

Siegel modular forms of small weight and the Witt operator

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ABSTRACT. To calculate dimensions of Siegel modular forms including non-cusp forms, we determine the image of Siegel Φ -operator for small weight which were unknown in general theory. We treat the case of the Hecke type group of prime level and also vector valued Siegel modular forms of level one of degree two. For this purpose we propose a new basis problem on theta functions related with the Witt operator. We also show the surjectivity of the Witt operator in case of vector valued Siegel modular forms of level one for big weight by giving certain new dimension formulas of Siegel modular forms. We also give new upper and lower bounds of unknown dimensions of vector valued Siegel modular forms of small weight.

1. Introduction

In this paper, we are interested in the dimensions of the whole Siegel modular forms which are not necessarily cusp forms. The dimensions of holomorphic Siegel cusp forms of degree two are explicitly known for many discrete subgroups if the weight k is big enough. There are also some results for small k (cf. [12], [11]), though there are no general ways to calculate dimensions for small weights. So to calculate the dimension including non-cusp forms, we must investigate the difference from cusp forms. Since cusp forms are defined to be modular forms which vanish at the boundary of the Satake compactification of the Siegel modular variety, this difference is the same as the dimension of the image of the restriction of the Siegel modular forms to the boundary. This restriction operator is called the (generalized) Siegel Φ -operator. Since irreducible components of this boundary consists of Siegel modular varieties of smaller dimensions, the image of the Siegel Φ -operator consists of vectors of functions whose components are Siegel modular forms of smaller degree on each irreducible component of the boundary which coincide at the intersections of the components. If the Φ -operator is surjective to the space of modular forms on the boundary, then the difference of dimensions is reduced to the calculation of the dimensions of modular forms of lower degrees. The surjectivity of this operator for scalar valued modular forms is known for any degree in Satake [14] when the weight

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is big enough. But for small weights, this operator is not necessarily surjective and the image is not known in general.

Now to determine the image of the Siegel Φ operator in case of degree two, we noticed that it is useful to consider the so-called Witt operator defined as follows. For $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = {}^t Z \in H_2$ (where H_2 is the Siegel upper half space of degree two) and any function $F(Z)$ of H_2 , we write

$$(WF)(\tau, \omega) = F \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}.$$

This Witt operator W was often used to determine the ring structure of Siegel modular forms of degree two. In this paper, first we treat the discrete subgroup $\Gamma_0(p)$ of Hecke type in $Sp(2, \mathbb{R})$ (matrix size four). We investigate the image of the Witt operator of scalar valued Siegel modular forms belonging to $\Gamma_0(p)$ by investigating a variant of the basis problem, asking if some modular forms are spanned by theta functions. As an application, we shall show that for weight $k = 2$ the Siegel Φ -operator is not surjective but the dimension of the image is exactly obtained (Theorem 3.2, 4.1).

Secondly we treat vector valued Siegel modular forms of $Sp(2, \mathbb{Z})$. The surjectivity of the vector valued Siegel Φ -operator is known when the weight is $\det^k Sym_j$ with $k \geq 5$ by Arakawa [1]. We shall determine the image for all weights $k \leq 4$ unknown before (cf. Theorem 5.1). We also determine the image of the Witt operator for weight $k \geq 10$ by showing that W is surjective to a certain space well described by modular forms of one variable (cf. Theorem 6.3). This result is interesting as itself, and it is also interesting to ask to what extent the same sort of theorem holds for weight $k \leq 10$, since if we can determine the image of W , we can get a dimension formula for small weight. Although this is still unknown, we can give upper or lower bounds of still unknown dimensions of vector valued Siegel cusp forms of weight $\det^k Sym_j$ with $k = 2$ or 3 in the last section. In particular, we have an existence theorem for non-zero Siegel modular forms of weight $\det^3 Sym_j$ for big j . This theorem is completely new since no such examples were known before for any j .

2. Review on Siegel Modular Forms

We denote by H_n the Siegel upper half space of degree n ,

$$H_n = \{Z = {}^t Z \in M_n(\mathbb{C}); Im(Z) > 0\}$$

where $Im(Z) > 0$ means that the imaginary part of Z is a positive definite matrix. We denote by $Sp(n, \mathbb{R})$ the real symplectic group of rank n .

$$Sp(n, \mathbb{R}) = \{g \in M_{2n}(\mathbb{R}); gJ^t g = J\}$$

where $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and 1_n is the $n \times n$ unit matrix. The group $Sp(n, \mathbb{R})$ acts on H_n in the usual way by $gZ = (AZ + B)(CZ + D)^{-1}$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$. We fix a rational irreducible representation (ρ, V) of $GL_n(\mathbb{C})$. For every

$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and a V -valued mapping F on H_n , we write

$$(F|_\rho[g])(Z) = \rho(CZ + D)^{-1}F(gZ).$$

This defines an action of $Sp(n, \mathbb{R})$ on the space of holomorphic functions on H_n . We take a discrete subgroup Γ of $Sp(n, \mathbb{R})$ with $\text{vol}(\Gamma \backslash H_n) < \infty$. A holomorphic function F on H_n is said to be a Siegel modular form of weight ρ belonging to Γ if $F|_\rho[\gamma] = F$ for all $\gamma \in \Gamma$ (with the holomorphy condition at cusps of Γ when $n = 1$). When F vanishes at the boundary of the Satake compactification of $\Gamma \backslash H_n$, then F is said to be a cusp form. We denote by $A_\rho(\Gamma)$ or $S_\rho(\Gamma)$ the space of Siegel modular forms or Siegel cusp forms, respectively. When $\rho(u) = \det(u)^k$ for $u \in GL_n(\mathbb{C})$, we say that F is of weight k and we write $A_k(\Gamma) = A_\rho(\Gamma)$ and $S_k(\Gamma) = S_\rho(\Gamma)$. In this paper, we mainly consider the case $n = 2$. In this case, the polynomial representations of $GL_2(\mathbb{C})$ are written as $\rho_{k,j} = \det^k \text{Sym}_j$ for some integers $k \geq 0$ and $j \geq 0$ where Sym_j is the symmetric tensor representation of degree j , and we write $A_{\rho_{k,j}}(\Gamma) = A_{k,j}(\Gamma)$ and $S_{\rho_{k,j}}(\Gamma) = S_{k,j}(\Gamma)$.

We put $Sp(n, \mathbb{Z}) = Sp(n, \mathbb{R}) \cap M_{2n}(\mathbb{Z})$. For any integer N , we define the Hecke type congruence subgroup $\Gamma_0^{(n)}(N)$ of $Sp(n, \mathbb{Z})$ by

$$\Gamma_0^{(n)}(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z}); C \equiv 0 \pmod N \right\}.$$

When $n = 2$, we simply write $\Gamma_0(N) = \Gamma_0^{(2)}(N)$.

3. The Siegel Φ -operator and the Witt operator

For a function F on H_2 , we define a function ΦF on H_1 by

$$(\Phi F)(\tau) = \lim_{\lambda \rightarrow \infty} F \begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}$$

when it converges. This ΦF is well defined for every $F \in A_k(\Gamma_0(N))$. By definition, F is a cusp form if and only if $\Phi(F|_k[g]) = 0$ for any $g \in Sp(2, \mathbb{Z})$. For any prime p , the structure of the boundary of the Satake compactification of $\Gamma_0(p) \backslash H_2$ is well known (e.g. see [9]) and it has two one-dimensional cusps isomorphic to the compactification of $\Gamma_0^{(1)}(p) \backslash H_1$ crossing at a zero-dimensional cusp among three zero-dimensional cusps. So the space $\partial A_k(\Gamma_0(p))$ of modular forms on the boundary is a vector space of pairs of modular forms on $A_k(\Gamma_0^{(1)}(p))$ which takes the same value at the crossing point cusp. The compactification of $\Gamma_0^{(1)}(p)$ has two cusps $i\infty$ and 0 and by a suitable realization of cusps of $\Gamma_0(p) \backslash H_2$, the above space can be identified with

$$\partial A_k(\Gamma_0(p)) := \{(f, g); f, g \in A_k(\Gamma_0^{(1)}(p)), f, g \text{ have the same value at the cusp } 0\}.$$

Here we understand as usual that f and g have the same value at 0 if

$$\lim_{\lambda \rightarrow \infty} (f|_k[\pi_1])(i\lambda) = \lim_{\lambda \rightarrow \infty} (g|_k[\pi_1])(i\lambda),$$

where for any natural number n , we put

$$\pi_n = \begin{pmatrix} 0 & -\sqrt{p}^{-1}1_n \\ \sqrt{p}1_n & 0 \end{pmatrix}.$$

For $F \in A_k(\Gamma_0(p))$, the generalized Φ -operator is identified with the mapping of $A_k(\Gamma_0(p))$ to $\partial A_k(\Gamma_0(p))$ defined by

$$F \rightarrow \tilde{\Phi}(F) = (\Phi(F), \Phi(F|_k[\pi_2])).$$

If k is odd, then $A_k(\Gamma_0^{(1)}(p)) = \{0\}$ and $\tilde{\Phi}$ is always surjective since it is a mapping to $\{0\}$. So obviously we have $A_k(\Gamma_0(p)) = S_k(\Gamma_0(p))$. When k is even, if $k \geq 4$, for any $(c_1, c_2) \in \mathbb{C}^2$, there exists a modular form $E \in A_k(\Gamma_0^{(1)}(p))$ such that the value of E is c_1 at $i\infty$ and c_2 at the cusp 0, so $\dim A_k(\Gamma_0^{(1)}(p))/S_k(\Gamma_0^{(1)}(p)) = 2$. But if $k = 2$, then $\dim A_k(\Gamma_0^{(1)}(p))/S_2(\Gamma_0(p)) = 1$. These are well known classical results proved by dimension formula or by the theory of Eisenstein series. So the dimension of $\partial A_k(\Gamma_0(p))$ for even k is given by

$$\dim \partial A_k(\Gamma_0(p)) = 2 \dim S_k(\Gamma_0^{(1)}(p)) + \begin{cases} 1 & k = 2, \\ 3 & k \geq 4. \end{cases}$$

For even k , the surjectivity of $\tilde{\Phi}$ is known for big k as follows.

THEOREM 3.1 (Satake [14]). *Notation being as above, assume that $k \geq 6$. Then $\tilde{\Phi}$ is surjective to $\partial A_k(\Gamma_0(p))$.*

Actually the surjectivity holds also for $k = 4$. This result was obtained after the conference in a joint work with Böcherer and will be reported elsewhere. Here in this paper, we would like to describe the image of $\tilde{\Phi}$ in the case $k = 2$. In this case, we shall show later that $\tilde{\Phi}$ is not surjective to $\partial A_2(\Gamma_0(p))$ but $\dim \tilde{\Phi}(A_2(\Gamma_0(p))) = \dim(A_2(\Gamma_0(p))/S_2(\Gamma_0(p))) = \dim A_2(\Gamma_0^{(1)}(p))$ for any prime p .

Before proving this, we shall explain the Witt operator and a variant of the basis problem which is used for the proof. For every holomorphic function F on H_2 , the Witt operator W is defined by

$$(WF)(\tau, \omega) = F \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}.$$

This operator was first introduced in Witt [20] and later used, for example by Igusa, Hammond, Freitag, to determine the structure of the ring of Siegel modular forms.

For any $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_0^{(1)}(N)$ ($i = 1, 2$), we put

$$\iota(g_1, g_2) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$

Then we have $\iota(g_1, g_2) \in \Gamma_0(N)$. So if $F \in A_k(\Gamma_0(N))$, then for each variable τ or ω , WF is a modular form of weight k of one variable belonging to $\Gamma_0^{(1)}(N)$. In particular, for odd k we have $WF = 0$. So we assume that k is even now. By the action of the matrix

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \Gamma_0(N)$$

on F , we see $F \begin{pmatrix} \omega & z \\ z & \tau \end{pmatrix} = (-1)^k F \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$. Hence we have $WF(\tau, \omega) = WF(\omega, \tau)$ for even k . This means that we can regard WF as an element of the vector space $Sym^2(A_k(\Gamma_0^{(1)}(N)))$ of the symmetric tensors of degree two of $A_k(\Gamma_0^{(1)}(N))$.

We can determine the image of $\tilde{\Phi}$ by the image of W at least when $N = p$ is a prime. This can be seen as follows. When $k \geq 4$, for a cusp $\kappa = i\infty$ or 0 , we denote by E_κ an Eisenstein series in $A_k(\Gamma_0^{(1)}(p))$ whose value is non-zero at κ and 0 at the other cusp. We can assume that $(E_{i\infty}|_k[\pi_1])(\tau) = E_0(\tau)$ and then we have $(E_0|_k[\pi_1])(\tau) = E_{i\infty}(\tau)$. When $k = 2$, we denote by E the unique Eisenstein series (up to constant) in $A_2(\Gamma_0^{(1)}(p))$, which does not vanish at any cusps. In this case, we have $E|_k[\pi_1] = -E(\tau)$. For any $F \in A_k(\Gamma_0(p))$, we have

$$WF(\tau, \omega) = \sum_{j=1}^d (f_j(\tau)g_j(\omega) + f_j(\omega)g_j(\tau))$$

where $f_j, g_j \in A_k(\Gamma_0^{(1)}(p))$. When $k \geq 4$, $WF(\tau, \omega)$ is given modulo $Sym^2(S_k(\Gamma_0^{(1)}(p)))$ by

$$\begin{aligned} & \sum_{\kappa} (f_\kappa(\omega)E_\kappa(\tau) + f_\kappa(\tau)E_\kappa(\omega)) \\ & + c_1(E_{i\infty}(\tau)E_0(\omega) + E_0(\tau)E_{i\infty}(\omega)) + c_2E_{i\infty}(\tau)E_{i\infty}(\omega) + c_3E_0(\tau)E_0(\omega) \end{aligned}$$

where f_κ are cusp forms of weight k and c_i are constants. We may assume that $E_{i\infty}(i\infty) = 1$ and then we have

$$\Phi(F) = f_\infty(\tau) + c_1E_0(\tau) + c_2E_{i\infty}(\tau).$$

On the other hand, we have $W(F|_k[\pi_2]) = p^{-k}(\tau\omega)^{-k}(WF)(-(p\tau)^{-1}, -(p\omega)^{-1})$. If $f(\tau) \in A_k(\Gamma_0^{(1)}(p))$, then we also have $f|_k[\pi_1] \in A_k(\Gamma_0^{(1)}(p))$ and if $f \in S_k(\Gamma_0^{(1)}(p))$, then the latter is also a cusp form. So for the same F as above, we have

$$\begin{aligned} & (p\tau\omega)^{-k}(WF)(-(p\tau)^{-1}, -(p\omega)^{-1}) = \\ & g_0(\omega)E_{i\infty}(\tau) + g_0(\tau)E_{i\infty}(\omega) + g_{i\infty}(\omega)E_0(\tau) + g_{i\infty}(\tau)E_0(\omega) \\ & + c_1(E_{i\infty}(\tau)E_0(\omega) + E_0(\tau)E_{i\infty}(\omega)) + c_2E_0(\tau)E_0(\omega) + c_3E_{i\infty}(\tau)E_{i\infty}(\omega). \end{aligned}$$

modulo $Sym^2(S_k(\Gamma_0^{(1)}(p)))$ where $g_\kappa = f_\kappa|_k[\pi_1]$. So we have

$$\Phi(F|_k[\pi_2]) = g_0(\tau) + c_1E_0(\tau) + c_3E_{i\infty}(\tau).$$

Similarly, when $k = 2$, there exist a cusp form $f \in S_2(\Gamma_0^{(1)}(p))$ and a constant c such that

$$WF(\tau, \omega) = f(\tau)E(\omega) + E(\tau)f(\omega) + cE(\tau)E(\omega)$$

modulo $Sym^2(S_k(\Gamma_0^{(1)}(p)))$. So we have

$$(p\tau\omega)^{-2}WF(-(p\tau)^{-1}, -(p\omega)^{-1}) = -(f|_2[\pi_1])(\tau)E(\omega) - (f|_2[\pi_1])(\omega)E(\tau) + cE(\tau)E(\omega).$$

Hence assuming $E(i\infty) = 1$, we have

$$\begin{aligned} \Phi F &= f(\tau) + cE(\tau) \\ \Phi(F|_k[\pi_2]) &= -g(\tau) + cE(\tau). \end{aligned}$$

where $g = f|_2[\pi_1]$. Of course, here, both $\Phi(F)$ and $\Phi(F|_k[\pi_2])$ are modular forms of $A_k(\Gamma_0^{(1)}(p))$ which have the same value at the cusp 0 .

The Witt operator is not surjective to $Sym^2(A_k(\Gamma_0^{(1)}(N)))$ in general even when N is a prime. The author was informed of this fact by Cris Poor. Motivated by investigation of the image of $\tilde{\Phi}$, we define the mapping \overline{W} by composition of W and the natural projection $Sym^2(A_k(\Gamma_0^{(1)}(p))) \rightarrow Sym^2(A_k(\Gamma_0^{(1)}(p)))/Sym^2(S_k(\Gamma_0^{(1)}(p)))$. We saw above that the image of $\tilde{\Phi}$ is determined by the image of \overline{W} .

THEOREM 3.2. *For any prime p , we have*

- (1) *The mapping \overline{W} is surjective.*
- (2) *If $k \geq 4$, then the Siegel operator $\tilde{\Phi}$ is surjective to $\partial A_k(\Gamma_0(p))$.*
- (3) *If $k = 2$, the operator $\tilde{\Phi}$ is not surjective, but we have*

$$\dim \tilde{\Phi}(A_2(\Gamma_0(p))) = \dim A_2(\Gamma_0^{(1)}(p)).$$

Proof. If we assume that (1) is true, then the proof of (2) and (3) is obvious from the above consideration. When $k \geq 6$, then we know by Satake [14] that (2) is true. This also implies (1). So the problem is to show (1) for $k = 2$ and $k = 4$. The proof for $k = 4$ is a joint work with Boecherer and will be written elsewhere. We shall prove the case $k = 2$ in the next section.

4. A Variant of the basis problem

To prove Theorem 3.2, we consider some more general problem. For a natural number k , let S be a $2k \times 2k$ positive definite integral symmetric matrix. If all the diagonal components of S are even, S is said to be even. The minimum of natural numbers N such that NS^{-1} is also even is called the level of S . For a natural number n , we write

$$\theta_S^{(n)}(Z) = \sum_{X \in M_{2k,n}(\mathbb{Z})} \exp(\pi i \operatorname{Tr}({}^t X S X Z)).$$

If $\det(S)$ is a square, we have $\theta_S^{(n)}(Z) \in A_k(\Gamma_0^{(n)}(N))$. The usual basis problem asks if $A_k(\Gamma_0(N))$ is spanned by θ_S for various S . We consider the following variant of this problem.

A variant of the basis problem. *Is the space*

$Sym^2(A_k(\Gamma_0^{(1)}(N)))/Sym^2(S_k(\Gamma_0^{(1)}(N)))$ spanned by the images of theta functions $\theta_S^{(1)}(\tau)\theta_S^{(1)}(\omega)$ associated with $2k \times 2k$ positive definite even integral symmetric matrices S of level N ?

A numerical example. Assume that $N = 5$. We denote by E_8 the 8×8 even unimodular symmetric matrix which is unique up to isomorphism. We put

$$S_0 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 10 & 5 \\ 0 & 1 & 5 & 4 \end{pmatrix} \quad S_1 = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 4 & -1 & -1 \\ 1 & -1 & 4 & -1 \\ 2 & -1 & -1 & 4 \end{pmatrix} \quad S_2 = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

The levels of S_i are all 5 and we have $\det(S_0) = 5^2$, $\det(S_1) = 5$, and $\det(S_2) = 5^3$. Then theta functions associated with $E_8, 5E_8, S_0 + S_0, S_1 + S_1, S_1 + S_2, S_2 + S_2$ are in $A_4(\Gamma_0(5))$ and the answer is affirmative for $k = 4$.

THEOREM 4.1. *When N is a prime p , then for any natural number k , the answer to the above variant of the basis problem is affirmative.*

Again when $k = 4$, this is a joint work with Boecherer and will be reported elsewhere. When $k \geq 6$, then by [3], the space $A_k(\Gamma_0(p))$ is spanned by $\theta_S^{(2)}(Z)$. Since $\tilde{\Phi}$ is surjective for $k \geq 6$ by virtue of Satake (loc.cit.), \overline{W} is also surjective as shown in the last section. For $n = 2$, we have $W(\theta_S^{(2)}) = \theta_S^{(1)}(\tau)\theta_S^{(1)}(\omega)$ and we are done. When $k = 2$, then by the result of Eichler [4], the space $A_2(\Gamma_0(p))$ is spanned by theta functions $\theta_S^{(1)}(\tau)$. We have $\Phi(\theta_S^{(2)}(Z)) = \theta_S^{(1)}(\tau)$ for any S so Φ is surjective to $A_2(\Gamma_0(p))$. By the results in the last section, we see that $\dim \text{Sym}^2(A_2(\Gamma_0^{(1)}(p)))/\text{Sym}^2(S_2(\Gamma_0^{(1)}(p))) = \dim A_2(\Gamma_0(p))$ and the surjectivity of (single) Φ to $A_2(\Gamma_0(p))$ and the surjectivity of \overline{W} are equivalent. So we are done.

Hence we also proved Theorem 3.2.

It seems interesting to ask if we can generalize the above results to more general N .

5. The image of Φ -operator in the vector valued case.

Now we consider the space $A_{k,j}(Sp(2, \mathbb{Z}))$ for $j > 0$. Since we can see easily that $A_{k,j}(Sp(2, \mathbb{Z})) = 0$ for any odd j by writing down the action of -1_4 , we assume from now on that j is even. To fix an idea, we give a realization of Sym_j . We denote by u_1, u_2 two independent variables. For $g \in GL_2(\mathbb{C})$, we write $(v_1, v_2) = (u_1, u_2)g$. We define the $(j + 1) \times (j + 1)$ matrix $\text{Sym}_j(g)$ by

$$(v_1^j, v_1^{j-1}v_2, \dots, v_2^j) = (u_1^j, u_1^{j-1}u_2, \dots, u_2^j)\text{Sym}_j(g).$$

This gives the symmetric tensor representation of $GL_2(\mathbb{C})$ of degree j . For $F \in A_{k,j}(Sp(2, \mathbb{Z}))$, the Siegel Φ -operator is defined as usual by

$$\Phi(F) = \lim_{\lambda \rightarrow \infty} F \begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}.$$

Since $Sp(2, \mathbb{Z})$ has the unique one-dimensional cusp, this coincides with the operator to the boundary. By definition, $\Phi(F)$ is a \mathbb{C}^{j+1} -valued function. But as shown in Arakawa [1], the first component of $\Phi(F)$ belongs to $S_{k+j}(SL_2(\mathbb{Z}))$ (under the assumption that $j > 0$) and all the other components vanish. So we can identify $\Phi(F)$ as an element of $S_{k+j}(SL_2(\mathbb{Z}))$. Then since $S_{k+j}(SL_2(\mathbb{Z})) = 0$ for odd k , Φ is of course surjective to $S_{k+j}(SL_2(\mathbb{Z}))$ in this case and we see that $A_{k,j}(Sp(2, \mathbb{Z})) = S_{k,j}(Sp(2, \mathbb{Z}))$ if k is odd. Arakawa proved the surjectivity of Φ for even $k \geq 6$ in [1] by constructing Klingen type Eisenstein series. We determine the image of Φ for $k \leq 4$ here.

THEOREM 5.1. *If $k \geq 4$ and j is even with $j > 0$, then we have $\Phi(A_{k,j}(Sp(2, \mathbb{Z}))) = S_{k+j}(SL_2(\mathbb{Z}))$. If $k = 2$, then $\Phi(A_{2,j}(Sp(2, \mathbb{Z}))) = 0$ and $A_{2,j}(Sp(2, \mathbb{Z})) = S_{2,j}(Sp(2, \mathbb{Z}))$.*

Proof. Since the case $k \geq 6$ is known by Arakawa [1], we prove the case $k = 4$ and $k = 2$. First we prove the case $k = 4$. We write the inner product of $x, y \in \mathbb{R}^8$ by (x, y) . Let $E_8 \subset \mathbb{R}^8$ be the lattice of rank 8 which is even unimodular with respect to $(*, *)$. This is unique up to isomorphism. We take a vector $\mathfrak{a} \in \mathbb{C}^8$ such that $(\mathfrak{a}, \mathfrak{a}) = 0$ and define a vector valued theta function $\theta_{\mathfrak{a}}(Z)$ ($Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$)

associated with a harmonic polynomial by

$$\theta_{\mathfrak{a}}(Z) = \begin{pmatrix} \theta_{\mathfrak{a},0}(Z) \\ \vdots \\ \theta_{\mathfrak{a},j}(Z) \end{pmatrix}$$

where we put

$$\theta_{\mathfrak{a},\nu}(Z) = \binom{j}{\nu} \sum_{x,y \in E_8} (x, \mathfrak{a})^{j-\nu} (y, \mathfrak{a})^{\nu} \exp(\pi i((x, x)\tau + 2(x, y)z + (y, y)\omega)).$$

It is easy to see that $\theta_{\mathfrak{a}}(Z) \in A_{4,j}(Sp(2, \mathbb{Z}))$. (For example, see Freitag [6] p. 161.) We see that in the Fourier expansion of $\Phi(\theta_{\mathfrak{a}})$, the only terms with $y = 0$ remain and we have

$$\Phi(\theta_{\mathfrak{a}}) = \sum_{x \in E_8} (x, \mathfrak{a})^j \exp(\pi i(x, x)\tau) \in S_{j+4}(SL_2(\mathbb{Z})).$$

(This can be regarded as a special case of Freitag [5] Hilfssatz 1.4). It is classically well known that every harmonic polynomial of 8 variables of degree j is a linear combination of $(x, \mathfrak{a})^j$ with $\mathfrak{a} \in \mathbb{C}^8$ with $(\mathfrak{a}, \mathfrak{a}) = 0$ (cf. e.g. Takeuchi [15]). On the other hand, the space $S_{4+j}(SL_2(\mathbb{Z}))$ ($j > 0$) is spanned by theta functions associated with harmonic polynomials by virtue of Waldspurger [18] or Theorem 5 in Boecherer [2]. So the functions $\Phi(\theta_{\mathfrak{a}})$ associated with $\mathfrak{a} \in \mathbb{C}^8$ with $(\mathfrak{a}, \mathfrak{a}) = 0$ spans the whole $S_{4+j}(SL_2(\mathbb{Z}))$. This implies that Φ is surjective to this space $S_{4+j}(SL_2(\mathbb{Z}))$. This proves the case $k = 4$. Now we prove the case $k = 2$. We define the Witt operator as before. This time, WF is a vector valued function on $H_1 \times H_1$. If we write down the condition that F is invariant by the action of $\iota(g_1, g_2) \in Sp(2, \mathbb{Z})$ where $g_1, g_2 \in SL_2(\mathbb{Z})$, we can see that the first component of $WF(\tau, \omega)$ is a modular form of weight $2+j$ of $SL_2(\mathbb{Z})$ with respect to the variable τ and of weight 2 of $SL_2(\mathbb{Z})$ with respect to the variable ω . Since $A_2(SL_2(\mathbb{Z})) = \{0\}$, we have $WF = 0$. As we explained, $\Phi(F)$ can be identified with the first component, which is equal to the first component of $\lim_{\lambda \rightarrow \infty} WF(\tau, i\lambda)$, which is zero. So we prove the assertion.

6. The Witt operator in the vector valued case

In the argument of the last section, to show $A_{2,j}(Sp(2, \mathbb{Z})) = S_{2,j}(Sp(2, \mathbb{Z}))$, we used the Witt operator W for the vector valued case. Now it is interesting to investigate the image $W(A_{k,j}(Sp(2, \mathbb{Z})))$ for general k by several reasons. For $F \in A_{k,j}(Sp(2, \mathbb{Z}))$, we write the ν -th component of F by $F_{\nu-1}$.

$$F = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_j \end{pmatrix}$$

By the action of

$$\iota(g_1, g_2) = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in Sp(2, \mathbb{Z})$$

with $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(\mathbb{Z})$, we see that

$$WF_\nu(g_1\tau, g_2\omega) = (c_1\tau + d_1)^{k+j-\nu}(c_2\omega + d_2)^{k+\nu}WF(\tau, \omega),$$

so we have $WF_\nu(\tau, \omega) = \sum_{j=1}^t f_j(\tau)g_j(\omega)$ for some $f_j \in A_{k+j-\nu}(SL_2(\mathbb{Z}))$ and $g_j \in A_{k+\nu}(SL_2(\mathbb{Z}))$. In other words, we can regard WF_ν as an element of $A_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes A_{k+\nu}(SL_2(\mathbb{Z}))$. Besides, by automorphy with respect to $U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, we

have

$$WF_\nu(\omega, \tau) = (-1)^k WF_{j-\nu}(\tau, \omega).$$

In particular, if we write $j = 2m$ with $m \in \mathbb{Z}$, then $WF_m(\omega, \tau) = (-1)^k WF_m(\tau, \omega)$. This means that WF_m belongs to the space of symmetric tensors $Sym^2(A_{k+m}(SL_2(\mathbb{Z})))$ or alternating tensors $Alt^2(A_{k+m}(SL_2(\mathbb{Z})))$ according to k is even or odd. We have some more conditions on WF . Since the first component of $\Phi(F)$ is a cusp form and all the other components are zero, we should have $WF_0 \in S_{k+j}(SL_2(\mathbb{Z})) \otimes A_k(SL_2(\mathbb{Z}))$ and $WF_\nu \in A_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes S_{k+\nu}(SL_2(\mathbb{Z}))$, and besides, since $WF_\nu(\omega, \tau) = (-1)^k WF_{j-\nu}(\tau, \omega)$, we have $WF_\nu \in S_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes S_{k+\nu}(SL_2(\mathbb{Z}))$ for any $\nu \neq 0$ or j . In particular, if F is a cusp form, then we have $WF_\nu \in S_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes S_{k+\nu}(SL_2(\mathbb{Z}))$ for all ν with $0 \leq \nu \leq j$.

We denote by $V_{k,j}$ the \mathbb{C}^{j+1} -valued functions

$$f(\tau, \omega) = \begin{pmatrix} f_0(\tau, \omega) \\ f_1(\tau, \omega) \\ \vdots \\ f_j(\tau, \omega) \end{pmatrix}$$

on $H_1 \times H_1$ defined by

$$V_{k,j} = \{f(\tau, \omega) = (f_\nu(\tau, \omega)) \quad ; \quad f_\nu(\tau, \omega) \in S_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes S_{k+\nu}(SL_2(\mathbb{Z})), \\ f_\nu(\omega, \tau) = (-1)^k f_{j-\nu}(\tau, \omega)\}$$

We denote by $\tilde{V}_{k,j}$ the space spanned by $V_{k,j}$ and functions

$$\begin{pmatrix} f(\tau)E_k(\omega) \\ 0 \\ \vdots \\ 0 \\ (-1)^k E_k(\tau)f(\omega) \end{pmatrix}$$

where E_k is the unique normalized Eisenstein series of weight k of $SL_2(\mathbb{Z})$ and f is any element in $S_{k+j}(SL_2(\mathbb{Z}))$. We see in the above that for every $F \in A_{k,j}(Sp(2, \mathbb{Z}))$ or $S_{k,j}(Sp(2, \mathbb{Z}))$, we have $WF \in \tilde{V}_{k,j}$ or $V_{k,j}$, respectively. Now we ask if W is surjective. If $k = 2$ and $j > 0$ for example, this is not surjective as we have seen already, but for big k , we have the following theorem.

THEOREM 6.1. *If $k \geq 10$, then we have $W(A_{k,j}(Sp(2, \mathbb{Z}))) = \tilde{V}_{k,j}$ and $W(S_{k,j}(Sp(2, \mathbb{Z}))) = V_{k,j}$.*

The assertion for cusp forms is obtained if we can show

$$\dim \ker(W|_{S_{k,j}(Sp(2, \mathbb{Z}))}) = \dim S_{k,j}(Sp(2, \mathbb{Z})) - \dim V_{k,j}.$$

The assertion for $A_{k,j}(Sp(2, \mathbb{Z}))$ easily follows from this result and the surjectivity of the Φ -operator. So, the first thing we should do is to characterize the kernel of W . This is well known in the scalar valued case and similarly done for the vector valued case as follows. We define the theta constant on H_2 associated with characteristic $m = (m', m'') \in \mathbb{Z}$ by

$$\theta_m(Z) = \sum_{p \in \mathbb{Z}^2} e\left(\frac{1}{2} {}^t \left(p + \frac{m'}{2}\right) Z \left(p + \frac{m'}{2}\right) + {}^t \left(p + \frac{m'}{2}\right) \frac{m''}{2}\right)$$

where $e(x) = \exp(2\pi i x)$ for any x . We define the holomorphic function $\chi_5(Z)$ on H_2 by the product of ten theta constants with even characteristics.

$$\chi_5 = \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111}.$$

It is known that $\chi_5|_5[\gamma] = \text{sgn}(\gamma)\chi_5$ for any $\gamma \in Sp(2, \mathbb{Z})$ where $\text{sgn}(\gamma) = 1$ or -1 depending on γ . This sgn defines a character of $Sp(2, \mathbb{Z})$ of order two. The kernel of sgn is a normal subgroup of $Sp(2, \mathbb{Z})$ of index two and denoted by $\Gamma_e(1)$ in Igusa [13], so we use the same notation here. The group $\Gamma_e(1)$ contains the principal congruence subgroup $\Gamma(2)$ of level 2 and we have $Sp(2, \mathbb{Z})/\Gamma(2) \cong S_6$ (the permutation group on six letters). The above sgn is nothing but the sign character on S_6 with the alternative group A_6 as the kernel and this corresponds to $\Gamma_e(1)$ (See Igusa loc. cit.). If we denote by $A_{k,j}(Sp(2, \mathbb{Z}), \text{sgn})$ the space of holomorphic functions F on H_2 such that $F|_k[\gamma] = \text{sgn}(\gamma)F$ for any $\gamma \in Sp(2, \mathbb{Z})$, then we have the direct sum decomposition

$$A_{k,j}(\Gamma_e(1)) = A_{k,j}(Sp(2, \mathbb{Z})) + A_{k,j}(Sp(2, \mathbb{Z}), \text{sgn}).$$

It is well known that χ_5 is a cusp form, $W\chi_5 = 0$, and that $\chi_5/(e^{\pi i \tau} + 1)(e^{\pi i \tau} - 1)$ is non-vanishing holomorphic function on the fundamental domain of $Sp(2, \mathbb{Z})$ (cf. Freitag [6] p. 145). This means that if $WF = 0$ for $F \in A_{k,j}(Sp(2, \mathbb{Z}))$, then F/χ_5 is also holomorphic and $F/\chi_5 \in A_{k-5,j}(Sp(2, \mathbb{Z}), \text{sgn})$. If $j = 0$ and k is even, then this implies automatically that F/χ_5^2 is also holomorphic, but in our case $j > 0$, this does not hold in general. Anyway, we have

$$\ker(W|_{S_{k,j}(Sp(2, \mathbb{Z}))}) = A_{k-5,j}(Sp(2, \mathbb{Z}), \text{sgn}).$$

Now we show that $A_{k,j}(Sp(2, \mathbb{Z}), \text{sgn}) = S_{k,j}(Sp(2, \mathbb{Z}), \text{sgn})$. For $F = (F_\nu) \in A_{k,j}(Sp(2, \mathbb{Z}), \text{sgn})$, we have

$$WF_0(\tau, \gamma\omega) = \text{sgn}_1(\gamma)(c\omega + d)^k WF(\tau, \omega)$$

for any $\gamma \in SL_2(\mathbb{Z})$ where sgn_1 is defined by the character of $SL_2(\mathbb{Z})$ which gives the sign character of S_3 through the isomorphism $SL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$. Any such function is a multiple of $\eta^{12}(\omega) = \Delta^{1/2}(\omega)$ where η is the Dedekind eta function and Δ is the Ramanujan Delta function. So WF is a cusp form with respect to ω . This implies that $\Phi(F) = 0$, so F is a cusp form. Hence we can show the surjectivity of W if we can calculate the dimensions of $S_{k,j}(Sp(2, \mathbb{Z}))$, $S_{k-5,j}(Sp(2, \mathbb{Z}), \text{sgn})$ and $V_{k,j}$. The dimension of $V_{k,j}$ is calculated by classical dimension formulas for $SL_2(\mathbb{Z})$. The dimension of $S_{k,j}(Sp(2, \mathbb{Z}))$ is known for all k if $j = 0$ by Igusa and for $k \geq 5$, $j > 0$ by Tsushima [16]. The dimension of $S_{k,j}(\Gamma_e(1))$ is given below.

In [16], the dimension formula of $S_{k,j}(Sp(2, \mathbb{Z}))$ were obtained by the Riemann-Roch-Hirzebruch theorem and holomorphic Lefschetz theorem. In [19], the same dimensions have been obtained by Selberg trace formula and we use this method here. In the Selberg trace formula, the dimensions are given as a sum of complicated integrals defined over conjugacy classes of elements in the discrete group and we shall call each such value in the summand as a contribution of the conjugacy classes. In the dimension formula, a contribution is non-zero only when the semi-simple parts of the Jordan decomposition of elements are torsion. From [8, Section 5-1], the principal polynomials of the torsion elements of $Sp(2, \mathbb{Q})$ are as follows:

$$\begin{array}{l}
 f_1(x) = (x - 1)^4, f_1(-x) \\
 f_2(x) = (x - 1)^2(x + 1)^2 \\
 f_3(x) = (x - 1)^2(x^2 + 1), f_3(-x) \\
 f_4(x) = (x - 1)^2(x^2 + x + 1), f_4(-x) \\
 f_5(x) = (x - 1)^2(x^2 - x + 1), f_5(-x) \\
 f_6(x) = (x^2 + 1)^2
 \end{array}
 \left|
 \begin{array}{l}
 f_7(x) = (x^2 + x + 1)^2 \\
 f_8(x) = (x^2 + 1)(x^2 + x + 1), f_8(-x) \\
 f_9(x) = (x^2 + x + 1)(x^2 - x + 1) \\
 f_{10}(x) = (x^4 + x^3 + x^2 + x + 1), f_{10}(-x) \\
 f_{11}(x) = x^4 + 1 \\
 f_{12}(x) = x^4 - x^2 + 1.
 \end{array}
 \right.$$

So for each l with $1 \leq l \leq 12$, we denote by H_l the contribution to the dimensions $\dim S_{k,j}(\Gamma_e(1))$ of elements such that the principal polynomial of the semi-simple part is $f_l(x)$ or $f_l(-x)$. The contribution H_l is a sum of contributions of semi-simple elements, unipotent elements or quasi-unipotent elements (i.e. elements such that some power are unipotent), and we denote by H_l^e, H_l^u or H_l^{qu} for each such subcontribution.

In the theorem below, we use notation $t = [t_0, t_1, \dots, t_{l-1}; l]_m$ which means that $t = t_n$ if $m \equiv n \pmod{l}$. We note that $\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) = 0$ if j is odd. In case of $j = 0$, the dimensions of $S_{k,0}(\Gamma_e(1))$ were calculated by Igusa [13]. In case of $j > 0$, the following result is new.

THEOREM 6.2. *Assume that $k \geq 5$ and j is even with $j \geq 0$. Then we have*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) = \sum_{i=1}^{12} H_i,$$

where H_i are the total contribution of elements of $\Gamma_e(1)$ with principal polynomial $f_i(\pm x)$ and given below:

$$\begin{aligned}
 H_1 &= H_1^e + H_1^u, & H_1^e &= 2^{-6}3^{-3}5^{-1}(j + 1)(k - 2)(j + k - 1)(j + 2k - 3), \\
 & & H_1^u &= -2^{-6}3^{-2}(j + 1)(j + 2k - 3) + 2^{-4}3^{-1}(j + 1).
 \end{aligned}$$

$$\begin{aligned}
 H_2 &= H_2^e + H_2^{qu}, & H_2^e &= 2^{-6}3^{-2}(-1)^k(j + k - 1)(k - 2), \\
 & & H_2^{qu} &= -2^{-4}3^{-1}(-1)^k(j + 2k - 3) + 2^{-6} \cdot 3 \cdot (-1)^k.
 \end{aligned}$$

$$\begin{aligned}
 H_3 &= H_3^e + H_3^{qu}, & H_3^e &= 0, \\
 H_3^{qu} &= -2^{-3}[(-1)^{j/2}, -1, (-1)^{j/2+1}, 1; 4]_k + 2^{-4}[1, (-1)^{j/2}, -1, (-1)^{j/2+1}; 4]_k.
 \end{aligned}$$

$$\begin{aligned}
 H_4 &= H_4^e + H_4^{qu}, \\
 H_4^e &= 2^{-2}3^{-3}([(j + k - 1), -(j + k - 1), 0; 3]_k + [(k - 2), 0, -(k - 2); 3]_{j+k}), \\
 H_4^{qu} &= -2^{-2}3^{-2}([1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k}) - 3^{-2}([0, -1, -1; 3]_k + [1, 1, 0; 3]_{j+k}).
 \end{aligned}$$

$$\begin{aligned}
 H_5 &= H_5^e + H_5^{qu}, & H_5^{qu} &= -2^{-2}3^{-1}([-1, -1, 0, 1, 1, 0; 6]_k + [1, 0, -1, -1, 0, 1; 6]_{j+k}), \\
 H_5^e &= 2^{-2}3^{-2}([-(j+k-1), -(j+k-1), 0, (j+k-1), (j+k-1), 0; 6]_k \\
 &\quad + [(k-2), 0, -(k-2), -(k-2), 0, (k-2); 6]_{j+k}).
 \end{aligned}$$

$$\begin{aligned}
 H_6 &= H_6^e + H_6^{qu}, & H_6^{qu} &= -2^{-3}(-1)^{j/2}, \\
 H_6^e &= 2^{-6}(-1)^{j/2}(j+2k-3) + 2^{-6}(-1)^{j/2+k}(j+1).
 \end{aligned}$$

$$\begin{aligned}
 H_7 &= H_7^e + H_7^{qu}, & H_7^{qu} &= -2^{-1}3^{-1}[1, -1, 0; 3]_j, \\
 H_7^e &= 3^{-3}(j+2k-3)[1, -1, 0; 3]_j + 2^{-1}3^{-3}(j+1)[0, 1, -1; 3]_{j+2k}.
 \end{aligned}$$

$$H_8 = 0.$$

$$H_9 = 2^{-1}3^{-2} \begin{cases} [1, 0, 0, -1, 0, 0; 6]_k & (j = 6n) \\ [-1, 1, 0, 1, -1, 0; 6]_k & (j = 6n + 2) \\ [0, -1, 0, 0, 1, 0; 6]_k & (j = 6n + 4) \end{cases}$$

$$H_{10} = 2 \cdot 5^{-1} \begin{cases} [1, 0, 0, -1, 0; 5]_k & (j = 10n) \\ [-1, 1, 0, 0, 0; 5]_k & (j = 10n + 2) \\ 0 & (j = 10n + 4) \\ [0, 0, 0, 1, -1; 5]_k & (j = 10n + 6) \\ [0, -1, 0, 0, 1; 5]_k & (j = 10n + 8) \end{cases}$$

$$H_{11} = 2^{-3} \begin{cases} [1, 1, -1, -1; 4]_k & (j = 8n) \\ [-1, 1, 1, -1; 4]_k & (j = 8n + 2) \\ [-1, -1, 1, 1; 4]_k & (j = 8n + 4) \\ [1, -1, -1, 1; 4]_k & (j = 8n + 6) \end{cases}$$

$$H_{12} = 0.$$

PROOF. If we classify the $Sp(2, \mathbb{Z})$ -conjugacy classes which belong to $\Gamma_e(1)$, then we can calculate the dimension formula by the general arithmetic formula [19, Theorem 4.2] and the data of the conjugacy classes and the centralizers of $Sp(2, \mathbb{Z})$ (cf. [7]). Hence we only explain the classification of the $Sp(2, \mathbb{Z})$ -conjugacy classes inside $\Gamma_e(1)$.

We denote by $(i_1 i_2 \cdots i_j)$ the cyclic permutation of i_1, i_2, \dots, i_j in the symmetric group S_6 . We identify $Sp(2, \mathbb{Z})/\Gamma(2)$ with S_6 as in [13]. Since this isomorphism is obtained by the action of $Sp(2, \mathbb{Z})$ on 6 odd characteristics, it is easy to calculate the image for each element of $Sp(2, \mathbb{Z})$. We write the projection of $Sp(2, \mathbb{Z})$ to S_6 simply by $Sp(2, \mathbb{Z}) \ni \gamma \rightarrow \sigma \in S_6$. Then we have

$$Sp(2, \mathbb{Z}) \ni \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow (12), \quad Sp(2, \mathbb{Z}) \ni \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix} \rightarrow (123456).$$

We give a list of representatives of the $Sp(2, \mathbb{Z})$ -conjugacy classes and their images in S_6 . We put $\tau_1 = (13)(24)(56)$, $\tau_2 = (12)(36)(45)$, $\tau_3 = (12)(34)(56)$. We denote by E_{ij} the 4×4 matrix unit such that the (i, j) component is one and the other components are zero. We denote by I_4 the 4×4 unit matrix. The notations α_* ,

$\varepsilon(S)$, $\hat{\delta}(n)$, \dots below are representatives of $Sp(2, \mathbb{Z})$ conjugacy classes used in [7]. Sometimes we use an abbreviated notation $\pm\gamma \rightarrow \tau$. This means that γ and $-\gamma$ are not conjugate and must be considered separately but projected in the same element in S_6 . Below we give a list of conjugacy classes which are projected to A_6 . The contributions to the dimensions are given by the sum of contributions of such conjugacy classes known in [19]. The details of the calculations are omitted here.

The contribution H_1 .

$$H_1^e. \pm\alpha_0 = I_4 \rightarrow 1 \in A_6.$$

H_1^u . $\pm\varepsilon(S) = I_4 + s_1E_{13} + s_{12}(E_{14} + E_{23}) + s_2E_{24}$, $((s_1, s_{12}, s_2) \in \mathbb{Z}^3, (s_1, s_{12}, s_2) \neq (0, 0, 0))$. If $(s_1, s_{12}, s_2) \equiv (0, 0, 0) \pmod{2}$, $\varepsilon(S) \rightarrow 1 \in A_6$. If $(s_1, s_{12}, s_2) \equiv (1, 0, 0) \pmod{2}$, $\varepsilon(S) \rightarrow (12) \notin A_6$. If $(s_1, s_{12}, s_2) \equiv (0, 1, 0) \pmod{2}$, $\varepsilon(S) \rightarrow \tau_3 \notin A_6$. If $(s_1, s_{12}, s_2) \equiv (0, 0, 1) \pmod{2}$, $\varepsilon(S) \rightarrow (34) \notin A_6$. If $(s_1, s_{12}, s_2) \equiv (1, 0, 1) \pmod{2}$, $\varepsilon(S) \rightarrow (12)(34) \in A_6$. If $(s_1, s_{12}, s_2) \equiv (1, 1, 0) \pmod{2}$, $\varepsilon(S) \rightarrow (12)\tau_3 \in A_6$. If $(s_1, s_{12}, s_2) \equiv (0, 1, 1) \pmod{2}$, $\varepsilon(S) \rightarrow (34)\tau_3 \in A_6$. If $(s_1, s_{12}, s_2) \equiv (1, 1, 1) \pmod{2}$, $\varepsilon(S) \rightarrow (56) \notin A_6$.

The contribution H_2 .

$$H_2^e. \delta_1 = E_{11} - E_{22} + E_{33} - E_{44} \rightarrow 1 \in A_6. \delta_2 = \delta_1 - E_{14} + E_{23} \rightarrow \tau_3 \notin A_6.$$

H_2^{qu} . $\pm\hat{\delta}_1(n) = \delta_1 + nE_{13}$, $(n \in \mathbb{Z}, n \neq 0)$. If $n \equiv 0 \pmod{2}$, $\hat{\delta}_1(n) \rightarrow 1 \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\delta}_1(n) \rightarrow (12) \notin A_6$. $\pm\hat{\delta}_2(n) = \delta_1 + nE_{13} - E_{14} + E_{23}$, $(n \in \mathbb{Z}, n \neq 0)$.

If $n \equiv 0 \pmod{2}$, $\hat{\delta}_2(n) \rightarrow \tau_3 \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\delta}_2(n) \rightarrow \tau_3(12) \in A_6$. $\hat{\delta}_1(m, n) = \delta_1 + mE_{13} + nE_{24}$, $((m, n) \in \mathbb{Z}^2, m \neq 0, n \neq 0)$. If $(m, n) \equiv (0, 0) \pmod{2}$, $\hat{\delta}_1(m, n) \rightarrow 1 \in A_6$. If $(m, n) \equiv (1, 0) \pmod{2}$, $\hat{\delta}_1(m, n) \rightarrow (12) \notin A_6$. If $(m, n) \equiv (0, 1) \pmod{2}$, $\hat{\delta}_1(m, n) \rightarrow (34) \notin A_6$. If $(m, n) \equiv (1, 1) \pmod{2}$, $\hat{\delta}_1(m, n) \rightarrow (12)(34) \in A_6$. $\hat{\delta}_2(m, n) = \delta_1 + mE_{13} - E_{14} + E_{23} + nE_{24}$, $((m, n) \in \mathbb{Z}^2, m \neq 0, n \neq 0)$. If $(m, n) \equiv (0, 0) \pmod{2}$, $\hat{\delta}_2(m, n) \rightarrow \tau_3 \notin A_6$. If $(m, n) \equiv (1, 0) \pmod{2}$, $\hat{\delta}_2(m, n) \rightarrow \tau_3(12) \in A_6$. If $(m, n) \equiv (0, 1) \pmod{2}$, $\hat{\delta}_2(m, n) \rightarrow \tau_3(34) \in A_6$. If $(m, n) \equiv (1, 1) \pmod{2}$, $\hat{\delta}_2(m, n) \rightarrow \tau_3(12)(34) \notin A_6$. $\hat{\delta}_3(m, n) = \delta_1 + 2mE_{13} + (m+2)E_{14} + E_{21} + (m-2)E_{23} + nE_{24} + E_{34}$, $((m, n) \in \mathbb{Z}^2, m \neq 0, 2m-n \neq 0)$. If $(m, n) \equiv (0, 0) \pmod{2}$, $\hat{\delta}_3(m, n) \rightarrow \tau_1\tau_2\tau_1 \notin A_6$. If $(m, n) \equiv (1, 0) \pmod{2}$, $\hat{\delta}_3(m, n) \rightarrow \tau_1\tau_2\tau_1\tau_3(34) \notin A_6$. If $(m, n) \equiv (0, 1) \pmod{2}$, $\hat{\delta}_3(m, n) \rightarrow \tau_1\tau_2\tau_1(34) \in A_6$. If $(m, n) \equiv (1, 1) \pmod{2}$, $\hat{\delta}_3(m, n) \rightarrow \tau_1\tau_2\tau_1\tau_3 \in A_6$. $\hat{\delta}_4(m, n) = \delta_1 + (2m-1)E_{13} + mE_{14} + E_{21} + (m-1)E_{23} + nE_{24} + E_{34}$, $((m, n) \in \mathbb{Z}^2)$. If $(m, n) \equiv (0, 0) \pmod{2}$, $\hat{\delta}_4(m, n) \rightarrow \tau_1\tau_2\tau_1(12) \in A_6$. If $(m, n) \equiv (1, 0) \pmod{2}$, $\hat{\delta}_4(m, n) \rightarrow \tau_1\tau_2\tau_1\tau_3(34)(56)(34) \in A_6$. If $(m, n) \equiv (0, 1) \pmod{2}$, $\hat{\delta}_4(m, n) \rightarrow \tau_1\tau_2\tau_1(12)(34) \notin A_6$. If $(m, n) \equiv (1, 1) \pmod{2}$, $\hat{\delta}_4(m, n) \rightarrow \tau_1\tau_2\tau_1(12)\tau_3 \notin A_6$.

The contribution H_3 .

$$H_3^e. \pm\beta_5 = -E_{13} + E_{22} + E_{31} + E_{44} \rightarrow (25) \notin A_6. \pm\beta_6 = E_{13} + E_{22} - E_{31} + E_{44} \rightarrow (25) \notin A_6.$$

H_3^{qu} . $\pm\hat{\beta}_7(n) = \beta_5 + nE_{24}$, $(n \in \mathbb{Z}, n \neq 0)$. If $n \equiv 0 \pmod{2}$, $\hat{\beta}_7(n) \rightarrow (25) \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_7(n) \rightarrow (25)(34) \in A_6$. $\pm\hat{\beta}_8(n) = \beta_6 + nE_{24}$, $(n \in \mathbb{Z}, n \neq 0)$. If $n \equiv 0 \pmod{2}$, $\hat{\beta}_8(n) \rightarrow (25) \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_8(n) \rightarrow (25)(34) \in A_6$. $\pm\hat{\beta}_9(n) = \beta_6 - E_{14} + E_{21} + nE_{24}$, $(n \in \mathbb{Z})$. If $n \equiv 0 \pmod{2}$, $\hat{\beta}_9(n) \rightarrow \tau_3(25) \in A_6$. If

$n \equiv 1 \pmod{2}$, $\hat{\beta}_9(n) \rightarrow \tau_3(25)(34) \notin A_6$. $\pm\hat{\beta}_{10}(n) = \beta_5 + E_{23} + nE_{24} + E_{34}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\beta}_{10}(n) \rightarrow (25)\tau_3 \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_{10}(n) \rightarrow (25)\tau_3(34) \notin A_6$.

The contribution H_4 .

H_4^e . $\pm\beta_1 = E_{13} + E_{22} - E_{31} - E_{33} + E_{44} \rightarrow (15)(25) \in A_6$. $\pm\beta_2 = -E_{11} - E_{13} + E_{22} + E_{31} + E_{44} \rightarrow (12)(25) \in A_6$.

H_4^{qu} . $\pm\hat{\beta}_3(n) = -E_{13} + E_{22} + nE_{24} + E_{31} - E_{33} + E_{44}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\beta}_3(n) \rightarrow (15)(25) \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_3(n) \rightarrow (15)(25)(34) \notin A_6$.
 $\pm\hat{\beta}_4(n) = -E_{11} + E_{13} + E_{22} + nE_{24} - E_{31} + E_{44}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\beta}_4(n) \rightarrow (12)(25) \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_4(n) \rightarrow (12)(25)(34) \notin A_6$.
 $\pm\hat{\beta}_5(n) = \hat{\beta}_4(n) + E_{14} - E_{21}$, ($n \in \mathbb{Z}$). If $n \equiv 0 \pmod{2}$, $\hat{\beta}_5(n) \rightarrow (12)(25)\tau_1\tau_2\tau_1 \notin A_6$.
If $n \equiv 1 \pmod{2}$, $\hat{\beta}_5(n) \rightarrow (12)(25)\tau_1\tau_2\tau_1(34) \in A_6$. $\pm\hat{\beta}_6(n) = \hat{\beta}_3(n) - E_{23} - E_{34}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\beta}_6(n) \rightarrow \tau_1\tau_2\tau_1(15)(25) \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_6(n) \rightarrow \tau_1\tau_2\tau_1(15)(25)(34) \in A_6$.

The contribution H_5 .

H_5^e . $\pm\beta_3 = -E_{13} + E_{22} + E_{31} + E_{33} + E_{44} \rightarrow (15)(25) \in A_6$. $\pm\beta_4 = E_{11} + E_{13} + E_{22} - E_{31} + E_{44} \rightarrow (12)(25) \in A_6$.

H_5^{qu} . $\pm\hat{\beta}_1(n) = E_{13} + E_{22} + nE_{24} - E_{31} + E_{33} + E_{44}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\beta}_1(n) \rightarrow (15)(25) \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_1(n) \rightarrow (15)(25)(34) \notin A_6$.
 $\pm\hat{\beta}_2(n) = E_{11} - E_{13} + E_{22} + nE_{24} + E_{31} + E_{44}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\beta}_2(n) \rightarrow (12)(25) \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\beta}_2(n) \rightarrow (12)(25)(34) \notin A_6$.

The contribution H_6 .

H_6^e . $\pm\alpha_1 = E_{13} + E_{24} - E_{31} - E_{42} \rightarrow (25)(46) \in A_6$. $\gamma_1 = -E_{12} + E_{21} - E_{34} + E_{43} \rightarrow \tau_1 \notin A_6$. $\gamma_2 = \gamma_1 + E_{13} - E_{24} \rightarrow \tau_1\tau_3 \in A_6$.

H_6^{qu} . $\hat{\gamma}_1(n) = \gamma_1 - nE_{14} + nE_{23}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_1(n) \rightarrow \tau_1 \notin A_6$.
If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_1(n) \rightarrow \tau_1(12)(34) \notin A_6$. $\hat{\gamma}_2(n) = \gamma_1 - nE_{14} + (n+1)E_{23}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_2(n) \rightarrow \tau_1(12) \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_2(n) \rightarrow \tau_1(34) \in A_6$.
 $\hat{\gamma}_3(n) = \gamma_2 - nE_{14} + nE_{23}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_3(n) \rightarrow \tau_1\tau_3 \in A_6$.
If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_3(n) \rightarrow \tau_1\tau_3(12)(34) \in A_6$. $\hat{\gamma}_4(n) = \gamma_2 - nE_{14} + (n+1)E_{23}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_4(n) \rightarrow \tau_1\tau_3(12) \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_4(n) \rightarrow \tau_1\tau_3(34) \notin A_6$.

The contribution H_7 .

H_7^e . $\pm\alpha_2 = \alpha_1 - E_{33} - E_{44} \rightarrow (354)\tau_1(354)\tau_1 \in A_6$. $\pm\alpha_3 = -\alpha_1 - E_{11} - E_{22} \rightarrow (345)\tau_1(345)\tau_1 \in A_6$. $\pm\gamma_3 = \gamma_1 - E_{22} - E_{33} \rightarrow \tau_1\tau_2 \in A_6$.

H_7^{qu} . $\hat{\gamma}_5(n) = \gamma_3 - nE_{13} - 2nE_{14} + nE_{23} - nE_{24}$, ($n \in \mathbb{Z}$, $n \neq 0$). If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_5(n) \rightarrow \tau_1\tau_2 \in A_6$.
If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_5(n) \rightarrow \tau_1\tau_2\tau_3 \notin A_6$. $\hat{\gamma}_6(n) = \gamma_3 - nE_{13} - 2nE_{14} + (n+1)E_{23} - nE_{24}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_6(n) \rightarrow \tau_1\tau_2(12) \notin A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_6(n) \rightarrow \tau_1\tau_2\tau_3(12) \in A_6$.
 $\hat{\gamma}_7(n) = \gamma_3 - nE_{13} - 2nE_{14} + (n+2)E_{23} - nE_{24}$, ($n \in \mathbb{Z}$).
If $n \equiv 0 \pmod{2}$, $\hat{\gamma}_7(n) \rightarrow \tau_1\tau_2 \in A_6$. If $n \equiv 1 \pmod{2}$, $\hat{\gamma}_7(n) \rightarrow \tau_1\tau_2\tau_3 \notin A_6$.

The contribution H_8 .

$\pm\alpha_{19} = -E_{11} - E_{13} - E_{24} + E_{31} + E_{42} \rightarrow (12)(25)(46) \notin A_6$. $\pm\alpha_{20} = E_{13} + E_{24} - E_{31} - E_{33} - E_{42} \rightarrow (15)(25)(46) \notin A_6$.
 $\pm\alpha_{21} = -E_{11} - E_{13} + E_{24} + E_{31} - E_{42} \rightarrow (12)(25)(46) \notin A_6$. $\pm\alpha_{22} = E_{13} - E_{24} - E_{31} - E_{33} + E_{42} \rightarrow (15)(25)(46) \notin A_6$.

The contribution H_9 .

$\pm\alpha_7 = -E_{13} - E_{23} - E_{24} + E_{31} - E_{32} + E_{42} \rightarrow (25)(46)\tau_2 \notin A_6$. $\pm\alpha_8 = -E_{13} - E_{22} - E_{24} + E_{31} + E_{33} + E_{42} \rightarrow (15)(25)(46)(36) \in A_6$. $\alpha_9 = -E_{12} - E_{14} + E_{21} + E_{32} +$

$E_{43} \rightarrow (12)(25)\tau_1 \notin A_6$. $\alpha_{10} = E_{12} + E_{23} + E_{34} - E_{41} - E_{43} \rightarrow (36)\tau_1(25) \notin A_6$.
 $\alpha_{11} = -E_{11} - E_{13} + E_{22} + E_{24} + E_{31} - E_{42} \rightarrow (345)\tau_1(345)\tau_1 \in A_6$. $\alpha_{12} = E_{13} - E_{24} - E_{31} - E_{33} + E_{42} + E_{44} \rightarrow (354)\tau_1(354)\tau_1 \in A_6$.

The contribution H_{10} .

$\pm\alpha_{15} = E_{12} + E_{13} + E_{14} + E_{23} + E_{34} - E_{41} - E_{44} \rightarrow \tau_1(25)(56)(12) \in A_6$. $\pm\alpha_{16} = -E_{11} + E_{13} + E_{24} - E_{31} - E_{34} + E_{41} - E_{42} - E_{43} \rightarrow (25)(46)\tau_3(15) \in A_6$. $\pm\alpha_{17} = -E_{12} - E_{13} - E_{21} - E_{24} + E_{31} - E_{32} - E_{33} + E_{42} \rightarrow (15)\tau_3(25)(46) \in A_6$. $\pm\alpha_{18} = -E_{13} - E_{14} + E_{21} - E_{22} - E_{23} + E_{32} + E_{43} \rightarrow (12)(56)(25)\tau_1 \in A_6$.

The contribution H_{11} .

$\alpha_4 = -E_{14} + E_{21} + E_{32} + E_{43} \rightarrow (25)\tau_1 \in A_6$. $\alpha_5 = -E_{12} - E_{23} - E_{34} + E_{41} \rightarrow \tau_1(25) \in A_6$. $\pm\alpha_6 = -E_{12} - E_{13} - E_{21} + E_{22} - E_{24} + E_{31} - E_{32} - E_{33} + E_{42} \rightarrow \tau_3(25)(46)(12)\tau_1 \notin A_6$.

The contribution H_{12} .

$\alpha_{13} = -E_{14} + E_{21} + E_{32} + E_{34} + E_{43} \rightarrow (15)(25)\tau_1 \notin A_6$. $\alpha_{14} = E_{12} + E_{21} + E_{23} + E_{34} - E_{41} \rightarrow (34)\tau_1(25) \notin A_6$. □

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1))$.

$j \setminus k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	2	3
2	0	0	0	0	0	1	0	1	0	1	1	2	3	3	2	3	4
4	0	0	0	0	0	1	1	1	1	2	3	4	4	5	6	7	9
6	0	1	0	1	2	2	2	4	4	5	5	8	9	11	12	13	16
8	1	1	0	1	2	4	4	4	5	7	9	11	13	15	16	19	23
10	0	1	0	1	1	4	4	5	6	8	10	14	15	18	20	24	29
12	1	2	3	3	5	7	8	11	12	14	18	21	25	29	33	36	42

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(Sp(2, \mathbb{Z}))$.

$j \setminus k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	3
2	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	0	3
4	0	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	6
6	0	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4	9
8	0	0	0	0	1	1	2	1	3	2	5	4	7	5	9	7	13
10	0	0	0	0	0	1	2	1	3	2	5	5	8	6	11	9	15
12	0	0	1	1	2	2	4	4	6	5	9	8	13	11	17	15	22

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(Sp(2, \mathbb{Z}), \text{sgn})$.

$j \setminus k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	0	0	0	1	0	1	0	1	0	2	0	3	0	2	0
2	0	0	0	0	0	1	0	1	0	1	0	2	1	3	0	3	1
4	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	5	3
6	0	1	0	1	1	2	1	3	2	4	2	6	4	8	5	9	7
8	1	1	0	1	1	3	2	3	2	5	4	7	6	10	7	12	10
10	0	1	0	1	1	3	2	4	3	6	5	9	7	12	9	15	14
12	1	2	2	2	3	5	4	7	6	9	9	13	12	18	16	21	20

(*) In our theorem, we assumed $k > 4$. The above value for $(k, j) = (4, 0)$ is valid by virtue of Igusa [13]. As for $k = 4, j > 0$, the above values are conjectural.

THEOREM 6.3. *For every $k \geq 10$ and j is even, we have*

$$\dim_{\mathbb{C}} S_{k,j}(Sp(2, \mathbb{Z})) - \dim_{\mathbb{C}} S_{k-5,j}(Sp(2, \mathbb{Z}), \text{sgn}) = \dim_{\mathbb{C}} V_{k,j}.$$

The Witt operator W from $S_{k,j}(Sp(2, \mathbb{Z}))$ to $V_{k,j}$ is surjective for $k \geq 10$.

PROOF. Here, let H_i (resp. H_i^{sgn}) be the total contribution of elements with principal polynomial $f_i(\pm x)$ to $\dim_{\mathbb{C}} S_{k,j}(Sp(2, \mathbb{Z}))$ (resp. $\dim_{\mathbb{C}} S_{k-5,j}(Sp(2, \mathbb{Z}), \text{sgn})$). Let $k \geq 10$. First we assume that k is even and put $k = 2l$ and $j = 2m$. Then we have

$$\begin{aligned} & H_1 + H_2 + H_6 - H_1^{\text{sgn}} - H_2^{\text{sgn}} - H_6^{\text{sgn}} \\ &= 2^{-4}3^{-3}(m-1)m(m+1) + 2^{-5}(-1)^m(m+1) \\ &+ 2^{-5}3^{-2}(m+1)(2l-7)(2l+2m-7) + \begin{cases} 2^{-3}3^{-1}(m+2l-7) & m \equiv 0(2) \\ 0 & m \equiv 1(2) \end{cases} \\ H_3 - H_3^{\text{sgn}} &= -2^{-4}(-1)^l - 2^{-2}3^{-1}(-1)^{m+l} + 2^{-4}3^{-1}(-1)^l\{l+m+(-1)^ml\}, \\ & H_4 + H_5 - H_4^{\text{sgn}} - H_5^{\text{sgn}} \\ &= -2^{-2}3^{-3}(m+2l-7)([-1, 2, -1; 3]_l - [2, -1, -1; 3]_{m+l}) \\ &- 2^{-2}3^{-3}m([-1, 2, -1; 3]_l + [2, -1, -1; 3]_{m+l}) \\ &- 2^{-1}3^{-3}([-1, 1, 0; 3]_l + [1, -1, 0; 3]_{m+l}), \\ & H_7 + H_9 + H_{12} - H_7^{\text{sgn}} - H_9^{\text{sgn}} - H_{12}^{\text{sgn}} \\ &= -2^{-1}3^{-3}(m+1)[-1, 2, -1; 3]_{m-l} \\ &+ 3^{-3}[1, -1, 0; 3]_m + \begin{cases} 2^{-1}3^{-1}[1, 0, -1; 3]_{m+2l} & m \equiv 0(2) \\ 0 & m \equiv 1(2) \end{cases}, \\ H_8 - H_8^{\text{sgn}} &= 2^{-2}3^{-1}(-1)^l\{[0, -1, 1; 3]_{l+m} + (-1)^m[1, 0, -1; 3]_l\}, \\ & H_{10} - H_{10}^{\text{sgn}} = 0, \\ & H_{11} - H_{11}^{\text{sgn}} = \begin{cases} 2^{-3}(-1)^{l+m/2} & m \equiv 0(2) \\ 0 & m \equiv 1(2) \end{cases}. \end{aligned}$$

Next, if k is odd and $k = 2l + 1$ and $j = 2m$, then we have

$$\begin{aligned} & H_1 + H_2 + H_6 - H_1^{\text{sgn}} - H_2^{\text{sgn}} - H_6^{\text{sgn}} \\ &= 2^{-4}3^{-3}(m-2)m(m-1) + 2^{-5}(-1)^{m-1}m \\ &+ 2^{-5}3^{-2}m(2l-5)(2l+2m-7) - \begin{cases} 0 & m \equiv 0(2) \\ 2^{-3}3^{-1}(m+2l-6) & m \equiv 1(2) \end{cases}, \\ H_3 - H_3^{\text{sgn}} &= -2^{-4}(-1)^{l+1} - 2^{-2}3^{-1}(-1)^{m+l} + 2^{-4}3^{-1}(-1)^{l+1}\{l+m+(-1)^{m-1}(l+1)\}, \\ & H_4 + H_5 - H_4^{\text{sgn}} - H_5^{\text{sgn}} \\ &= -2^{-2}3^{-3}(m+2l-6)([-1, 2, -1; 3]_{l+1} - [2, -1, -1; 3]_{m+l}) \\ &- 2^{-2}3^{-3}(m-1)([-1, 2, -1; 3]_{l+1} + [2, -1, -1; 3]_{m+l}) \\ &- 2^{-1}3^{-3}([-1, 1, 0; 3]_{l+1} + [1, -1, 0; 3]_{m+l}), \end{aligned}$$

$$\begin{aligned}
 &H_7 + H_9 + H_{12} - H_7^{\text{sgn}} - H_9^{\text{sgn}} - H_{12}^{\text{sgn}} \\
 &= -2^{-1}3^{-3}m[-1, 2, -1; 3]_{m+2l+1} \\
 &+ 3^{-3}[1, -1, 0; 3]_{m-1} - \begin{cases} 0 & m \equiv 0(2) \\ 2^{-1}3^{-1}[1, 0, -1; 3]_{m+2l+1} & m \equiv 1(2) \end{cases},
 \end{aligned}$$

$$\begin{aligned}
 H_8 - H_8^{\text{sgn}} &= 2^{-2}3^{-1}(-1)^{l+1}\{[0, -1, 1; 3]_{l+m} + (-1)^{m-1}[1, 0, -1; 3]_{l+1}\}, \\
 H_{10} - H_{10}^{\text{sgn}} &= 0,
 \end{aligned}$$

$$H_{11} - H_{11}^{\text{sgn}} = \begin{cases} 0 & m \equiv 0(2) \\ -2^{-3}(-1)^{l+(m+1)/2} & m \equiv 1(2) \end{cases}.$$

On the other hand, for any non negative k we have

$$\begin{aligned}
 \dim_{\mathbb{C}} V_{k,j} &= 2^{-1}(-1)^k \dim_{\mathbb{C}} S_{k+j/2}(SL_2(\mathbb{Z})) \\
 &+ 2^{-1} \sum_{a=0}^j \dim_{\mathbb{C}} S_{k+j-a}(SL_2(\mathbb{Z})) \times \dim_{\mathbb{C}} S_{k+a}(SL_2(\mathbb{Z})).
 \end{aligned}$$

We set

$$h_1(k) = 2^{-2}3^{-1}(k - 7), \quad h_2(k) = 2^{-2}(-1)^{k/2} \text{ and } h_3(k) = 3^{-1}[1, 0, -1; 3]_k.$$

Then $\dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z})) = h_1(k) + h_2(k) + h_3(k)$ if k is even and $k \geq 4$ and $\dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z})) = 0$ if k is odd. If $k = 2l$ and $j = 2m$, we have

$$\begin{aligned}
 \dim_{\mathbb{C}} V_{2l,2m} &= 2^{-1} \dim_{\mathbb{C}} S_{2l+m}(SL_2(\mathbb{Z})) \\
 &+ 2^{-1} \sum_{a=0}^m \dim_{\mathbb{C}} S_{2l+2m-2a}(SL_2(\mathbb{Z})) \times \dim_{\mathbb{C}} S_{2l+2a}(SL_2(\mathbb{Z})).
 \end{aligned}$$

To give formulas of $\dim V_{k,j}$, we put

$$\begin{aligned}
 J_1 &= 2^{-1} \sum_{a=0}^m (h_1(2l + 2m - 2a)h_1(2l + 2a) + h_2(2l + 2m - 2a)h_2(2l + 2a)), \\
 J_2 &= 2^{-1} \sum_{a=0}^m (h_1(2l + 2m - 2a)h_2(2l + 2a) + h_2(2l + 2m - 2a)h_1(2l + 2a)), \\
 J_3 &= 2^{-1} \sum_{a=0}^m (h_1(2l + 2m - 2a)h_3(2l + 2a) + h_3(2l + 2m - 2a)h_1(2l + 2a)), \\
 J_4 &= 2^{-1} \sum_{a=0}^m h_3(2l + 2m - 2a)h_3(2l + 2a), \\
 J_5 &= 2^{-1} \sum_{a=0}^m (h_2(2l + 2m - 2a)h_3(2l + 2a) + h_3(2l + 2m - 2a)h_2(2l + 2a)).
 \end{aligned}$$

Then we have

$$2^{-1} \sum_{a=0}^m \dim_{\mathbb{C}} S_{2l+2m-2a}(SL_2(\mathbb{Z})) \times \dim_{\mathbb{C}} S_{2l+2a}(SL_2(\mathbb{Z})) = J_1 + J_2 + J_3 + J_4 + J_5,$$

. Here J_i are explicitly given as follows.

$$\begin{aligned}
 J_1 &= 2^{-4}3^{-3}(m-1)m(m+1) + 2^{-5}(-1)^m(m+1) \\
 &\quad + 2^{-5}3^{-2}(m+1)(2l-7)(2l+2m-7), \\
 J_2 &= -2^{-4}(-1)^l - 2^{-2}3^{-1}(-1)^{m+l} + 2^{-4}3^{-1}(-1)^l\{l+m+(-1)^ml\}, \\
 J_3 &= -2^{-2}3^{-3}(m+2l-7)([-1, 2, -1; 3]_l - [2, -1, -1; 3]_{m+l}) \\
 &\quad - 2^{-2}3^{-3}m([-1, 2, -1; 3]_l + [2, -1, -1; 3]_{m+l}) \\
 &\quad - 2^{-1}3^{-3}([-1, 1, 0; 3]_l + [1, -1, 0; 3]_{m+l}), \\
 J_4 &= -2^{-1}3^{-3}(m+1)[-1, 2, -1; 3]_{m-l} + 3^{-3}[1, -1, 0; 3]_m, \\
 J_5 &= 2^{-2}3^{-1}(-1)^l\{[0, -1, 1; 3]_{l+m} + (-1)^m[1, 0, -1; 3]_l\}.
 \end{aligned}$$

We also have

$$\begin{aligned}
 &2^{-1} \dim_{\mathbb{C}} S_{2l+m}(SL_2(\mathbb{Z})) = \\
 &\begin{cases} 2^{-3}3^{-1}(m+2l-7) + 2^{-3}(-1)^{l+m/2} + 2^{-1}3^{-1}[1, 0, -1; 3]_{m+2l} & m \equiv 0(2) \\ 0 & m \equiv 1(2) \end{cases}.
 \end{aligned}$$

Hence we have the equality in the theorem when $k = 2l$ and $j = 2m$. If $k = 2l + 1$ and $j = 2m$, we have

$$\begin{aligned}
 \dim_{\mathbb{C}} V_{2l+1,2m} &= -2^{-1} \dim_{\mathbb{C}} S_{2l+m+1}(SL_2(\mathbb{Z})) \\
 &\quad + 2^{-1} \sum_{a=0}^{m-1} \dim_{\mathbb{C}} S_{2l+2m-2a}(SL_2(\mathbb{Z})) \times \dim_{\mathbb{C}} S_{2l+2+2a}(SL_2(\mathbb{Z})),
 \end{aligned}$$

and the proof in this case can be given similarly. □

7. Some bounds of dimensions for small weights

In this section, by using the Witt operator, we give some estimates for the dimension of small k which is unknown yet. We have dimension formulas for $A_{k,j}(Sp(2, \mathbb{Z}))$ if $k \geq 5$. We know that $\dim A_{0,j}(Sp(2, \mathbb{Z})) = A_{1,j}(Sp(2, \mathbb{Z})) = 0$ for $j > 0$. We are interested in $A_{3,j}(Sp(2, \mathbb{Z})) = S_{3,j}(Sp(2, \mathbb{Z}))$ and $A_{2,j}(Sp(2, \mathbb{Z})) = S_{2,j}(Sp(2, \mathbb{Z}))$. An exact conjecture on the dimensions for $S_{3,j}(Sp(2, \mathbb{Z}))$ was given in [10] but here we give an upper bound and a lower bound for those dimensions. First we give an upper bound. As we explained, if $WF = 0$ for $F \in S_{k,j}(Sp(2, \mathbb{Z}))$, then F/χ_5 is holomorphic and belongs to $S_{k-5,j}(Sp(2, \mathbb{Z}), sgn)$. If $k < 5$, then we have $S_{k-5,j}(Sp(2, \mathbb{Z}), sgn) = 0$, so we have $F = 0$. This means that if $k < 5$, then we have

$$\dim S_{k,j}(Sp(2, \mathbb{Z})) \leq \dim V_{k,j}.$$

For example. $V_{2,j} = 0$ for $j \leq 18$ and $V_{3,j} = 0$ for $j \leq 20$. So we have

$$S_{2,j}(Sp(2, \mathbb{Z})) = 0 \text{ for } j \leq 18 \qquad S_{3,j}(Sp(2, \mathbb{Z})) = 0 \text{ for } j \leq 20$$

We have

$$\begin{aligned}
 \sum_{j=0}^{\infty} \dim V_{2,j} s^j &= \frac{s^{20}(1+s^{10})}{(1-s^4)(1-s^6)(1-s^8)(1-s^{12})}, \\
 \sum_{j=0}^{\infty} \dim V_{3,j} s^j &= \frac{s^{22}}{(1-s^2)(1-s^6)(1-s^8)(1-s^{12})}.
 \end{aligned}$$

Numerical examples of the upper bounds are given in the following table.

j	20	22	24	26	28	30	32	34	36	38	40
$k = 2$	1	0	1	1	2	2	4	3	6	6	8
$k = 3$	0	1	1	1	2	3	3	5	6	7	9

Next we give a lower bound. The idea is to consider $\chi_5 S_{3,j}(Sp(2, \mathbb{Z}))$ inside $S_{8,j}(Sp(2, \mathbb{Z}), sgn)$. If we define the Witt operator on $S_{8,j}(Sp(2, \mathbb{Z}), sgn)$ in the same way, then we have $\dim S_{3,j}(Sp(2, \mathbb{Z})) = \dim \text{Ker}(W|_{S_{8,j}(Sp(2, \mathbb{Z}), sgn)})$. We consider the image of W on $S_{8,j}(Sp(2, \mathbb{Z}), sgn) = A_{8,j}(Sp(2, \mathbb{Z}), sgn)$. We define the character sgn_1 of $SL_2(\mathbb{Z})$ by $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \rightarrow S_3/A_3 \cong \{\pm 1\}$. For each integer k , we denote by $A_k(SL_2(\mathbb{Z}), sgn_1)$ the space of holomorphic functions f on H_1 such that $f|_k[\gamma] = sgn_1(\gamma)f$ for all $\gamma \in SL_2(\mathbb{Z})$ and holomorphic also at $i\infty$. For each k and j , we define the space $W_{k,j}$ of \mathbb{C}^{j+1} -valued functions on $H_1 \times H_1$ by

$$W_{k,j} = \{f(\tau, \omega) = (f_{j-\nu}(\tau, \omega))_{0 \leq \nu \leq j}; f_{j-\nu}(\tau, \omega) = (-1)^{k+1} f_\nu(\omega, \tau), \\ f_{j-\nu}(\tau, \omega) \in S_{k+j-\nu}(SL_2(\mathbb{Z}), sgn_1) \otimes S_{k+\nu}(SL_2(\mathbb{Z}), sgn_1)\}.$$

We see easily that $A_k(SL_2(\mathbb{Z}), sgn_1) = S_k(SL_2(\mathbb{Z}), sgn_1) = \Delta^{1/2} A_{k-6}(SL_2(\mathbb{Z}))$ for any k . In the same way as in the last section, we see that for $F \in A_k(Sp(2, \mathbb{Z}), sgn)$, we have $WF \in W_{k,j}$. In general, we see from numerical examples that W is not surjective. But since $\text{ker}(W|_{S_{8,j}(Sp(2, \mathbb{Z}), sgn)}) = S_{3,j}(Sp(2, \mathbb{Z}))$, we have

$$\dim S_{8,j}(Sp(2, \mathbb{Z}), sgn) - \dim S_{3,j}(Sp(2, \mathbb{Z})) \leq \dim W_{8,j}.$$

In other words, we have

$$\dim S_{8,j}(Sp(2, \mathbb{Z}), sgn) - \dim W_{8,j} \leq \dim S_{3,j}(Sp(2, \mathbb{Z})).$$

We have a formula for $\dim S_{8,j}(Sp(2, \mathbb{Z}), sgn)$ as given in the last section and

$$\sum_{j=0}^{\infty} \dim S_{8,j}(Sp(2, \mathbb{Z}), sgn) s^j = \frac{s^6 + s^8 + s^{10} + 2s^{12} + s^{14} + s^{16} - s^{22}}{(1 - s^6)(1 - s^8)(1 - s^{10})(1 - s^{12})}$$

It is also easy to calculate $\dim W_{8,j}$ and we have

$$\sum_{j=0}^{\infty} \dim W_{8,j} s^j = \frac{s^6(1 + s^4 - s^{12} - s^{16} + s^{18})}{(1 - s^2)(1 - s^6)(1 - s^8)(1 - s^{12})}$$

We give numerical examples of lower bounds of $\dim S_{3,j}(Sp(2, \mathbb{Z}))$ in the following table.

j	60	62	64	66	68	70	72	74	76	78
<i>lowerbound</i>	1	0	0	2	2	1	5	3	5	7

We have a conjecture on $\dim S_{3,j}(Sp(2, \mathbb{Z}))$ in [10] for any j given by

$$\sum_{j=0}^{\infty} \text{Conj. dim } S_{3,j}(Sp(2, \mathbb{Z})) s^j = \frac{s^{36}}{(1 - s^6)(1 - s^8)(1 - s^{10})(1 - s^{12})}.$$

Compared with this, which predicts the existence of a Siegel cusp form for $j = 36$ already, the above estimate is not so sharp. But we would like to emphasize here that as far as the author knows, no example of non-zero Siegel modular form in $S_{3,j}(Sp(2, \mathbb{Z}))$ was known before for any j since there were no known way to construct such modular forms, and our results assure at least the existence of such forms.

Now as for $S_{2,j}(Sp(2, \mathbb{Z}))$, we have also the inequality

$$\dim S_{7,j}(Sp(2, \mathbb{Z}), \text{sgn}) - \dim W_{7,j} \leq \dim S_{2,j}(Sp(2, \mathbb{Z})).$$

Unfortunately the left hand side is always zero or negative, so we cannot get any non-trivial lower bound of the dimension. But the space $S_{k,j}(Sp(2, \mathbb{Z}))$ can be in principle obtained explicitly in the following steps.

- (1) Construct a basis of $S_{k+10}(Sp(2, \mathbb{Z}))$, for example by theta functions with pluri-harmonic polynomials.
- (2) Get the kernel of $W|S_{k+10}(Sp(2, \mathbb{Z}))$ and divide F in the kernel by χ_5 .
- (3) Give basis of the space of $F/\chi_5 \in S_{k+5,j}(Sp(2, \mathbb{Z}), \text{sgn})$ which are in the kernel of W again.
- (4) For any element of such kernel, divide it again by χ_5 .

Then we get the space $S_{k,j}(Sp(2, \mathbb{Z}))$. If we can give a basis of $S_{k+5,j}(Sp(2, \mathbb{Z}), \text{sgn})$ directly, then we can skip (1) and (2) but this is often more difficult. By excuting these steps, R. Uchida has shown that $S_{2,20}(Sp(2, \mathbb{Z})) = 0$ in [17], the first non-trivial case for $k = 2$.

Problem. *Is there any non-zero Siegel modular forms in $S_{2,j}(Sp(2, \mathbb{Z}))$ for some j ?*

As for large k , we can prove the following theorem by an argument similar to the proof of Theorem 6.3.

THEOREM 7.1. *For $k \geq 10$ and even $j \geq 0$, we have*

$$\begin{aligned} & \dim_{\mathbb{C}} S_{k,j}(Sp(2, \mathbb{Z}), \text{sgn}) - \dim_{\mathbb{C}} S_{k-5,j}(Sp(2, \mathbb{Z})) = \dim_{\mathbb{C}} W_{k,j} \\ & + \begin{cases} [1, 0, 0; 3; j] - \dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z}), \text{sgn}) - \dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \text{sgn}) & k \equiv 0 \pmod{2} \\ [1, 0, 0; 3; j] - \dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \text{sgn}) + \dim_{\mathbb{C}} S_{k+j-5}(SL_2(\mathbb{Z})) & k \equiv 1 \pmod{2} \end{cases} \end{aligned}$$

where we put formally $\dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \text{sgn}) = 2^{-2}3^{-1}(j - 1) + 3^{-1}[1, 0, -1; 3]_j - 2^{-2}(-1)^{j/2}$ for any j .

We omit the proof here. We do not know the meaning of the above mysterious equality.

Problem. *Give more intrinsic proof of Theorems 6.3 and 7.1 e.g. by constructing Siegel modular forms which behave well under W .*

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