot p_1^{\nu_1} \cdots p_6^{\nu_6}$ and $s_{<\lambda>} = \sum_{n_1,n_2} \beta^\lambda_{n_1,n_2} \cdot p_1^{n_1} p_2^{n_2}$ for some rational numbers α^μ_ν and $\beta^\lambda_{n_1,n_2}$. The trace of Frobenius on $e_{\acute{e}t,\mu}(\mathcal{A}_2[2] \otimes \overline{\mathbb{F}}_q, \mathbb{V}_\lambda)$ is then equal to (compare [2, Equation (3.1)] or [4, Section 4.2])

$$\sum_{n_1,n_2} \sum_\nu \alpha^\mu_\nu \beta^\lambda_{n_1,n_2} \cdot (a(\mathcal{M}_2, \nu, n_1, n_2) + a(\mathcal{A}_{1,1}, \nu, n_1, n_2)) \cdot q^{(|\lambda|-n_1-n_2)/2}.$$

Using these results, we have been able to (conjecturally) identify the non-Eisenstein pieces of $e_c(\mathcal{A}_2[2], \mathbb{V}_{l,m})$. In this process, we have greatly benefited from William Stein's tables of modular forms [21].

6 A lifting to Vector-Valued Modular Forms

The Saito–Kurokawa lifting (see e.g. [18, 24]) gives a way to transform a cusp form f that is a normalized eigenform of weight $2k$ (k odd) on $\mathrm{SL}(2,\mathbb{Z})$ into a scalar-valued cusp (eigen) form of weight $k+1$ on $\mathrm{Sp}(4,\mathbb{Z})$. In terms of L-factors, the reciprocal of the characteristic polynomial of Frobenius at a prime p is in the Saito–Kurokawa case equal to

$$(1 - p^{k-1}X)(1 - a(p)X + p^{2k-1}X^2)(1 - p^k X),$$

with $a(p)$ the Hecke eigenvalue of f at p.

Based on our calculations of the cohomology of local systems $\mathbb{V}_{l,m}$ on $\mathcal{A}_2[2]$, we conjecture the following (Yoshida type) lifting from pairs of elliptic modular forms to vector-valued Siegel modular forms. Recall the notion of spinor L-function (see [1]) and that the Atkin–Lehner involution w_2 acts on $S_k(\Gamma_0(2))$ with eigenspaces $S_k^+(\Gamma_0(2))$ and $S_k^-(\Gamma_0(2))$ for the eigenvalues $+1$ and -1.

Conjecture 6.1. For an eigenform $f \in S_{l+m+4}(\Gamma_0(2))^{\mathrm{new}}$ and an eigenform $g \in S_{l-m+2}(\Gamma_0(2))^{\mathrm{new}}$, there is a Siegel modular form $F \in S_{l-m,m+3}(\Gamma_2[2])$, an eigenform for the Hecke algebra, with spinor L-function

$$L(F, s) = L(f, s)L(g, s - m - 1).$$

The form F generates an $\mathbb{S}_6$-subrepresentation of $S_{l-m,m+3}(\Gamma_2[2])$ of dimension 5 if f and g have the same eigenvalue $\pm$ under w_2 and of dimension 1 if they have opposite eigenvalues under w_2.

Similarly, for an eigenform $f \in S_{l+m+4}(\Gamma_0(4))^{\mathrm{new}}$ and an eigenform $g \in S_{l-m+2}(\Gamma_0(4))^{\mathrm{new}}$, there is a Siegel modular form $F \in S_{l-m,m+3}(\Gamma_2[2])$ with spinor L-function

$$L(F, s) = L(f, s)L(g, s - m - 1)$$

and it will generate an $\mathbb{S}_6$-subrepresentation of $S_{l-m,m+3}(\Gamma_2[2])$ of dimension 5. □

The $\mathbb{S}_6$-subrepresentations are described below.

Remark 6.2. The first part of this conjecture is consistent with work of Böcherer and Schulze-Pillot who constructed a Yoshida-type lifting for a pair (f, g) of newforms on $\Gamma_0(2)$ with the same sign under w_2 to Siegel modular forms on the congruence subgroup $\Gamma_0^{(2)}(2) \subset \mathrm{Sp}(4, \mathbb{Z})$; see [6, Theorem 5.1 and the ensuing remark on p. 99]. □

Note that by Tsushima (see [23]), we know the dimensions of the spaces $S_{j,k}(\Gamma_2[2])$ of Siegel modular forms of weight (j, k) on the group $\Gamma_2[2]$. In the cases $(j, k) = (4, 4)$, $(6, 3)$, and $(8, 3)$, it seems that the conjectured lifts generate all of $S_{j,k}(\Gamma_2[2])$.

Let us define $S^{\mathrm{lift}}_{j,k}(\Gamma_2[2])$ to be the subspace in $S_{j,k}(\Gamma_2[2])$ consisting of the cusp forms obtained by the lifting described above. The following conjecture tells us the action of $\mathbb{S}_6$ on the space of lifted cusp forms.

Notation 6.3. Put $\tau_k^+ := \dim S_k^+(\Gamma_0(2))^{\mathrm{new}}$ and $\tau_k^- := \dim S_k^-(\Gamma_0(2))^{\mathrm{new}}$. □

Conjecture 6.4. If we assume that $l \neq m$ and let $k := l + m + 4$, $k' := l - m + 2$, then $S^{\mathrm{lift}}_{l-m,m+3}(\Gamma_2[2])$ decomposes as a representation of $\mathbb{S}_6$ as

$$\tau_{4,k}\, \tau_{4,k'} \cdot s[2, 1^4] + (\tau_k^+ \tau_{k'}^+ + \tau_k^- \tau_{k'}^-) \cdot s[2^3] + (\tau_k^+ \tau_{k'}^- + \tau_k^- \tau_{k'}^+) \cdot s[1^6].$$

□

Remark 6.5. Note that there are no vector-valued lifts of level 1. □

We also give a corresponding conjecture for the Saito–Kurokawa lifts.

Conjecture 6.6. If we assume that $l = m$ and let $k := l + m + 4$, then $S^{\mathrm{lift}}_{l-m,m+3}(\Gamma_2[2])$ decomposes as a representation of $\mathbb{S}_6$ for m odd as

$$(\tau_k^+ + \tau_{1,k}) \cdot s[4, 2] + (\tau_k^- + \tau_{1,k}) \cdot s[2^3] + \tau_{1,k} \cdot s[6],$$

and for m even as

$$\tau_{4,k} \cdot s[3^2] + \tau_k^+ \cdot s[1^6] + \tau_k^- \cdot s[5, 1].$$

□

7 Decomposing the Endoscopic Contribution

The Hecke algebra of $\mathrm{GSp}(4,\mathbb{Q})$ acts on the inner cohomology of the local system $\mathbb{V}_{l,m}$, cf. e.g. [10, p. 249], [17], and [22]. This inner cohomology on $\mathcal{A}_2(w^n)$ also has a Hodge filtration $(0) \subseteq F^{l+m+3} \subseteq F^{l+2} \subseteq F^{m+1} \subseteq F^0$. The action of the Hecke algebra respects this Hodge filtration. We now consider the irreducible representations H of the Hecke algebra occurring in the inner cohomology that have the property that $F^{l+m+3} \cap H = (0)$. We define the *middle endoscopic part* of the inner cohomology of our local system $\mathbb{V}_{l,m}$ on $\mathcal{A}_2(w^n)$ to be the direct sum of the representations with that property, i.e., the part that contains no contribution from holomorphic (vector-valued) Siegel modular cusp forms of weight $(j,k) = (l-m, m+3)$. Here, we use that $h^{l+m+3,0} = h^{0,l+m+3}$, which follows from the fact that eigenforms have totally real eigenvalues and all representations of $\mathrm{GSp}(4,\mathbb{F}_2)$ are defined over a totally real field, or even $\mathbb{Q}$. This endoscopic part should come from the group $\mathrm{GL}(2,\mathbb{Q}) \times \mathrm{GL}(2,\mathbb{Q})/\mathbb{G}_m$.

The Saito-Kurokawa lift for level 1 associates, for odd $l = m$, to the space $S_{l+m+4}(\mathrm{SL}(2,\mathbb{Z}))$ of cusp forms on $\mathrm{SL}(2,\mathbb{Z})$ the motive

$$-S[\mathrm{SL}(2,\mathbb{Z}), l+m+4] - s_{l+m+4}(L^{l+2} + L^{m+1})$$

in the cohomology of the local system $\mathbb{V}_{l,m}$ on $\mathcal{A}_2$, the minus sign indicating that it lands in odd degree cohomology. In [11, Conjecture 4.1], we conjecture the existence of a (strict) endoscopic part

$$-s_{l+m+4} S[\mathrm{SL}(2,\mathbb{Z}), l-m+2] L^{m+1} = s_{l+m+4}(L^{l+2} + L^{m+1}).$$

Assuming this and adding the two contributions, the net result would be the existence of $S[\mathrm{SL}(2,\mathbb{Z}), l+m+4]$ in the inner cohomology. In level 2, we see a similar phenomenon that becomes clear if one takes the action of $\mathbb{S}_6$ into account.

For $l \neq m$ and $f \in S_{l+m+4}(\Gamma_0(N))^{\mathrm{new}}$ and $g \in S_{l-m+2}(\Gamma_0(N))^{\mathrm{new}}$, the motive of the corresponding lifting is of the form $M_f + L^{m+1} M_g$ where M_f and M_g denote the motives associated to the cusp forms f and g. Let us call M_f the "leading" part of the vector-valued lift and $L^{m+1} M_g$ the "trailing" one. Note that in the cohomology where this lift appears, the trailing part contributes to the middle endoscopy. Let us define the *strict* *endoscopic part* of the cohomology to be the middle endoscopy minus the contribution from the trailing terms coming from the lifts described in Conjecture 6.1.

Conjecture 7.1. Assume that $l \neq m$ and let $k := l + m + 4$, $k' := l - m + 2$. The middle endoscopic part of the inner cohomology of $\mathbb{V}_{l,m}$ on $\mathcal{A}_2[2]$ is given by

$$\begin{aligned} -L^{m+1}\Big(&\big(\tau_{4,k}\cdot s[3,1^3]+\tau_{1,k}\cdot s[3^2]+(\tau_{1,k}+\tau_{2,k})\cdot s[4,1^2]\big)S[\Gamma_0(4),k']^{\text{new}}\\ &+\big((\tau_{1,k}+\tau_{2,k})\cdot s[3,2,1]+\tau_{4,k}\cdot s[4,1^2]+\tau_{1,k}\cdot s[4,2]+\tau_{1,k}\cdot s[5,1]\big)S[\Gamma_0(2),k']^{\text{new}}\\ &+\big(\tau_k^{+}\cdot s[4,2]+\tau_k^{-}\cdot s[5,1]\big)S^{+}[\Gamma_0(2),k']^{\text{new}}+\big(\tau_k^{-}\cdot s[4,2]+\tau_k^{+}\cdot s[5,1]\big)S^{-}[\Gamma_0(2),k']^{\text{new}}\\ &+\big(\tau_{1,k}\cdot s[2^3]+(\tau_{1,k}+\tau_{2,k})\cdot s[3,2,1]+\tau_{4,k}\cdot s[3^2]+\tau_{4,k}\cdot s[4,1^2]+(\tau_{2,k}+2\tau_{1,k})\cdot s[4,2]\\ &+(\tau_{1,k}+\tau_{2,k})\cdot s[5,1]+\tau_{1,k}\cdot s[6]\big)S[\Gamma_0(1),k']\Big). \end{aligned}$$

□

Conjecture 7.2. Let $k := l + m + 4$, $k' := l - m + 2$, then the strict endoscopic part of the inner cohomology of $\mathbb{V}_{l,m}$ on $\mathcal{A}_2[2]$ is given by

$$-5L^{m+1}\cdot \dim S_k(\Gamma_0(4))\cdot S[\Gamma_0(4),k'],$$

where we interpret $S[\Gamma_0(4),2]$ as $-L-1$. □

Remark 7.3. In Conjecture 7.1, both the strict endoscopy and the lifts from Conjecture 6.1 contribute to the terms $-L^{m+1}\cdot\tau_{4,k}\cdot s[3,1^3]S[\Gamma_0(4),k']^{\text{new}}$, $-L^{m+1}(\tau_k^{+}\cdot s[4,2]+\tau_k^{-}\cdot s[5,1])S^{+}[\Gamma_0(2),k']^{\text{new}}$ and $-L^{m+1}(\tau_k^{-}\cdot s[4,2]+\tau_k^{+}\cdot s[5,1])S^{-}[\Gamma_0(2),k']^{\text{new}}$. □

Conjecture 7.4. Assume that $l = m$ and let $k := 2m + 4$. The middle endoscopic part of the inner cohomology of $\mathbb{V}_{m,m}$ on $\mathcal{A}_2[2]$ is given by

$$L^{m+1}(L+1)\begin{cases}\tau_{4,k}\cdot s[3^2]+\tau_k^{+}\cdot s[1^6]+\tau_k^{-}\cdot s[5,1] & \text{if } m \text{ odd}\\ (\tau_k^{+}+\tau_{1,k})\cdot s[4,2]+(\tau_k^{-}+\tau_{1,k})\cdot s[2^3]+\tau_{1,k}\cdot s[6] & \text{if } m \text{ even}\end{cases}$$

□

8 Dimension Checks

In the case of one Weierstrass point, we have computed the numerical Euler characteristic $\sum(-1)^i \dim H_c^i(\mathcal{A}_2(w^1),\mathbb{V}_{l,m})\in\mathbb{Z}$ for any (l,m) using methods as in [3] and [5]. The conjectural results agree for (l,m) with $l+m\leq 10$ with these numerical Euler characteristics of the local systems and moreover for larger values of (l, m), e.g. for $l + m \leq 20$, we find that the numerical Euler characteristic minus the conjectured Eisenstein and endoscopic part is always a nonpositive multiple of 4. For all $l + m \leq 20$, this number equals -4 times the dimension of the space of Siegel modular cusp forms $S_{l-m,m+3}(\Gamma_2(w^1))$ as calculated by a program provided to us by R. Tsushima.

9 Examples of Eigenvalues of Hecke Eigenforms

We will now give a number of examples. We first write out some (conjectural) results for the first few local systems. Needless to say they are based on ample numerical evidence. Recall that the cohomology has the following parts,

$$e_c(\mathcal{A}, \mathbb{V}_{l,m}) = e_{\mathrm{Eis}}(\mathcal{A}, \mathbb{V}_{l,m}) + e_{\mathrm{End}^s}(\mathcal{A}, \mathbb{V}_{l,m}) - S[\Gamma_2[2], (l-m, m+3)],$$

where the third part has dimension $4 \dim S_{l-m,m+3}(\Gamma_2[2])$. Here, we will write $\Phi_{N,k} := S[\Gamma_0(N), k]^{\mathrm{new}}$ and in all of the following cases, this will be a motive associated to a single newform.

(l, m)	$e_c(\mathcal{A}_2[2], \mathbb{V}_{l,m})$
$(0, 0)$	$L^3 + L^2 - 14L + 16$
$(2, 0)$	$-30L + 30$
$(1, 1)$	$5L^3 - 10L^2$
$(4, 0)$	$-45L + 45 - 10L\Phi_{4,6}$
$(3, 1)$	$-30L^2 - 15\Phi_{4,6}$
$(2, 2)$	$9L^4 - 21L^3 - \Phi_{2,8}$
$(6, 0)$	$-60L + 60 - 31L\Phi_{2,8} - \Phi_{2,10}$
$(5, 1)$	$-45L^2 + 15 - 30\Phi_{2,8} - 20L\Phi_{4,6} - 5\Phi_{4,10}$
$(4, 2)$	$-45L^3 + 45 - S[\Gamma_2[2], (2, 5)]$
$(3, 3)$	$10L^5 - 35L^4 - 15\Phi_{4,6} - 5\Phi_{2,10}$
$(8, 0)$	$-75L + 75 - 25L\Phi_{4,10} - 40L\Phi_{2,10} - 5\Phi_{4,12}$
$(7, 1)$	$-60L^2 + 30 - 15\Phi_{4,10} - 30\Phi_{2,10} - 40L^2\Phi_{2,8} - S[\Gamma_2[2], (6, 4)]$
$(6, 2)$	$-60L^3 + 60 - 20L^3\Phi_{4,6} - S[\Gamma_2[2], (4, 5)]$
$(5, 3)$	$-60L^4 - 30\Phi_{2,8} - S[\Gamma_2[2], (2, 6)]$
$(4, 4)$	$15L^6 - 45L^5 + 30 - 15\Phi_{4,6} - 5\Phi_{4,12}$

In a number of cases, we can identify "genuine" Siegel modular forms, i.e. not lifts of the type described in Conjecture 6.1. The space $S_{j,k}(\Gamma_2[2])$ can be decomposed under the action of $\mathbb{S}_6$ into a sum of spaces $S_{j,k}(\Gamma_2[2])^{\mu}$ corresponding to the $\mathbb{S}_6$-representation given by the partition μ. For the cases appearing in the table above, we have

$$\begin{aligned}
S_{2,5}(\Gamma_2[2]) &= S_{2,5}(\Gamma_2[2])^{[2^2,1^2]}\\
S_{6,4}(\Gamma_2[2]) &= S_{6,4}(\Gamma_2[2])^{[2^2,1^2]} \oplus S_{6,4}(\Gamma_2[2])^{[3,1^3]}\\
S_{4,5}(\Gamma_2[2]) &= S_{4,5}(\Gamma_2[2])^{[2,1^4]} \oplus S_{4,5}(\Gamma_2[2])^{[2^2,1^2]} \oplus S_{4,5}(\Gamma_2[2])^{[3,2,1]}\\
S_{2,6}(\Gamma_2[2]) &= S_{2,6}(\Gamma_2[2])^{[3,1^3]} \oplus S_{2,6}(\Gamma_2[2])^{[3,2,1]},
\end{aligned}$$

and each of these subspaces is generated by one vector-valued Siegel modular form. For instance for $(l,m) = (4,2)$, we have one vector-valued Siegel modular form appearing with the representation $s[2^2, 1^2]$, i.e. with multiplicity 9, which agrees with the result of Tsushima that $S_{2,5}(\Gamma_2[2])$ is nine-dimensional, see [23]. Moreover, according to our data, $S_{4,5}(\Gamma_2[2])^{[2,1^4]}$ is generated by a lift with corresponding motive $\Phi_{4,12} + L^3\Phi_{4,6}$.

The trace of Frobenius, for a prime $p > 2$, on the space $S[\Gamma_2[2],(j,k)]$ is equal to the trace of the Hecke operator $T(p)$ on $S_{j,k}(\Gamma_2[2])$. In the following two tables, we write the (conjectural) Hecke eigenvalues for the generating Siegel modular form for $3 \le p \le 23$ in four cases when $S_{j,k}(\Gamma_2[2])^{\mu}$ is generated by a single vector-valued Siegel modular form. We are assuming here the conjectures on the endoscopy given above. Note that all these eigenvalues have many small prime factors.

p	$S_{2,5}(\Gamma_2[2])^{[2^2,1^2]}$	$S_{6,4}(\Gamma_2[2])^{[2^2,1^2]}$
3	$-2^3 \cdot 5$	$-2^3 \cdot 5 \cdot 7$
5	$-2^2 \cdot 5^2 \cdot 13$	$-2^2 \cdot 5 \cdot 149$
7	$2^4 \cdot 3 \cdot 5 \cdot 13$	$-2^4 \cdot 3 \cdot 5 \cdot 401$
11	$2^3 \cdot 11 \cdot 13 \cdot 31$	$2^3 \cdot 36383$
13	$-2^2 \cdot 5 \cdot 3469$	$2^2 \cdot 5 \cdot 37 \cdot 251$
17	$-2^2 \cdot 5 \cdot 11 \cdot 13 \cdot 197$	$2^2 \cdot 5 \cdot 19 \cdot 6983$
19	$2^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17$	$-2^3 \cdot 5 \cdot 29 \cdot 6287$
23	$-2^4 \cdot 5 \cdot 13 \cdot 311$	$-2^4 \cdot 5 \cdot 43 \cdot 2267$

p	$S_{6,4}(\Gamma_2[2])^{[3,1^3]}$	$S_{10,3}(\Gamma_2[2])^{[2^2,1^2]}$
3	$-2^3 \cdot 3$	$2^3 \cdot 5^2$
5	$2^2 \cdot 3^2 \cdot 7 \cdot 41$	$2^2 \cdot 5 \cdot 127$
7	$2^4 \cdot 5^2 \cdot 73$	$-2^4 \cdot 3 \cdot 5^2 \cdot 13$
11	$-2^3 \cdot 3^2 \cdot 4793$	$-2^3 \cdot 439 \cdot 1123$
13	$-2^2 \cdot 7 \cdot 21563$	$2^2 \cdot 5^2 \cdot 47 \cdot 4457$
17	$-2^2 \cdot 3^2 \cdot 2351$	$2^2 \cdot 5^2 \cdot 799441$
19	$-2^3 \cdot 7 \cdot 11 \cdot 37 \cdot 383$	$2^3 \cdot 5 \cdot 7 \cdot 461 \cdot 1723$
23	$-2^4 \cdot 3^2 \cdot 11 \cdot 17 \cdot 29 \cdot 43$	$2^4 \cdot 5^2 \cdot 3653483$

We compute the slopes for the single Siegel modular cusp form generating the space $S_{2,6}(\Gamma_2[2])^{[3,1^3]}$ and for the one generating $S_{2,6}(\Gamma_2[2])^{[3,2,1]}$. Recall that the reciprocal of the characteristic polynomial of Frobenius is

$$1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{l+m+2})X^2 - \lambda(p)p^{l+m+3}X^3 + p^{2l+2m+6}X^4$$

and the "slope" refers to the slopes of the Newton polygon.

p	$\lambda(p)$	$\lambda(p^2)$	slopes
3	$2^3 \cdot 3^3$	$-2^2 \cdot 3^6 \cdot 107$	$3, 4, 7, 8$
5	$-2^2 \cdot 3^4 \cdot 17$	$2^2 \cdot 181 \cdot 26161$	$0, 4, 7, 11$
3	$-2^3 \cdot 3^2 \cdot 5$	$3^4 \cdot 1753$	$2, 4, 7, 9$
5	$2^2 \cdot 3 \cdot 5 \cdot 7^2$	$5^2 \cdot 117119$	$1, 4, 7, 10$

10 Harder's Congruences

Harder predicts a congruence between an elliptic modular form f and a Siegel modular form whenever a "large" prime ℓ divides a critical value $L(f, s)$ of the L-series of the elliptic modular form, see [13, 16, 17]. In cooperation with Harder, we checked a few cases. This lends at the same time credibility to our computations and conjectures and to Harder's conjecture.

For example, if f is a newform in the one-dimensional space $S_{20}^{+}(\Gamma_0(2))$, then 61 divides $L(f, 12)$ and one expects the congruence

$$p^8 + a(p) + p^{11} \equiv \lambda(p) \pmod{61}$$

for the Hecke eigenvalues $\lambda(p)$ of an eigenform $F \in S_{2,10}(\Gamma_2[2])$ for every prime $p \neq 2$. By using dimension formulas of R. Tsushima for $S_{j,k}(\Gamma_2(w^0))$ and $S_{j,k}(\Gamma_2(w^1))$ (see also Section 8), we find that $\dim S_{2,10}(\Gamma_2(w^0)) = 0$ and $\dim S_{2,10}(\Gamma_2(w^1)) = 1$. For $p \leq 37$ the eigenvalues $\lambda(p)$ we have calculated for a nonzero $F \in S_{2,10}(\Gamma_2(w^1))$ satisfy the required congruence, e.g. $\lambda(3) = 18360$ and $3^8 - 13092 + 3^{11} \equiv 18360 \pmod{61}$.

In the following table, we list a few congruences that are valid for the eigenvalues that we find. Also, in these cases, the corresponding spaces of modular forms are one-dimensional and the Siegel modular forms do not come from level 1.

$\langle f \rangle$	$\langle F \rangle$	s	ℓ
$S_{20}^{+}(\Gamma_0(2))$	$S_{2,10}(\Gamma_2(w^1))$	12	61
$S_{20}^{+}(\Gamma_0(2))$	$S_{10,6}(\Gamma_2(w^1))$	16	109
$S_{18}^{-}(\Gamma_0(2))$	$S_{6,7}(\Gamma_2(w^1))$	13	29
$S_{20}^{-}(\Gamma_0(2))$	$S_{12,5}(\Gamma_2(w^1))$	17	79
$S_{22}^{+}(\Gamma_0(2))$	$S_{16,4}(\Gamma_2(w^1))$	20	37

Moreover, for a newform $f \in S_{16}(\Gamma_0(4))$, we find that our traces of Frobenius on both $S_{8,5}(\Gamma_2[2])^{[4,1^2]}$ and $S_{8,5}(\Gamma_2[2])^{[3^2]}$ satisfy the expected congruence modulo $\ell = 37$, which divides $L(f, 13)$.

As explained to us by Harder, using the character table of $\mathbb{S}_6$ and some arguments from the representation theory of $\mathrm{GSp}(4, \mathbb{Q}_2)$, one finds: if the level of f is 2, then the Siegel modular form appears with each of the representations $s[5,1]$, $s[4,2]$, and $s[3,2,1]$. If f is of level 4, then the Siegel modular form appears with both $s[4,1^2]$ and $s[3^2]$. This is compatible with our examples.

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