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Multiplicity formulas for discrete series representations in $L^2(\Gamma\backslash \mathrm{Sp}(2,\mathbb{R}))$

Satoshi Wakatsuki [*,1]

Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakumamachi, Kanazawa, Ishikawa, 920-1192, Japan

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ABSTRACT

Let Sp(2) denote the split symplectic group of rank 2 over $\mathbb{Q}$. Fix a prime p. Let K_p be a parahoric subgroup of $\mathrm{Sp}(2,\mathbb{Q}_p)$. An arithmetic subgroup Γ is defined by $\Gamma = \mathrm{Sp}(2,\mathbb{Q}) \cap (\mathrm{Sp}(2,\mathbb{R})K_0)$, where $K_0 = K_p \prod_{v<\infty,\, v\neq p} \mathrm{Sp}(2,\mathbb{Z}_v)$. In this paper, we calculate Arthur's L^2-Lefschetz trace formula for Sp(2) in order to obtain an explicit formula for multiplicities of discrete series and some non-tempered unitary representations in the discrete spectrum of $L^2(\Gamma\backslash \mathrm{Sp}(2,\mathbb{R}))$ for each such Γ. From them we derive explicit multiplicity formulas for large discrete series, which are our main results. The multiplicity formulas are applied to a study on numbers of cuspidal automorphic representations of PGSp(2).

1. Introduction

Let Sp(2) denote the split symplectic group of rank 2 over $\mathbb{Q}$. Fix a prime number p. Let K_p be a parahoric subgroup of $\mathrm{Sp}(2,\mathbb{Q}_p)$. We set $K_v = \mathrm{Sp}(2,\mathbb{Z}_v)$ for each finite place $v \neq p$ of $\mathbb{Q}$. An arithmetic subgroup Γ is defined by $\Gamma = \mathrm{Sp}(2,\mathbb{Q}) \cap (\mathrm{Sp}(2,\mathbb{R})K_0)$, where $K_0 = \prod_{v<\infty} K_v$. The purpose of this paper is to give a computable explicit multiplicity formula for large discrete series in the discrete spectrum of $L^2(\Gamma\backslash \mathrm{Sp}(2,\mathbb{R}))$ for each such Γ. Let μ be a finite-dimensional irreducible rational representation of Sp(2). A single L-packet consists of discrete series whose infinitesimal characters are the same as that of the contragradient representation of μ. Any such L-packet has four

* Fax: +81 76 264 5738.
E-mail address: wakatsuk@staff.kanazawa-u.ac.jp.

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discrete series representations. One is called a holomorphic discrete series, another is called an anti-holomorphic discrete series, and the others are called large discrete series. Moreover, the multiplicity of the holomorphic discrete series is equal to that of the anti-holomorphic one and the two large discrete series have the same multiplicity. Multiplicity formulas for holomorphic discrete series are already known (cf. [12,14,20,38,41] and Appendix A), but those for large discrete series were not. Therefore, we require multiplicity formulas for large discrete series. We calculate the geometric side of Arthur's L^2-Lefschetz trace formula (cf. [3,9]) for Sp(2) with respect to any μ and the characteristic function of K_0 by using data about Γ-conjugacy classes (cf. [10,12–14,20,39]). Then, we obtain an explicit formula for a linear combination of multiplicities of discrete series and some non-tempered unitary representations of $\mathrm{Sp}(2,\mathbb{R})$. If the highest weight of μ is regular, then the non-tempered unitary representations do not appear in the formula. Hence, the half of the linear combination is the sum of the multiplicities of the holomorphic discrete series and the large discrete series. Thus, we can derive an explicit multiplicity formula for the large discrete series with respect to Γ from them.

Using the explicit formulas we can compute differences between multiplicities of holomorphic discrete series and large ones. The differences are expressed by multiplicities for discrete series of $\mathrm{SL}(2,\mathbb{R})$. It is obvious that the differences can be associated with endoscopic groups of GSp(2) from the point of view of Arthur's conjecture. Hence, we have a conjecture on an explicit form for a stable trace formula with respect to multiplicities of discrete series. This study is closely related to Spallone's works [34,36] for a stable version of Arthur's L^2-Lefschetz trace formula.

Set $H=\mathrm{PGSp}(2)$ and $G=\mathrm{Sp}(2)$. Let $\widehat{H(\mathbb{A})}$ (resp. $\widehat{G(\mathbb{R})}$) denote the set of equivalence classes of irreducible unitary representations of $H(\mathbb{A})$ (resp. $G(\mathbb{R})$), where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. For each $\pi\in\widehat{H(\mathbb{A})}$ (resp. $\sigma_\infty\in\widehat{G(\mathbb{R})}$), we denote by m_π (resp. $m(\sigma_\infty,\Gamma)$) the multiplicity of π (resp. σ_∞) in the discrete spectrum of $L^2(H(\mathbb{Q})\backslash H(\mathbb{A}))$ (resp. $L^2(\Gamma\backslash G(\mathbb{R}))$). We set $K_0'=K_p'\prod_{v<\infty,\ v\neq p}\mathrm{GSp}(2,\mathbb{Z}_v)$ where K_p' is a parahoric subgroup of $\mathrm{GSp}(2,\mathbb{Q}_p)$ satisfying $K_p=K_p'\cap\mathrm{Sp}(2,\mathbb{Q}_p)$. Let $N_{\pi,K_0'}$ denote the space of K_0'-fixed vectors in $\bigotimes_{v<\infty}\pi_v$ for $\pi=\bigotimes_v\pi_v\in\widehat{H(\mathbb{A})}$. Assume that σ_∞ is a discrete series of $G(\mathbb{R})$. Then, we have the equality

$$m(\sigma_\infty,\Gamma)=\sum_{\pi}m_\pi\cdot\dim_{\mathbb{C}}N_{\pi,K_0'}$$

where $\pi=\bigotimes_v\pi_v$ runs over all representations of $\widehat{H(\mathbb{A})}$ such that σ_∞ is a constituent of $\pi_\infty|_{G(\mathbb{R})}$. The dimensions $\dim_{\mathbb{C}}N_{\pi,K_0'}$ are determined by Schmidt in [32,33]. Our multiplicity formula provides the numerical value of $m(\sigma_\infty,\Gamma)$. Therefore, we can get numerical information about multiplicities and numbers of cuspidal automorphic representations. Furthermore, if we take linear combinations of multiplicities, then we obtain some formulas for numbers of cuspidal automorphic representations satisfying some conditions at p. For an earlier study related to this work, we refer to Gross and Pollack [11]. They have studied explicit computations for Euler characteristics associated with Steinberg representations by using trace formula in general. We find two interesting numerical experiments using such linear combinations. One is that a linear combination provides only even numbers for any parameter. In [42], we explained that it is related to class numbers and Arthur's conjecture. The other is that values of a linear combination for suitable parameters coincide with dimensions of spaces of Siegel cusp forms for $\mathrm{Sp}(2,\mathbb{Z})$. The reason is unclear.

This paper is organized as follows. In Section 2, we review multiplicities, discrete series, and some non-tempered unitary representations. In Section 3, we explicitly calculate Arthur's L^2-Lefschetz trace formula for Sp(2). In Section 4, we give multiplicity formulas for large discrete series. In Section 5, we consider some relations and numerical experiments on multiplicities. In Appendix A, we give some dimension formulas which are necessary to prove multiplicity formulas in Section 4. In Appendix B, we give two tables, which are due to Schmidt [32]. One is a classification for irreducible admissible representations of $\mathrm{GSp}(2,\mathbb{Q}_p)$ supported in the minimal parabolic subgroup and the other is a table for dimensions of spaces of parahori-invariant vectors.

2. Preliminaries

2.1. Notations

For a finite set X, the cardinality of X is denoted by $|X|$. Let R be a ring. We denote by $M(n,R)$ the ring of matrices of degree n over R. Let $\mathrm{diag}(a_1,a_2,\dots,a_n)$ denote the diagonal matrix whose entries are given by $a_1,a_2,\dots,a_n$. We denote by I_n the unit matrix of $M(n,R)$ and by O_n the zero matrix of $M(n,R)$. For a matrix x, ${}^t x$ denotes the transpose of x. Let $\mathrm{GL}(n,R)$ denote the group of invertible matrices in $M(n,R)$, and let $\mathrm{SL}(n,R)$ denote the subgroup of matrices with determinant one in $\mathrm{GL}(n,R)$.

Let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the field of rational, real, and complex numbers respectively. We denote by $\mathbb{Z}$ the ring of rational integers. Let i denote the complex number $\sqrt{-1}$. For each place v of $\mathbb{Q}$, we denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ at v. The real place of $\mathbb{Q}$ is denoted by ∞, that is, $\mathbb{Q}_\infty=\mathbb{R}$. Let $\mathbb{Z}_v$ denote the ring of integers of $\mathbb{Q}_v$ for each finite place v. Let G be an algebraic group over $\mathbb{Q}$. Let $G(\mathbb{Z})$, $G(\mathbb{Z}_v)$, $G(\mathbb{Q})$, and $G(\mathbb{Q}_v)$ denote the group of $\mathbb{Z}$-valued, $\mathbb{Z}_v$-valued, $\mathbb{Q}$-valued, and $\mathbb{Q}_v$-valued points of G respectively. Let $\mathbb{A}$ denote the adele ring of $\mathbb{Q}$. We denote by $G(\mathbb{A})$ the adelization of G. We put $G(\mathbb{A}_{\mathrm{fin}})=\{(x_v)\in G(\mathbb{A})\mid x_\infty=1\}$. We set

$$\mathrm{GSp}(2)=\left\{g\in\mathrm{GL}(4)\;\middle|\;{}^\exists\lambda(g)\in\mathrm{GL}(1)\text{ s.t. } g\begin{pmatrix}O_2 & I_2\\ -I_2 & O_2\end{pmatrix}{}^t g=\lambda(g)\begin{pmatrix}O_2 & I_2\\ -I_2 & O_2\end{pmatrix}\right\}.$$

Let Z denote the center of $\mathrm{GSp}(2)$. We set $\mathrm{Sp}(2)=\{g\in\mathrm{GSp}(2)\mid\lambda(g)=1\}$ and $\mathrm{PGSp}(2)=\mathrm{GSp}(2)/Z$.

2.2. Discrete spectrum

Let G be a connected semi-simple algebraic group over $\mathbb{Q}$ and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. The group $G(\mathbb{R})$ acts on $L^2(\Gamma\backslash G(\mathbb{R}))$ by the right regular representation. Let $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))$ be the subspace of $L^2(\Gamma\backslash G(\mathbb{R}))$ spanned by the closed $G(\mathbb{R})$-invariant irreducible subspaces of $L^2(\Gamma\backslash G(\mathbb{R}))$. The subspace $L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R}))$ is its orthogonal complement. $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))$ is called the discrete spectrum and $L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R}))$ is called the continuous spectrum. We have the orthogonal direct sum

$$L^2\big(\Gamma\backslash G(\mathbb{R})\big)=L^2_{\mathrm{disc}}\big(\Gamma\backslash G(\mathbb{R})\big)\oplus L^2_{\mathrm{cont}}\big(\Gamma\backslash G(\mathbb{R})\big).$$

It is well known that the discrete spectrum $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))$ decomposes into an orthogonal direct sum

$$L^2_{\mathrm{disc}}\big(\Gamma\backslash G(\mathbb{R})\big)\cong\bigoplus_{\sigma_\infty\in\widehat{G(\mathbb{R})}}m(\sigma_\infty,\Gamma)\cdot H_{\sigma_\infty},$$

where $\widehat{G(\mathbb{R})}$ is the unitary dual of $G(\mathbb{R})$, H_{σ_∞} is the Hilbert space of σ_∞, and $m(\sigma_\infty,\Gamma)$ is a non-negative integer. Note that there are only countably many representations $\sigma_\infty\in\widehat{G(\mathbb{R})}$ such that $m(\sigma_\infty,\Gamma)>0$. We call $m(\sigma_\infty,\Gamma)$ the multiplicity of σ_∞ in $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))$.

2.3. Discrete series representations of $\mathrm{Sp}(2,\mathbb{R})$

We explain the Harish-Chandra parameter of $\mathrm{Sp}(2,\mathbb{R})$. For details, we refer to [23]. We set

$$K_\infty=\left\{\begin{pmatrix}A & B\\ -B & A\end{pmatrix}\in\mathrm{Sp}(2,\mathbb{R})\;\middle|\;A,B\in M(2,\mathbb{R})\right\}\cong\mathrm{U}(2).$$

The group K_∞ is a maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$. We set

$$\Xi_1 = \{(l_1,l_2)\in\mathbb{Z}^2 \mid l_1>l_2>0\},\qquad \Xi_2 = \{(l_1,l_2)\in\mathbb{Z}^2 \mid l_1>-l_2>0\},$$
$$\Xi_3 = \{(l_1,l_2)\in\mathbb{Z}^2 \mid -l_2>l_1>0\},\qquad \Xi_4 = \{(l_1,l_2)\in\mathbb{Z}^2 \mid -l_2>-l_1>0\}.$$

There exists a one-to-one correspondence between the set $\Xi=\Xi_1\cup\Xi_2\cup\Xi_3\cup\Xi_4$ and the set of unitary equivalence classes of discrete series representations of $\mathrm{Sp}(2,\mathbb{R})$. The integral point (l_1,l_2) is called the Harish-Chandra parameter. We assume that

(1) $D(l_1,l_2)$ denotes the holomorphic discrete series representation with the minimal K-type $\det^{l_2+2}\otimes\mathrm{Sym}_{l_1-l_2-1}$ if $(l_1,l_2)\in\Xi_1$,
(2) $D(l_1,l_2)$ denotes the large discrete series representation with the minimal K-type $\det^{l_2}\otimes\mathrm{Sym}_{l_1-l_2+1}$ if $(l_1,l_2)\in\Xi_2$,
(3) $D(l_1,l_2)$ denotes the large discrete series representation with the minimal K-type $\det^{l_2-1}\otimes\mathrm{Sym}_{l_1-l_2+1}$ if $(l_1,l_2)\in\Xi_3$,
(4) $D(l_1,l_2)$ denotes the anti-holomorphic discrete series representation with the minimal K-type $\det^{l_2-1}\otimes\mathrm{Sym}_{l_1-l_2-1}$ if $(l_1,l_2)\in\Xi_4$,

where Sym_j means the j-th symmetric tensor representation of $\mathrm{GL}(2,\mathbb{C})$. For each $(l_1,l_2)\in\Xi$, we easily see that

$$m\big(D(l_1,l_2),\Gamma\big)=m\big(D(-l_2,-l_1),\Gamma\big).$$

Hence, we have only to consider Ξ_1 and Ξ_2 for multiplicities of discrete series. From now on, we assume

$$(l_1,l_2)\in\Xi_1,$$

that is, $l_1>l_2>0$. For the sake of simplification of description, we set

$$D^{\mathrm{Hol}}_{l_1,l_2}=D(l_1,l_2)\quad\text{and}\quad D^{\mathrm{Large}}_{l_1,l_2}=D(l_1,-l_2).$$

Note that the set $\{D^{\mathrm{Hol}}_{l_1,l_2}, D^{\mathrm{Large}}_{l_1,l_2}, D(l_2,-l_1), D(-l_2,-l_1)\}$ is a single L-packet.

2.4. Holomorphic Siegel cusp form

We define spaces of holomorphic Siegel cusp forms of degree two. Let $\rho_{k,j}=\det^k\otimes\mathrm{Sym}_j$. The representation $\rho_{k,j}$ is a $(j+1)$-dimensional irreducible representation of $\mathrm{GL}(2,\mathbb{C})$. We set

$$\mathfrak{H}_2=\{Z\in M_2(\mathbb{C}) \mid Z={}^tZ,\ \mathrm{Im}(Z)>0\}.$$

The group $\mathrm{Sp}(2,\mathbb{R})$ acts on $\mathfrak{H}_2$ as $g\cdot Z=(AZ+B)(CZ+D)^{-1}$ for $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{Sp}(2,\mathbb{R})$, $Z\in\mathfrak{H}_2$. Let $S_{k,j}(\Gamma)$ denote the space of Siegel cusp forms of weight $\rho_{k,j}$ for Γ, i.e. the space of holomorphic functions $f:\mathfrak{H}_2\to\mathbb{C}^{j+1}$ satisfying

(i) $f(\gamma\cdot Z)=\rho_{k,j}(CZ+D)f(Z)$ for all $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma$, $Z\in\mathfrak{H}_2$,
(ii) $|\rho_{k,j}(\mathrm{Im}(Z)^{1/2})f(Z)|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$, where $\mathrm{Im}(Z)^{1/2}$ is the real symmetric matrix such that $(\mathrm{Im}(Z)^{1/2})^2=\mathrm{Im}(Z)$.

Note that $\dim_{\mathbb{C}} S_{k,j}(\Gamma) = 0$ if $-I_4 \in \Gamma$ and j is odd. We set

$$(l_1, l_2) = (j + k - 1, k - 2).$$

Then, it is known that

$$m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \Gamma\big) = \dim_{\mathbb{C}} S_{k,j}(\Gamma)$$

(cf. [43]). We have already known dimension formulas for spaces of Siegel cusp forms of degree two in many cases (see e.g., Igusa [22], Hashimoto [12], Hashimoto and Ibukiyama [14], Ibukiyama [20,21], Tsushima [37,38], Wakatsuki [41]).

2.5. Some non-tempered unitary representations

We will introduce some non-tempered unitary representations of $\mathrm{Sp}(2,\mathbb{R})$, which are related to the L^2-cohomology of $\Gamma\backslash\mathfrak{H}_2$. We set

$$P_1(\mathbb{R}) = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R}) \right\} \quad \text{and} \quad P_2(\mathbb{R}) = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R}) \right\}.$$

Let $M_k(\mathbb{R})A_k(\mathbb{R})N_k(\mathbb{R})$ be the Langlands decomposition of $P_k(\mathbb{R})$ for $k = 1$ or 2. Hence, we have $M_1(\mathbb{R}) \cong \mathrm{SL}^{\pm}(2,\mathbb{R})$, $A_1(\mathbb{R}) \cong \mathbb{R}_{>0}$, $N_1(\mathbb{R}) \cong \mathbb{R}^3$, $M_2(\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$, $A_2(\mathbb{R}) \cong \mathbb{R}_{>0}$, and $N_2(\mathbb{R}) \cong \mathbb{R} \ltimes \mathbb{R}^2$.

Let D_k^+ (resp. D_k^-) denote the holomorphic (resp. anti-holomorphic) discrete series of $\mathrm{SL}(2,\mathbb{R})$ which has the minimal K-type $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{i(k+1)\theta}$ (resp. $e^{-i(k+1)\theta}$). The discrete series D_k of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is defined by $D_k|_{\mathrm{SL}(2,\mathbb{R})} \cong D_k^+ \oplus D_k^-$. The quasi-character ν_1 on $\mathbb{R}_{>0}$ is defined by $\nu_1(a) = a$ ($a \in \mathbb{R}_{>0}$). Let sgn denote the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$.

Let k be an integer which is larger than 2. We denote by σ_k^- the Langlands quotient of the induced representation

$$\mathrm{Ind}_{M_1(\mathbb{R})A_1(\mathbb{R})N_1(\mathbb{R})}^{\mathrm{Sp}(2,\mathbb{R})}(D_{2k-3} \otimes \nu_1 \otimes 1).$$

The Harish-Chandra module of σ_k^- is equivalent to a certain $A_{\mathfrak{q}}(\lambda)$-module (cf. [40, Section 6]). Hence, σ_k^- is unitarizable. The A-packet of σ_k^- is $\{\sigma_k^-, D^{\mathrm{Hol}}_{k-1,k-2}, D(-k+2,-k+1)\}$ (cf. [1]).

Let l be an integer which is larger than 1. Let $\omega_l^{\pm}$ denote the Langlands quotient of the induced representation

$$\mathrm{Ind}_{M_2(\mathbb{R})A_2(\mathbb{R})N_2(\mathbb{R})}^{\mathrm{Sp}(2,\mathbb{R})}\big((D_l^{\pm} \otimes \mathrm{sgn}) \otimes \nu_1 \otimes 1\big).$$

The Harish-Chandra module of ω_l^+ (or ω_l^-) is also equivalent to a certain $A_{\mathfrak{q}}(\lambda)$-module. The set $\{\omega_l^+, \omega_l^-\}$ is an A-packet (cf. [1]).

3. Explicit calculation for Arthur's L^2-Lefschetz trace formula

3.1. Arithmetic subgroups

We set

$$G = \mathrm{Sp}(2)$$

throughout this section. Fix a prime number p. Put

$$x_1 = \mathrm{diag}(1,1,p,p) \quad \text{and} \quad x_2 = \mathrm{diag}(1,1,p,1).$$

We set

$$\mathbf{K}_p = \mathrm{Sp}(2,\mathbb{Z}_p), \qquad \mathbf{K}_p^{\mathrm{par}} = x_2 M(4,\mathbb{Z}_p) x_2^{-1} \cap \mathrm{Sp}(2,\mathbb{Q}_p), \qquad \mathbf{K}_p^{\mathrm{iwa}} = \mathbf{K}_p^{\mathrm{kli}} \cap \mathbf{K}_p^{\mathrm{sie}},$$
$$\mathbf{K}_p^{\mathrm{kli}} = x_2 M(4,\mathbb{Z}_p) x_2^{-1} \cap \mathrm{Sp}(2,\mathbb{Z}_p), \qquad \mathbf{K}_p^{\mathrm{sie}} = x_2 M(4,\mathbb{Z}_p) x_1^{-1} \cap \mathrm{Sp}(2,\mathbb{Z}_p).$$

These comprise the five conjugacy classes of parahoric subgroups of $\mathrm{Sp}(2,\mathbb{Q}_p)$. Therefore for the purpose of computing multiplicity formulas, we may and will assume that K_p is one of $\mathbf{K}_p$, $\mathbf{K}_p^{\mathrm{par}}$, $\mathbf{K}_p^{\mathrm{kli}}$, $\mathbf{K}_p^{\mathrm{sie}}$, and $\mathbf{K}_p^{\mathrm{iwa}}$. We set $K_v = \mathrm{Sp}(2,\mathbb{Z}_v)$ for each finite place $v \neq p$ of $\mathbb{Q}$ and

$$K_0 = \prod_{v<\infty} K_v.$$

An arithmetic subgroup Γ is defined by

$$\Gamma = \mathrm{Sp}(2,\mathbb{Q}) \cap \big(\mathrm{Sp}(2,\mathbb{R}) K_0\big).$$

If $K_p = \mathbf{K}_p$, then $\Gamma = \mathrm{Sp}(2,\mathbb{Z})$. The arithmetic subgroup Γ is denoted by $\mathrm{K}(p)$ if $K_p = \mathbf{K}_p^{\mathrm{par}}$, $\mathrm{Kl}(p)$ if $K_p = \mathbf{K}_p^{\mathrm{kli}}$, $\mathrm{Si}(p)$ if $K_p = \mathbf{K}_p^{\mathrm{sie}}$, and $\mathrm{I}(p)$ if $K_p = \mathbf{K}_p^{\mathrm{iwa}}$. We call $\mathrm{Sp}(2,\mathbb{Z})$ the full modular group, $\mathrm{K}(p)$ the paramodular group, $\mathrm{Kl}(p)$ the Klingen congruence subgroup, $\mathrm{Si}(p)$ the Siegel congruence subgroup, and $\mathrm{I}(p)$ the Iwahori subgroup.

3.2. L^2-Lefschetz number

For each $l = (l_1,l_2) \in \Xi_1$, we denote by $\mu(l_1,l_2)$ a finite-dimensional irreducible rational representation of $\mathrm{Sp}(2)$ over $\mathbb{Q}$, whose contragradient representation has the same infinitesimal character as $D_{l_1,l_2}^{\mathrm{Hol}}$. We define a test function h_Γ on $G(\mathbb{A}_{\mathrm{fin}})$ by $h_\Gamma = \mathrm{vol}(K_0)^{-1} \times h_0$, where $\mathrm{vol}(K_0)$ is the volume of K_0 and h_0 is the characteristic function of K_0. A number $\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)$ is defined by

$$\begin{aligned}
&(-1) \times \mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma) \\
&\quad = 2\cdot m\big(D_{l_1,l_2}^{\mathrm{Hol}},\Gamma\big) + 2\cdot m\big(D_{l_1,l_2}^{\mathrm{Large}},\Gamma\big) \\
&\qquad + \begin{cases} 0 & \text{if } l_1-l_2>1 \text{ and } l_2>1, \\ -2\cdot m(\sigma_{l_2+2}^-,\Gamma) & \text{if } l_1-l_2=1 \text{ and } l_2>1, \\ -2\cdot m(\omega_{l_1}^+,\Gamma) - 2\cdot m(\omega_{l_1}^-,\Gamma) & \text{if } l_1-l_2>1 \text{ and } l_2=1, \\ -2\cdot m(\sigma_3^-,\Gamma) - 2\cdot m(\omega_2^+,\Gamma) - 2\cdot m(\omega_2^-,\Gamma) - 4 & \text{if } l_1-l_2=1 \text{ and } l_2=1. \end{cases}
\end{aligned}$$

By the result of Hiraga [16,17] we find that the number $\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)$ agrees with the L^2-Lefschetz number $\mathscr{L}_\mu(h)$ defined in [3] for $\mu = \mu(l_1,l_2)$ and $h = h_\Gamma$.

3.3. Explicit formulas

In order to simplify descriptions for multiplicity formulas, we define some notations first. In the following, $t=[t_0,t_1,\dots,t_{l-1};l]_m$ means that $t=t_n$ if $m\equiv n \bmod l$. We set

$$C_1(k,j)=(j+1)(k-2)(j+k-1)(j+2k-3),\qquad C_1'(k,j)=(j+1)(j+2k-3),$$

$$C_2(k,j)=(j+k-1)(k-2)(-1)^k,\qquad C_2'(k,j)=(j+2k-3)(-1)^k,$$

$$C_3(k,j)=\left[(k-2)(-1)^{j/2},-(j+k-1),-(k-2)(-1)^{j/2},(j+k-1);4\right]_k,$$

$$C_3'(k,j)=\left[(-1)^{j/2},-1,-(-1)^{j/2},1;4\right]_k,$$

$$C_4(k,j)=\left[(j+k-1),-(j+k-1),0;3\right]_k+\left[(k-2),0,-(k-2);3\right]_{j+k},$$

$$C_4'(k,j)=[1,-1,0;3]_k+[1,0,-1;3]_{j+k},$$

$$C_4''(k,j)=[0,-1,0;3]_k+[1,0,0;3]_{j+k},$$

$$C_5(k,j)=\left[-(j+k-1),-(j+k-1),0,(j+k-1),(j+k-1),0;6\right]_k$$
$$+\left[(k-2),0,-(k-2),-(k-2),0,(k-2);6\right]_{j+k},$$

$$C_5'(k,j)=[-1,-1,0,1,1,0;6]_k+[1,0,-1,-1,0,1;6]_{j+k},$$

$$C_6(k,j)=(j+2k-3)(-1)^{j/2},\qquad C_6'(k,j)=(j+1)(-1)^{j/2+k},$$

$$C_7(k,j)=(j+2k-3)[1,-1,0;3]_j,\qquad C_7'(k,j)=(j+1)[0,1,-1;3]_{j+2k},$$

$$C_8(k,j)=\begin{cases}[1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k & \text{if } j\equiv 0 \bmod 12,\\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k & \text{if } j\equiv 2 \bmod 12,\\ [1,-1,0,0,-1,1,-1,1,0,0,1,-1;12]_k & \text{if } j\equiv 4 \bmod 12,\\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k & \text{if } j\equiv 6 \bmod 12,\\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k & \text{if } j\equiv 8 \bmod 12,\\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k & \text{if } j\equiv 10 \bmod 12,\end{cases}$$

$$C_9(k,j)=\begin{cases}[1,0,0,-1,0,0;6]_k & \text{if } j\equiv 0 \bmod 6,\\ [-1,1,0,1,-1,0;6]_k & \text{if } j\equiv 2 \bmod 6,\\ [0,-1,0,0,1,0;6]_k & \text{if } j\equiv 4 \bmod 6,\end{cases}$$

$$C_{10}(k,j)=\begin{cases}[1,0,0,-1,0;5]_k & \text{if } j\equiv 0 \bmod 10,\\ [-1,1,0,0,0;5]_k & \text{if } j\equiv 2 \bmod 10,\\ 0 & \text{if } j\equiv 4 \bmod 10,\\ [0,0,0,1,-1;5]_k & \text{if } j\equiv 6 \bmod 10,\\ [0,-1,0,0,1;5]_k & \text{if } j\equiv 8 \bmod 10,\end{cases}$$

$$C_{11}(k,j)=\begin{cases}[1,0,0,-1;4]_k & \text{if } j\equiv 0 \bmod 8,\\ [-1,1,0,0;4]_k & \text{if } j\equiv 2 \bmod 8,\\ [-1,0,0,1;4]_k & \text{if } j\equiv 4 \bmod 8,\\ [1,-1,0,0;4]_k & \text{if } j\equiv 6 \bmod 8,\end{cases}$$

$$C_{12}(k,j)=(-1)^{j/2+k}[1,-1,0;3]_j,\qquad C_{12}'(k,j)=(-1)^{j/2}[0,1,-1;3]_{j+2k}.$$

We also set

$$B(a,b;p,m)=\begin{cases}a & \text{if } p\neq m,\\ b & \text{if } p=m.\end{cases}$$

The Legendre symbol is denoted by $(*|p)$. Especially we use

$$(-1|p)=\begin{cases}0 & \text{if } p=2,\\ 1 & \text{if } p\equiv 1 \bmod 4,\\ -1 & \text{if } p\equiv 3 \bmod 4,\end{cases}\qquad (-3|p)=\begin{cases}0 & \text{if } p=3,\\ 1 & \text{if } p\equiv 1 \bmod 3,\\ -1 & \text{if } p\equiv 2 \bmod 3.\end{cases}$$

From [13, Section 5-1] we know that the characteristic polynomials of the torsion elements of $\mathrm{Sp}(2,\mathbb{Q})$ are as follows:

$f_1(x)=(x-1)^4,\ f_1(-x),$	$f_7(x)=(x^2+x+1)^2,\ f_7(-x),$
$f_2(x)=(x-1)^2(x+1)^2,$	$f_8(x)=(x^2+1)(x^2+x+1),\ f_8(-x),$
$f_3(x)=(x-1)^2(x^2+1),\ f_3(-x),$	$f_9(x)=(x^2+x+1)(x^2-x+1),$
$f_4(x)=(x-1)^2(x^2+x+1),\ f_4(-x),$	$f_{10}(x)=(x^4+x^3+x^2+x+1),\ f_{10}(-x),$
$f_5(x)=(x-1)^2(x^2-x+1),\ f_5(-x),$	$f_{11}(x)=x^4+1,$
$f_6(x)=(x^2+1)^2,$	$f_{12}(x)=x^4-x^2+1.$

We denote by $H^{\mathrm{Lef}}_{t,\Gamma}$ the total contribution of elements of Γ with characteristic polynomial $f_t(\pm x)$ to $(-1)\times\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)$. If we examine the K-types (cf. [16]), then it is obvious that

$$m\bigl(D^{\mathrm{Hol}}_{l_1,l_2},\Gamma\bigr)=m\bigl(D^{\mathrm{Large}}_{l_1,l_2},\Gamma\bigr)=0\quad \text{if } l_1-l_2 \text{ is even},$$
$$m\bigl(\omega_l^+,\Gamma\bigr)=m\bigl(\omega_l^-,\Gamma\bigr)=0\quad \text{if } l \text{ is odd}.$$

Thus, $\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)=0$ if l_1-l_2 is even. Hence we may assume that l_1-l_2 is odd. We will use the parameter (k,j) to describe multiplicity formulas instead of (l_1,l_2) (cf. Section 2.4). It is better, because only j is related to the assumption that $l_1-l_2=j+1$ is odd.

Theorem 3.1. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd. Then, we have*

$$(-1)\times\mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{Sp}(2,\mathbb{Z})})=\sum_{t=1}^{12}H^{\mathrm{Lef}}_{t,\mathrm{Sp}(2,\mathbb{Z})},$$
$$H^{\mathrm{Lef}}_{1,\mathrm{Sp}(2,\mathbb{Z})}=2^{-5}3^{-3}5^{-1}C_1(k,j)-2^{-1}3^{-1}(k-2)-2^{-2}3^{-1}(j+1)+2^{-1},$$
$$H^{\mathrm{Lef}}_{2,\mathrm{Sp}(2,\mathbb{Z})}=2^{-5}3^{-2}C_2(k,j)\cdot 7-2^{-2}3^{-1}C_2'(k,j)+2^{-1}(-1)^k,$$
$$H^{\mathrm{Lef}}_{3,\mathrm{Sp}(2,\mathbb{Z})}=2^{-3}3^{-1}C_3(k,j)-2^{-1}C_3'(k,j),$$
$$H^{\mathrm{Lef}}_{4,\mathrm{Sp}(2,\mathbb{Z})}=2^{-1}3^{-3}C_4(k,j)-3^{-1}C_4'(k,j),$$
$$H^{\mathrm{Lef}}_{5,\mathrm{Sp}(2,\mathbb{Z})}=2^{-1}3^{-2}C_5(k,j)-3^{-1}C_5'(k,j),$$
$$H^{\mathrm{Lef}}_{6,\mathrm{Sp}(2,\mathbb{Z})}=2^{-5}3^{-1}\bigl\{C_6(k,j)-C_6'(k,j)\bigr\}+2^{-2}\bigl\{(-1)^{j/2+k}-(-1)^{j/2}\bigr\},$$
$$H^{\mathrm{Lef}}_{7,\mathrm{Sp}(2,\mathbb{Z})}=2^{-1}3^{-3}\bigl\{C_7(k,j)-C_7'(k,j)\bigr\}+3^{-1}\bigl\{[0,1,-1;3]_{j+2k}-[1,-1,0;3]_j\bigr\},$$
$$H^{\mathrm{Lef}}_{8,\mathrm{Sp}(2,\mathbb{Z})}=3^{-1}C_8(k,j),\qquad H^{\mathrm{Lef}}_{9,\mathrm{Sp}(2,\mathbb{Z})}=2^2\cdot 3^{-2}C_9(k,j),$$
$$H^{\mathrm{Lef}}_{10,\mathrm{Sp}(2,\mathbb{Z})}=2^2\cdot 5^{-1}C_{10}(k,j),\qquad H^{\mathrm{Lef}}_{11,\mathrm{Sp}(2,\mathbb{Z})}=2^{-1}C_{11}(k,j),$$
$$H^{\mathrm{Lef}}_{12,\mathrm{Sp}(2,\mathbb{Z})}=2^{-1}3^{-1}\bigl\{C_{12}(k,j)-C_{12}'(k,j)\bigr\}.$$

Theorem 3.2. *Let $(l_1, l_2) = (j+k-1, k-2) \in \Xi_1$. Assume that $l_1 - l_2 = j+1$ is odd. Then, we have*

$$(-1) \times \mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{K}(p)}) = \sum_{t=1}^{12} H^{\mathrm{Lef}}_{t,\mathrm{K}(p)},$$

$$H^{\mathrm{Lef}}_{1,\mathrm{K}(p)} = 2^{-5}3^{-3}5^{-1}C_1(k,j)\cdot\big(p^2+1\big) - 2^{-1}3^{-1}(k-2)\cdot(p+1) - 2^{-2}3^{-1}(j+1)\cdot 2 + 2^{-1}\cdot 2,$$

$$H^{\mathrm{Lef}}_{2,\mathrm{K}(p)} = 2^{-5}3^{-2}C_2(k,j)\cdot B(14,11;p,2) - 2^{-2}3^{-1}C'_2(k,j)\cdot 2 + 2^{-1}(-1)^k\cdot 2,$$

$$H^{\mathrm{Lef}}_{3,\mathrm{K}(p)} = 2^{-3}3^{-1}C_3(k,j)\cdot B(2,5;p,2) - 2^{-1}C'_3(k,j)\cdot 2,$$

$$H^{\mathrm{Lef}}_{4,\mathrm{K}(p)} = 2^{-1}3^{-3}C_4(k,j)\cdot B(2,10;p,3) - 3^{-1}C'_4(k,j)\cdot 2,$$

$$H^{\mathrm{Lef}}_{5,\mathrm{K}(p)} = 2^{-1}3^{-2}C_5(k,j)\cdot 2 - 3^{-1}C'_5(k,j)\cdot 2,$$

$$\begin{aligned} H^{\mathrm{Lef}}_{6,\mathrm{K}(p)} = {}& 2^{-5}3^{-1}\big\{C_6(k,j) - C'_6(k,j)\big\}\cdot\big\{p\,(-1|p)+1\big\} \\ & + 2^{-2}\big\{(-1)^{j/2+k} - (-1)^{j/2}\big\}\cdot\big\{1+(-1|p)\big\}, \end{aligned}$$

$$\begin{aligned} H^{\mathrm{Lef}}_{7,\mathrm{K}(p)} = {}& 2^{-1}3^{-3}\big\{C_7(k,j) - C'_7(k,j)\big\}\cdot\big\{p\,(-3|p)+1\big\} \\ & + 3^{-1}\big\{[0,1,-1;3]_{j+2k} - [1,-1,0;3]_j\big\}\cdot\big\{1+(-3|p)\big\}, \end{aligned}$$

$$H^{\mathrm{Lef}}_{8,\mathrm{K}(p)} = 3^{-1}C_8(k,j)\cdot 2, \qquad H^{\mathrm{Lef}}_{9,\mathrm{K}(p)} = 2^2\cdot 3^{-2}C_9(k,j)\cdot B\big(2,2^{-1};p,2\big),$$

$$H^{\mathrm{Lef}}_{10,\mathrm{K}(p)} = 2^2\cdot 5^{-1}C_{10}(k,j)\cdot \begin{cases} 2 & \text{if } p\equiv \pm 1 \bmod 5, \\ 0 & \text{if } p \equiv 2,3 \bmod 5, \\ 1 & \text{if } p = 5, \end{cases}$$

$$H^{\mathrm{Lef}}_{11,\mathrm{K}(p)} = 2^{-1}C_{11}(k,j)\cdot \begin{cases} 2 & \text{if } p\equiv \pm 1 \bmod 8, \\ 0 & \text{if } p \equiv 3,5 \bmod 8, \\ 1 & \text{if } p = 2, \end{cases}$$

$$H^{\mathrm{Lef}}_{12,\mathrm{K}(p)} = 2^{-1}3^{-1}\big\{C_{12}(k,j) - C'_{12}(k,j)\big\}\cdot \begin{cases} 2 & \text{if } p\equiv 1,11 \bmod 12, \\ 0 & \text{if } p \equiv 5,7 \bmod 12, \\ 1 & \text{if } p = 2,3. \end{cases}$$

Theorem 3.3. *Let $(l_1, l_2) = (j+k-1, k-2) \in \Xi_1$. Assume that $l_1 - l_2 = j+1$ is odd. Then, we have*

$$(-1) \times \mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{Kl}(p)}) = \sum_{t=1}^{12} H^{\mathrm{Lef}}_{t,\mathrm{Kl}(p)},$$

$$\begin{aligned} H^{\mathrm{Lef}}_{1,\mathrm{Kl}(p)} = {}& 2^{-5}3^{-3}5^{-1}C_1(k,j)\cdot(p+1)\big(p^2+1\big) - 2^{-1}3^{-1}(k-2)\cdot(2p+2) \\ & - 2^{-2}3^{-1}(j+1)\cdot(p+3) + 2^{-1}\cdot 4, \end{aligned}$$

$$H^{\mathrm{Lef}}_{2,\mathrm{Kl}(p)} = 2^{-5}3^{-2}C_2(k,j)\cdot B(14p+14,33;p,2) - 2^{-2}3^{-1}C'_2(k,j)\cdot(p+3) + 2^{-1}(-1)^k\cdot 4,$$

$$H^{\mathrm{Lef}}_{3,\mathrm{Kl}(p)} = 2^{-3}3^{-1}C_3(k,j)\cdot B\big(p+2+(-1|p),7;p,2\big) - 2^{-1}C'_3(k,j)\cdot\big\{3+(-1|p)\big\},$$

$$H^{\mathrm{Lef}}_{4,\mathrm{Kl}(p)} = 2^{-1}3^{-3}C_4(k,j)\cdot B\big(p+2+(-3|p),13;p,3\big) - 3^{-1}C'_4(k,j)\cdot\big\{3+(-3|p)\big\},$$

$$H^{\mathrm{Lef}}_{5,\mathrm{Kl}(p)} = 2^{-1}3^{-2}C_5(k,j)\cdot\big\{p+2+(-3|p)\big\} - 3^{-1}C'_5(k,j)\cdot\big\{3+(-3|p)\big\},$$

$$H^{\mathrm{Lef}}_{6,\mathrm{Kl}(p)} = 2^{-5}3^{-1}\{C_6(k,j) - C'_6(k,j)\} \cdot (p+1)\{1 + (-1|p)\} + 2^{-2}\{(-1)^{j/2+k} - (-1)^{j/2}\} \cdot \{2 + 2\,(-1|p)\},$$

$$H^{\mathrm{Lef}}_{7,\mathrm{Kl}(p)} = 2^{-1}3^{-3}\{C_7(k,j) - C'_7(k,j)\} \cdot (p+1)\{1 + (-3|p)\} + 3^{-1}\{[0,1,-1;3]_{j+2k} - [1,-1,0;3]_j\} \cdot \{2 + 2\,(-3|p)\},$$

$$H^{\mathrm{Lef}}_{8,\mathrm{Kl}(p)} = 3^{-1}C_8(k,j) \cdot \{2 + (-1|p) + (-3|p)\},$$

$$H^{\mathrm{Lef}}_{9,\mathrm{Kl}(p)} = 2^2 \cdot 3^{-2}C_9(k,j) \cdot \{2 + 2\,(-3|p)\},$$

$$H^{\mathrm{Lef}}_{10,\mathrm{Kl}(p)} = 2^2 \cdot 5^{-1}C_{10}(k,j) \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 5, \\ 1 & \text{if } p = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Lef}}_{11,\mathrm{Kl}(p)} = 2^{-1}C_{11}(k,j) \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 8, \\ 1 & \text{if } p = 2, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Lef}}_{12,\mathrm{Kl}(p)} = 2^{-1}3^{-1}\{C_{12}(k,j) - C'_{12}(k,j)\} \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 12, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 3.4. *Let $(l_1, l_2) = (j+k-1, k-2) \in \Xi_1$. Assume that $l_1 - l_2 = j+1$ is odd. Then, we have*

$$(-1) \times \mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{Si}(p)}) = \sum_{t=1}^{12} H^{\mathrm{Lef}}_{t,\mathrm{Si}(p)},$$

$$H^{\mathrm{Lef}}_{1,\mathrm{Si}(p)} = 2^{-5}3^{-3}5^{-1}C_1(k,j) \cdot (p+1)(p^2+1) - 2^{-1}3^{-1}(k-2) \cdot (p+3) - 2^{-2}3^{-1}(j+1) \cdot (2p+2) + 2^{-1} \cdot 4,$$

$$H^{\mathrm{Lef}}_{2,\mathrm{Si}(p)} = 2^{-5}3^{-2}C_2(k,j) \cdot B(7(p+1)^2, 57; p, 2) - 2^{-2}3^{-1}C'_2(k,j) \cdot (2p+2) + 2^{-1}(-1)^k \cdot 4,$$

$$H^{\mathrm{Lef}}_{3,\mathrm{Si}(p)} = 2^{-3}3^{-1}C_3(k,j) \cdot (p+1)\{1 + (-1|p)\} - 2^{-1}C'_3(k,j) \cdot \{2 + 2\,(-1|p)\},$$

$$H^{\mathrm{Lef}}_{4,\mathrm{Si}(p)} = 2^{-1}3^{-3}C_4(k,j) \cdot (p+1)\{1 + (-3|p)\} - 3^{-1}C'_4(k,j) \cdot \{2 + 2\,(-3|p)\},$$

$$H^{\mathrm{Lef}}_{5,\mathrm{Si}(p)} = 2^{-1}3^{-2}C_5(k,j) \cdot (p+1)\{1 + (-3|p)\} - 3^{-1}C'_5(k,j) \cdot \{2 + 2\,(-3|p)\},$$

$$H^{\mathrm{Lef}}_{6,\mathrm{Si}(p)} = 2^{-5}3^{-1}\{C_6(k,j) - C'_6(k,j)\} \cdot B(p + 2 + (-1|p), 7; p, 2) + 2^{-2}\{(-1)^{j/2+k} - (-1)^{j/2}\} \cdot \{3 + (-1|p)\},$$

$$H^{\mathrm{Lef}}_{7,\mathrm{Si}(p)} = 2^{-1}3^{-3}\{C_7(k,j) - C'_7(k,j)\} \cdot B(p + 2 + (-3|p), 13; p, 3) + 3^{-1}\{[0,1,-1;3]_{j+2k} - [1,-1,0;3]_j\} \cdot \{3 + (-3|p)\},$$

$$H^{\mathrm{Lef}}_{8,\mathrm{Si}(p)} = 3^{-1}C_8(k,j) \cdot \{1 + (-1|p)\}\{1 + (-3|p)\},$$

$$H^{\mathrm{Lef}}_{9,\mathrm{Si}(p)} = 2^2 \cdot 3^{-2} C_9(k,j) \cdot B\big(\{1+(-3|p)\}^2, 3/2; p, 2\big),$$

$$H^{\mathrm{Lef}}_{10,\mathrm{Si}(p)} = 2^2 \cdot 5^{-1} C_{10}(k,j) \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 5, \\ 1 & \text{if } p = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Lef}}_{11,\mathrm{Si}(p)} = 2^{-1} C_{11}(k,j) \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 8, \\ 2 & \text{if } p \equiv 3, 5 \bmod 8, \\ 0 & \text{if } p \equiv 7 \bmod 8, \\ 1 & \text{if } p = 2, \end{cases}$$

$$H^{\mathrm{Lef}}_{12,\mathrm{Si}(p)} = 2^{-1} 3^{-1} \{C_{12}(k,j) - C'_{12}(k,j)\} \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 12, \\ 2 & \text{if } p \equiv 5, 7 \bmod 12, \\ 0 & \text{if } p \equiv 11 \bmod 12, \\ 1 & \text{if } p = 2, 3. \end{cases}$$

Theorem 3.5. *Let $(l_1, l_2) = (j+k-1, k-2) \in \Xi_1$. Assume that $l_1 - l_2 = j+1$ is odd. Then, we have*

$$(-1) \times \mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{I}(p)}) = \sum_{t=1}^{12} H^{\mathrm{Lef}}_{t,\mathrm{I}(p)},$$

$$H^{\mathrm{Lef}}_{1,\mathrm{I}(p)} = 2^{-5} 3^{-3} 5^{-1} C_1(k,j) \cdot (p+1)^2 \big(p^2+1\big) - 2^{-1} 3^{-1} (k-2) \cdot (4p+4) - 2^{-2} 3^{-1} (j+1) \cdot (4p+4) + 2^{-1} \cdot 8,$$

$$H^{\mathrm{Lef}}_{2,\mathrm{I}(p)} = 2^{-5} 3^{-2} C_2(k,j) \cdot (p+1)^2 B(14, 11; p, 2) - 2^{-2} 3^{-1} C'_2(k,j) \cdot (4p+4) + 2^{-1} (-1)^k \cdot 8,$$

$$H^{\mathrm{Lef}}_{3,\mathrm{I}(p)} = 2^{-3} 3^{-1} C_3(k,j) \cdot B\big(2(p+1)\{1+(-1|p)\}, 9; p, 2\big) - 2^{-1} C'_3(k,j) \cdot \{4+4(-1|p)\},$$

$$H^{\mathrm{Lef}}_{4,\mathrm{I}(p)} = 2^{-1} 3^{-3} C_4(k,j) \cdot B\big(2(p+1)\{1+(-3|p)\}, 16; p, 3\big) - 3^{-1} C'_4(k,j) \cdot \{4+4(-3|p)\},$$

$$H^{\mathrm{Lef}}_{5,\mathrm{I}(p)} = 2^{-1} 3^{-2} C_5(k,j) \cdot 2(p+1)\{1+(-3|p)\} - 3^{-1} C'_5(k,j) \cdot \{4+4(-3|p)\},$$

$$H^{\mathrm{Lef}}_{6,\mathrm{I}(p)} = 2^{-5} 3^{-1} \{C_6(k,j) - C'_6(k,j)\} \cdot B\big(2(p+1)\{1+(-1|p)\}, 9; p, 2\big) + 2^{-2} \{(-1)^{j/2+k} - (-1)^{j/2}\} \cdot \{4+4(-1|p)\},$$

$$H^{\mathrm{Lef}}_{7,\mathrm{I}(p)} = 2^{-1} 3^{-3} \{C_7(k,j) - C'_7(k,j)\} \cdot B\big(2(p+1)\{1+(-3|p)\}, 16; p, 3\big) + 3^{-1} \{[0,1,-1;3]_{j+2k} - [1,-1,0;3]_j\} \cdot \{4+4(-3|p)\},$$

$$H^{\mathrm{Lef}}_{8,\mathrm{I}(p)} = 3^{-1} C_8(k,j) \cdot 2\{1+(-1|p)\}\{1+(-3|p)\},$$

$$H^{\mathrm{Lef}}_{9,\mathrm{I}(p)} = 2^2 \cdot 3^{-2} C_9(k,j) \cdot B\big(4+4(-3|p), 2; p, 3\big),$$

$$H^{\mathrm{Lef}}_{10,\mathrm{I}(p)} = 2^2 \cdot 5^{-1} C_{10}(k,j) \cdot \begin{cases} 8 & \text{if } p \equiv 1 \bmod 5, \\ 1 & \text{if } p = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Lef}}_{11,\mathrm{I}(p)} = 2^{-1} C_{11}(k,j) \cdot \begin{cases} 8 & \text{if } p \equiv 1 \bmod 8, \\ 1 & \text{if } p = 2, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Lef}}_{12,\mathrm{I}(p)} = 2^{-1} 3^{-1} \{C_{12}(k,j) - C'_{12}(k,j)\} \cdot \begin{cases} 8 & \text{if } p \equiv 1 \bmod 12, \\ 0 & \text{otherwise}. \end{cases}$$

Numerical examples of $(-1) \times \mathscr{L}_{\mu(j+k-1,k-2)}(h_{\mathrm{Sp}(2,\mathbb{Z})})$.

$j \diagdown k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	−4	0	0	0	−2	0	−2	2	−2	2	−4	2	−4	4	−4	4	−6
2	0	0	0	0	0	0	0	0	0	0	0	4	0	8	0	8	0
4	0	0	0	0	0	0	0	4	0	4	0	8	4	12	4	16	8
6	0	0	0	0	0	4	0	4	4	8	4	12	8	20	12	28	16
8	0	0	0	0	0	4	4	8	4	12	8	20	16	28	20	36	28
10	0	2	2	2	2	4	6	12	8	16	12	26	24	38	30	50	42

Numerical examples of $(-1) \times \mathscr{L}_{\mu(j+k-1,k-2)}(h_{\mathrm{K}(2)})$ $(p=2)$.

$j \diagdown k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	−4	0	−2	−2	−2	0	−4	0	−2	2	−6	0	−4	8	−6	6	−2
2	0	0	0	0	0	0	0	4	4	4	4	16	16	24	16	36	28
4	0	0	0	0	4	4	4	12	12	20	16	32	36	52	44	68	72
6	0	2	0	2	6	10	10	20	26	32	32	56	60	82	80	118	118

Numerical examples of $(-1) \times \mathscr{L}_{\mu(j+k-1,k-2)}(h_{\mathrm{K}(3)})$ $(p=3)$.

$j \diagdown k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	−4	−2	−2	0	−4	−2	−2	0	−4	6	−6	4	0	10	−2	24
2	0	0	0	0	0	4	4	8	8	16	16	32	32	52	44	72
4	0	0	2	4	2	10	14	22	24	42	36	68	72	96	102	144
6	0	2	4	10	10	22	30	40	50	76	76	112	128	166	176	242

Numerical examples of $(-1) \times \mathscr{L}_{\mu(2,1)}(h_{\mathrm{K}(p)})$ $((l_1, l_2) = (2,1),\ (k, j) = (3,0))$.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
Value	−4	−4	−6	−6	−8	−6	−8	−8	−10	−10	−10	−6	−12	−8	−14

Numerical examples of $(-1) \times \mathscr{L}_{\mu(2,1)}(h_{\mathrm{Kl}(p)})$ $((l_1, l_2) = (2,1),\ (k, j) = (3,0))$.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
Value	−4	−4	−6	−6	−4	−6	0	4	14	30	34	66	88	108	142	206

Numerical examples of $(-1) \times \mathscr{L}_{\mu(2,1)}(h_{\mathrm{Si}(p)})$ $((l_1, l_2) = (2,1),\ (k, j) = (3,0))$.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
Value	−4	−4	−6	−6	−8	−10	−8	−8	−6	−2	−2	14	20	32	46	86

Numerical examples of $(-1) \times \mathscr{L}_{\mu(2,1)}(h_{\mathrm{I}(p)})$ $((l_1, l_2) = (2,1),\ (k, j) = (3,0))$.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43
Value	−4	−4	−8	−8	4	20	96	164	368	972	1272	2628	3952	4804

3.4. *Proof of the theorems*

In this subsection, we explain our proof. First, we review Arthur's L^2-Lefschetz trace formula for $G(\mathbb{Q}) = \mathrm{Sp}(2,\mathbb{Q})$ (cf. Arthur [3] and Goresky, Kottwitz and MacPherson [9]). Next, we will state an explicit calculation for it.

Let μ be a finite-dimensional irreducible rational representation of G and let h be a locally constant, compactly supported, complex-valued function on $G(\mathbb{A}_{\mathrm{fin}})$. We denote by $\mathscr{L}_\mu(h)$ the L^2-Lefschetz number defined in [3, §1]. The formula for $G(\mathbb{Q})$ is

$$(-1) \times \mathscr{L}_\mu(h) = \sum_M (-1)^{\dim(A_M/A_G)} \left|W_M^G\right|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))} \left|\iota^M(\gamma)\right|^{-1} \chi(M_\gamma) \Phi_M(\gamma, \mu) h_M(\gamma), \tag{3.1}$$

where M runs over the Levi subgroups

$$M_0=\{\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})\mid a_1,a_2\in \mathrm{GL}(1)\},$$
$$M_1=\left\{\begin{pmatrix} A & O_2\\ O_2 & {}^tA^{-1}\end{pmatrix}\;\middle|\; A\in \mathrm{GL}(2)\right\},$$
$$M_2=\left\{\begin{pmatrix} a&0&b&0\\ 0&\alpha&0&0\\ c&0&d&0\\ 0&0&0&\alpha^{-1}\end{pmatrix}\;\middle|\;\begin{pmatrix} a&b\\ c&d\end{pmatrix}\in \mathrm{SL}(2),\ \alpha\in \mathrm{GL}(1)\right\},$$

and G. We have

$$A_{M_0}=M_0,\qquad A_{M_1}=\{\mathrm{diag}(a,a,a^{-1},a^{-1})\mid a\in \mathrm{GL}(1)\},$$
$$A_{M_2}=\{\mathrm{diag}(1,a,1,a^{-1})\mid a\in \mathrm{GL}(1)\},$$
$$\dim(A_{M_0}/A_G)=2,\qquad \dim(A_G/A_G)=0,\qquad \dim(A_{M_1}/A_G)=\dim(A_{M_2}/A_G)=1,$$
$$|W^G_{M_0}|=8,\qquad |W^G_{M_1}|=|W^G_{M_2}|=2,\qquad |W^G_G|=1.$$

The notation $(M(\mathbb{Q}))$ means the set of $M(\mathbb{Q})$-conjugacy classes in $M(\mathbb{Q})$. We may identify an element $\gamma\in M(\mathbb{Q})$ with the $M(\mathbb{Q})$-conjugacy class of γ in $M(\mathbb{Q})$, since every function in the formula is $M(\mathbb{Q})$-invariant. Let T_M be a maximal torus in M over $\mathbb{R}$ such that $T_M(\mathbb{R})/A_M(\mathbb{R})$ is compact. We call $\gamma\in M(\mathbb{R})$ an $\mathbb{R}$-elliptic element if γ is $M(\mathbb{R})$-conjugate to an element in $T_M(\mathbb{R})$. The function $\Phi_M(\gamma,\mu)$ vanishes unless γ is an $\mathbb{R}$-elliptic element (cf. [3, Remarks 1 in p. 261]). Hence, $\gamma\in(M(\mathbb{Q}))$ runs over only $\mathbb{R}$-elliptic conjugacy classes. We will later explain the functions $\Phi_M(\gamma,\mu)$ and $h_M(\gamma)$ in detail. Let $|\iota^M(\gamma)|$ denote the number of connected components in the centralizer of γ in M which contain rational points. We also denote by M_γ the connected component of 1 in the centralizer of γ in M. If we put $H=M_\gamma$, $\chi(H)$ is defined by

$$\chi(H)=(-1)^{q(H)}\,\mathrm{vol}\big(H(\mathbb{Q})A_H(\mathbb{R})^0\backslash H(\mathbb{A})\big)\,\mathrm{vol}\big(A_H(\mathbb{R})^0\backslash \bar{H}(\mathbb{A})\big)^{-1}\big|\mathscr{D}(H,B)\big|,$$

where $q(H)=\frac{1}{2}\dim(H(\mathbb{R})/(K_\infty\cap H(\mathbb{R}))A_H(\mathbb{R})^0)$, B is a maximal torus such that $B(\mathbb{R})$ is contained in $(K_\infty\cap H(\mathbb{R}))A_H(\mathbb{R})^0$, and $\mathscr{D}(H,B)=W(G(\mathbb{R}),B(\mathbb{R}))\backslash W(G,B)$ ($W(G,B)$ is the Weyl group of G on B and $W(G(\mathbb{R}),B(\mathbb{R}))$ is the subgroup of elements induced from $G(\mathbb{R})$). As for $\mathrm{vol}(A_H(\mathbb{R})^0\backslash\bar{H}(\mathbb{A}))$, we refer to [3, p. 265 and p. 276].

For the definition and properties of $\Phi_M(\gamma,\mu)$, we refer to [3, §4 and §5]. Let $\gamma\in M(\mathbb{R})$. If γ is regular and $\mathbb{R}$-elliptic, then we have

$$\Phi_M(\gamma,\mu)=(-1)^{q(G)}\big|D^G_M(\gamma)\big|^{1/2}\big\{\Theta_{l_1,l_2}(\gamma)+\Theta_{l_1,-l_2}(\gamma)+\Theta_{l_2,-l_1}(\gamma)+\Theta_{-l_2,-l_1}(\gamma)\big\}$$

where $(l_1,l_2)\in\Xi_1$, $D^G_M(\gamma)=\det((1-\mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{m}})$ ($\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{m}=\mathrm{Lie}(M)$), and Θ_{m_1,m_2} is the character of $D(m_1,m_2)$. Character formulas for discrete series are well known (see, e.g., Hirai [18] and Herb [15]). For singular $\mathbb{R}$-elliptic elements γ, the function $\Phi_M(\gamma,\mu)$ was studied by Spallone [35]. Hence, we can easily get explicit forms of $\Phi_M(\gamma,\mu)$ for each such γ.

Let $\gamma\in M(\mathbb{Q})$ and let v be a finite place of $\mathbb{Q}$. Let dk_v denote the Haar measure on $\mathrm{Sp}(2,\mathbb{Z}_v)$ normalized by $\int_{\mathrm{Sp}(2,\mathbb{Z}_v)}dk_v=1$. We set

$$N_0=\left\{\begin{pmatrix}1&0&0&0\\ *&1&0&0\\ *&*&1&*\\ *&*&0&1\end{pmatrix}\in G\right\},\qquad N_1=\left\{\begin{pmatrix}1&0&0&0\\ 0&1&0&0\\ *&*&1&0\\ *&*&0&1\end{pmatrix}\in G\right\},$$

$$N_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & * \\ * & 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

and $P_t = M_t N_t$ ($t = 0, 1, 2$). When $M = M_t$ ($t = 0, 1, 2$), we set $P = P_t$ and $N_P = N_t$. If $M = G$, then we put $P = G$ and $N_P = \{I_4\}$. Let $\mathrm{d}m_{1,v}$ (resp. $\mathrm{d}n_v$) denote the Haar measure on $M_\gamma(\mathbb{Q}_v)$ (resp. $N_P(\mathbb{Q}_v)$) normalized by $\int_{M_\gamma(\mathbb{Q}_v)\cap \mathrm{Sp}(2,\mathbb{Z}_v)} \mathrm{d}m_{1,v} = 1$ (resp. $\int_{N_P(\mathbb{Q}_v)\cap \mathrm{Sp}(2,\mathbb{Z}_v)} \mathrm{d}n_v = 1$). The measure $\mathrm{d}m_v$ on $M_\gamma(\mathbb{Q}_v)\backslash M(\mathbb{Q}_v)$ is induced from $\mathrm{d}m_{1,v}$ and the Haar measure on $M(\mathbb{Q}_v)$ normalized by $\mathrm{vol}(M(\mathbb{Q}_v)\cap \mathrm{Sp}(2,\mathbb{Z}_v)) = 1$. Then, $\mathrm{d}m = \prod_{v<\infty} \mathrm{d}m_v$ (resp. $\mathrm{d}n = \prod_{v<\infty} \mathrm{d}n_v$) is a Haar measure on $M_\gamma(\mathbb{A}_{\mathrm{fin}})\backslash M(\mathbb{A}_{\mathrm{fin}})$ (resp. $N_P(\mathbb{A}_{\mathrm{fin}})$). We set

$$K_{\mathrm{fin}} = \prod_{v<\infty} \mathrm{Sp}(2, \mathbb{Z}_v).$$

A Haar measure $\mathrm{d}k$ on K_{fin} is defined by $\mathrm{d}k = \prod_{v<\infty} \mathrm{d}k_v$. $h_M(\gamma)$ is defined by

$$h_M(\gamma) = \delta_P(\gamma)^{1/2} \int_{K_{\mathrm{fin}}} \int_{N_P(\mathbb{A}_{\mathrm{fin}})} \int_{M_\gamma(\mathbb{A}_{\mathrm{fin}})\backslash M(\mathbb{A}_{\mathrm{fin}})} h\big(k^{-1} m^{-1} \gamma m n k\big)\, \mathrm{d}m\, \mathrm{d}n\, \mathrm{d}k,$$

where $\delta_P(\gamma)$ is the modular function of $P(\mathbb{A}_{\mathrm{fin}})$. If $h = h_\Gamma$, then we have $\delta_P(\gamma) = 1$ for any γ satisfying $\Phi_M(\gamma, \mu) h_M(\gamma) \neq 0$. By the above-mentioned explanation, we see that it is sufficient to calculate explicitly $h_M(\gamma)$ in order to obtain the theorems.

From now on, we assume that

$$\mu = \mu(l_1, l_2) \quad \text{and} \quad h = h_\Gamma$$

(cf. Section 3.2). We start to explain our calculations for Arthur's formula. We set

$$\mathcal{T}_M(\gamma) = (-1)^{\dim(A_M/A_G)} \big|W_M^G\big|^{-1} \big|\iota^M(\gamma)\big|^{-1} \chi(M_\gamma) \Phi_M(\gamma, \mu) h_M(\gamma)$$

for $\gamma \in (M(\mathbb{Q}))$. Then, the right hand side of (3.1) is $\sum_M \sum_{\gamma \in (M(\mathbb{Q}))} \mathcal{T}_M(\gamma)$. We will give explicit forms of $\mathcal{T}_M(\gamma)$ for $M \neq G$. For an arithmetic subgroup D of $M(\mathbb{Q})$ and an element δ of $M(\mathbb{Q})$, we denote by D_δ the centralizer of δ in D and $\{\delta\}_D$ the D-conjugacy class of δ. Note that the characteristic polynomials of torsion elements of $\mathrm{SL}(2, \mathbb{Q})$ are $(x-1)^2$, $(x+1)^2$, x^2+1, x^2+x+1, and x^2-x+1.

Lemma 3.6. *Let M be one of M_0, M_1, and M_2. The $M(\mathbb{Q})$-conjugacy classes γ satisfying $\Phi_M(\gamma, \mu) h_M(\gamma) \neq 0$ are classified as below. In the following, the double sign of $\mathcal{T}_M(\gamma)$ corresponds to that of γ.*

(I) $M = M_0 \cong \mathrm{GL}(1) \times \mathrm{GL}(1)$.

(I-1) *Let $\gamma = \pm(1, 1)$. We have*

$$\mathcal{T}_M(\gamma) = 2^{-2} \times h_M(\gamma) \times (\pm 1)^{l_1 - l_2 - 1},$$

where $h_M(\gamma) = 1$ if $\Gamma = \mathrm{Sp}(2, \mathbb{Z})$, $h_M(\gamma) = 2$ if $\Gamma = \mathrm{K}(p)$, $h_M(\gamma) = 4$ if $\Gamma = \mathrm{Kl}(p)$, $h_M(\gamma) = 4$ if $\Gamma = \mathrm{Si}(p)$, $h_M(\gamma) = 8$ if $\Gamma = \mathrm{I}(p)$.

(I-2) *Let* $\gamma=\pm(1,-1)$*. We have*

$$\mathcal{T}_M(\gamma)=2^{-3}\times h_M(\gamma)\times\big\{(-1)^{l_2}-(-1)^{l_1}\big\},$$

where $h_M(\gamma)=1$ *if* $\Gamma=\mathrm{Sp}(2,\mathbb{Z})$, $h_M(\gamma)=2$ *if* $\Gamma=\mathrm{K}(p)$, $h_M(\gamma)=4$ *if* $\Gamma=\mathrm{Kl}(p)$, $h_M(\gamma)=4$ *if* $\Gamma=\mathrm{Si}(p)$, $h_M(\gamma)=8$ *if* $\Gamma=\mathrm{I}(p)$.

(II) $M=M_1\cong\mathrm{GL}(2)$.

(II-1) *Let* $\gamma=\pm I_2$*. We have*

$$\mathcal{T}_M(\gamma)=-2^{-2}3^{-1}\times h_M(\gamma)\times l_2\times(\pm1)^{l_1-l_2-1},$$

where $h_M(\gamma)=1$ *if* $\Gamma=\mathrm{Sp}(2,\mathbb{Z})$, $h_M(\gamma)=p+1$ *if* $\Gamma=\mathrm{K}(p)$, $h_M(\gamma)=2p+2$ *if* $\Gamma=\mathrm{Kl}(p)$, $h_M(\gamma)=p+3$ *if* $\Gamma=\mathrm{Si}(p)$, $h_M(\gamma)=4p+4$ *if* $\Gamma=\mathrm{I}(p)$.

(II-2) *Let* γ *be an* $M(\mathbb{Q})$*-conjugacy class whose characteristic polynomial is* x^2+1, x^2+x+1, *or* x^2-x+1*. We set* $\theta=\frac{\pi}{2}$, $\frac{2\pi}{3}$, *or* $\frac{\pi}{3}$ *respectively. Hence,* $e^{i\theta}$ *is a root of the characteristic polynomial of* γ*. Then, we obtain*

$$\mathcal{T}_M(\gamma)=\frac{e^{(l_1+l_2)i\theta}-e^{-(l_1+l_2)i\theta}-e^{(l_1-l_2)i\theta}+e^{-(l_1-l_2)i\theta}}{e^{i\theta}-e^{-i\theta}}\times\begin{cases}\sum_{\{\delta\}_D\subset D\cap\gamma}|D_\delta|^{-1} & \text{if } \Gamma=\mathrm{Sp}(2,\mathbb{Z}),\\ \sum_{\{\delta\}_{D'}\subset D'\cap\gamma}|D'_\delta|^{-1} & \text{if } \Gamma=\mathrm{K}(p),\\ 2\times\sum_{\{\delta\}_{D'}\subset D'\cap\gamma}|D'_\delta|^{-1} & \text{if } \Gamma=\mathrm{Kl}(p),\\ 2\times\sum_{\{\delta\}_D\subset D\cap\gamma}|D_\delta|^{-1}+\sum_{\{\delta\}_{D'}\subset D'\cap\gamma}|D'_\delta|^{-1} & \text{if } \Gamma=\mathrm{Si}(p),\\ 4\times\sum_{\{\delta\}_{D'}\subset D'\cap\gamma}|D'_\delta|^{-1} & \text{if } \Gamma=\mathrm{I}(p),\end{cases}$$

where

$$D=\mathrm{GL}(2,\mathbb{Z})\quad\text{and}\quad D'=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}(2,\mathbb{Z})\;\middle|\;c\equiv0\bmod p\right\}.$$

(III) $M=M_2\cong\mathrm{SL}(2)\times\mathrm{GL}(1)$.

(III-1) *Let* $\gamma=\pm(I_2,1)$*. We have*

$$\mathcal{T}_M(\gamma)=-2^{-3}3^{-1}\times h_M(\gamma)\times(l_1-l_2)\times(\pm1)^{l_1-l_2-1}$$

where $h_M(\gamma)=1$ *if* $\Gamma=\mathrm{Sp}(2,\mathbb{Z})$, $h_M(\gamma)=2$ *if* $\Gamma=\mathrm{K}(p)$, $h_M(\gamma)=p+3$ *if* $\Gamma=\mathrm{Kl}(p)$, $h_M(\gamma)=2p+2$ *if* $\Gamma=\mathrm{Si}(p)$, $h_M(\gamma)=4p+4$ *if* $\Gamma=\mathrm{I}(p)$.

(III-2) *Let* $\gamma=\pm(I_2,-1)$*. We have*

$$\mathcal{T}_M(\gamma)=-2^{-3}3^{-1}\times h_M(\gamma)\times\big\{l_1(-1)^{l_1}-l_2(-1)^{l_2}\big\}\times(\pm1)^{l_1-l_2-1},$$

where $h_M(\gamma)=1$ *if* $\Gamma=\mathrm{Sp}(2,\mathbb{Z})$, $h_M(\gamma)=2$ *if* $\Gamma=\mathrm{K}(p)$, $h_M(\gamma)=p+3$ *if* $\Gamma=\mathrm{Kl}(p)$, $h_M(\gamma)=2p+2$ *if* $\Gamma=\mathrm{Si}(p)$, $h_M(\gamma)=4p+4$ *if* $\Gamma=\mathrm{I}(p)$.

(III-3) *Let* γ *be an* $M(\mathbb{Q})$*-conjugacy class whose characteristic polynomial is* $(x^2+1)(x\mp1)$, $(x^2\pm x+1)(x\mp1)$, *or* $(x^2\mp x+1)(x\mp1)$*. For all* $M(\mathbb{Q})$*-conjugacy classes, we take representative elements* $\gamma=\pm\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),1\right)$, $\pm\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),1\right)$, $\pm\left(\left(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix}\right),1\right)$, $\pm\left(\left(\begin{smallmatrix}-1&1\\-1&0\end{smallmatrix}\right),1\right)$, $\pm\left(\left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right),1\right)$, *and* $\pm\left(\left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right),1\right)$*. We set* $\theta=\frac{\pi}{2}$, $-\frac{\pi}{2}$, $\frac{2\pi}{3}$, $-\frac{2\pi}{3}$, $\frac{\pi}{3}$, *or* $-\frac{\pi}{3}$ *respectively.*

$$\mathcal{T}_M(\gamma)=\frac{e^{il_1\theta}-e^{-il_1\theta}-e^{il_2\theta}+e^{-il_2\theta}}{e^{i\theta}-e^{-i\theta}}\times(\pm 1)^{l_1-l_2-1}$$
$$\times\begin{cases}\sum_{\{\delta\}_{D_1}\subset D_1\cap\gamma}|(D_1)_\delta|^{-1} & \text{if } \Gamma=\mathrm{Sp}(2,\mathbb{Z}),\\ 2\times\sum_{\{\delta\}_{D_1}\subset D_1\cap\gamma}|(D_1)_\delta|^{-1} & \text{if } \Gamma=\mathrm{K}(p),\\ 2\times\sum_{\{\delta\}_{D_1}\subset D_1\cap\gamma}|(D_1)_\delta|^{-1}+\sum_{\{\delta\}_{D_2}\subset D_2\cap\gamma}|(D_2)_\delta|^{-1} & \text{if } \Gamma=\mathrm{Kl}(p),\\ 2\times\sum_{\{\delta\}_{D_2}\subset D_2\cap\gamma}|(D_2)_\delta|^{-1} & \text{if } \Gamma=\mathrm{Si}(p),\\ 4\times\sum_{\{\delta\}_{D_2}\subset D_2\cap\gamma}|(D_2)_\delta|^{-1} & \text{if } \Gamma=\mathrm{I}(p),\end{cases}$$

where

$$D_1=\mathrm{SL}(2,\mathbb{Z})\times\{\pm 1\}\quad \textit{and}\quad D_2=\left\{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\alpha\right)\in D_1\ \middle|\ c\equiv 0 \bmod p\right\}.$$

Proof. It is sufficient to explain how to calculate $h_M(\gamma)$. For $\mathrm{K}(p)$, $\mathrm{Kl}(p)$, $\mathrm{Si}(p)$ and $\mathrm{I}(p)$, the calculations for $h_M(\gamma)$ are not trivial. Since our calculations for all cases are similar, we give a proof for only $\mathrm{K}(p)$.

We set

$$w_1=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\qquad w_2=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix},\qquad w_3=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&-1&0&0\end{pmatrix},$$

and $K_1=K_0\cap K_{\mathrm{fin}}$. If we put

$$x_{a,b,c}=\begin{pmatrix}1&0&0&0\\a&1&0&0\\b&c&1&0\\c&0&0&1\end{pmatrix},$$

then a set of representative elements of K_{fin}/K_1 is

$$R(K_{\mathrm{fin}}/K_1)=\{x_{a,b,c},\ w_1x_{0,b,c},\ w_1w_2w_3x_{a,0,0},\ w_2\mid 0\leqslant a,b,c\leqslant p-1\}.$$

We can easily show this result by a simple calculation.

First, we treat the cases (I-1) and (I-2). Let $\gamma=\pm(1,1)$ or $\pm(1,-1)$. Then, we have

$$h_M(\gamma)=\frac{1}{p+1}\sum_{g\in R(K_{\mathrm{fin}}/K_1)}h'_M(\gamma,g),\qquad h'_M(\gamma,g)=\int_{N_P(\mathbb{Q}_p)}h_{\mathbf{K}_p^{\mathrm{par}}}\big(g^{-1}\gamma ng\big)\,dn_v,$$

where $h_{\mathbf{K}_p^{\mathrm{par}}}$ is the characteristic function of $\mathbf{K}_p^{\mathrm{par}}$. By direct calculation we have

$$h'_M(\gamma,g)=\begin{cases}p^{-3} & \text{if } g=x_{a,b,c},\\ p^{-2} & \text{if } g=w_1x_{0,b,c},\\ 1 & \text{if } g=w_1w_2w_3x_{a,0,0},\\ p & \text{if } g=w_2.\end{cases}$$

Hence, we have proved the results of (I-1) and (I-2). As for (II-1), (III-1), (III-2), we can compute them by the same argument as (I-1) and (I-2). Hence, we omit the proofs for the cases (II-1), (III-1) and (III-2).

Next, we treat the case (II-2). We assume that γ is an $\mathbb{R}$-elliptic semi-simple element of $M(\mathbb{Q})$ such that $\det(\gamma)=1$ and $\gamma \neq \pm I_2$. Then, we have

$$h_M(\gamma)=\frac{1}{p+1}\sum_{g\in R(K_{\mathrm{fin}}/K_1)} h''_M(\gamma,g),$$

$$h''_M(\gamma,g)=\int\limits_{N_P(\mathbb{Q}_p)}\int\limits_{M_\gamma(\mathbb{Q}_p)\backslash M(\mathbb{Q}_p)} h_{\mathbf{K}_p^{\mathrm{par}}}\bigl(g^{-1}m^{-1}\gamma mng\bigr)\,\mathrm{d}m\,\mathrm{d}n.$$

By direct calculation we obtain

$$h''_M(\gamma,g)=\int\limits_{M_\gamma(\mathbb{Q}_p)\backslash M(\mathbb{Q}_p)} h_{K_{0,p}}\bigl(m^{-1}\gamma m\bigr)\,\mathrm{d}m\times\begin{cases} p^{-2} & \text{if } g=x_{a,b,c},\\ p^{-2} & \text{if } g=w_1x_{0,b,c},\\ p & \text{if } g=w_1w_2w_3x_{a,0,0},\\ p & \text{if } g=w_2,\end{cases}$$

where $h_{K_{0,p}}$ is the characteristic function of

$$K_{0,p}=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z}_p)\ \middle|\ c\in p\mathbb{Z}_p\right\}.$$

Therefore, the result of (II-2) follows. We can also prove the result of (III-3) by the same argument as (II-2). Hence, we omit the proof for (III-3). □

We can easily compute numerical values of $\mathcal{T}_M(\gamma)$ by this lemma, because data of conjugacy classes are well known (cf. [24]). Besides, we can easily reduce an explicit calculation for the term of $M=G$ to those of [12–14,20,21,41], since $h_M(\gamma)$ is nothing but an orbital integral in this case. Thus, we have completed the proofs of Theorems 3.1, 3.2, 3.3, 3.4, and 3.5.

4. Large discrete series

In this section, we give our main results which are explicit multiplicity formulas for large discrete series.

If we assume $l_2>1$, it follows from the definition of $\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)$ that

$$m\bigl(D^{\mathrm{Large}}_{l_1,l_2},\Gamma\bigr)=\frac{(-1)}{2}\mathscr{L}_{\mu(l_1,l_2)}(h_\Gamma)-m\bigl(D^{\mathrm{Hol}}_{l_1,l_2},\Gamma\bigr)+\begin{cases}0 & \text{if } l_1-l_2>0,\\ m(\sigma^-_{l_2+2},\Gamma) & \text{if } l_1-l_2=0.\end{cases}$$

Recall that $m(D^{\mathrm{Hol}}_{l_1,l_2},\Gamma)=\dim_{\mathbb{C}} S_{k,j}(\Gamma)$, where $(l_1,l_2)=(j+k-1,k-2)$. Dimension formulas for $\mathrm{Sp}(2,\mathbb{Z})$, $\mathrm{K}(p)$, and $\mathrm{Si}(p)$ are known if $k\geqslant 5$ (cf. [12,20–22,37,38,41]). We will explain dimension formulas for $\mathrm{Kl}(p)$ and $\mathrm{I}(p)$ in Appendix A. Thus, multiplicity formulas for $m(D^{\mathrm{Large}}_{l_1,l_2},\Gamma)$ are derived from the theorems of Section 3.3 and the dimension formulas if $k\geqslant 5$ and $j>0$.

We assume that $l_1-l_2=j+1$ is odd, since $m(D^{\mathrm{Large}}_{l_1,l_2},\Gamma)=0$ if l_1-l_2 is even. In the following theorems, $H^{\mathrm{Large}}_{t,\Gamma}$ means the total contribution of elements of Γ with the characteristic polynomial $f_t(\pm x)$ to

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\Gamma\big)-\begin{cases} m(\sigma_k^-,\Gamma) & \text{if } j=0,\\ 0 & \text{if } j>0.\end{cases}$$

The multiplicity formulas are as follows.

Theorem 4.1. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd and $l_2=k-2>2$. Then, we have*

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\mathrm{Sp}(2,\mathbb{Z})\big)=\sum_{t=1}^{12}H^{\mathrm{Large}}_{t,\mathrm{Sp}(2,\mathbb{Z})}+\begin{cases} m(\sigma_k^-,\mathrm{Sp}(2,\mathbb{Z})) & \textit{if } j=0,\\ 0 & \textit{if } j>0,\end{cases}$$

$$H^{\mathrm{Large}}_{1,\mathrm{Sp}(2,\mathbb{Z})}=2^{-7}3^{-3}5^{-1}C_1(k,j)+2^{-5}3^{-2}C_1'(k,j)$$
$$-2^{-3}3^{-1}(j+2k-3)-2^{-4}3^{-1}(j+1)+2^{-2},$$

$$H^{\mathrm{Large}}_{2,\mathrm{Sp}(2,\mathbb{Z})}=2^{-7}3^{-2}C_2(k,j)\cdot 7-2^{-4}3^{-1}C_2'(k,j)+2^{-5}(-1)^k\cdot 5,$$

$$H^{\mathrm{Large}}_{3,\mathrm{Sp}(2,\mathbb{Z})}=2^{-5}3^{-1}C_3(k,j)-2^{-3}C_3'(k,j),$$

$$H^{\mathrm{Large}}_{4,\mathrm{Sp}(2,\mathbb{Z})}=2^{-3}3^{-3}C_4(k,j)-2^{-2}3^{-2}C_4'(k,j)\cdot 5+3^{-2}C_4''(k,j),$$

$$H^{\mathrm{Large}}_{5,\mathrm{Sp}(2,\mathbb{Z})}=2^{-3}3^{-2}C_5(k,j)-2^{-2}3^{-1}C_5'(k,j),$$

$$H^{\mathrm{Large}}_{6,\mathrm{Sp}(2,\mathbb{Z})}=-2^{-7}3^{-1}C_6(k,j)\cdot 3-2^{-7}3^{-1}C_6'(k,j)\cdot 5+2^{-3}(-1)^{j/2+k},$$

$$H^{\mathrm{Large}}_{7,\mathrm{Sp}(2,\mathbb{Z})}=-2^{-2}3^{-3}C_7(k,j)-2^{-2}3^{-3}C_7'(k,j)\cdot 2+2^{-1}3^{-1}[0,1,-1;3]_{j+2k},$$

$$H^{\mathrm{Large}}_{8,\mathrm{Sp}(2,\mathbb{Z})}=2^{-2}3^{-1}C_8(k,j),\qquad H^{\mathrm{Large}}_{9,\mathrm{Sp}(2,\mathbb{Z})}=3^{-2}C_9(k,j),\qquad H^{\mathrm{Large}}_{10,\mathrm{Sp}(2,\mathbb{Z})}=5^{-1}C_{10}(k,j),$$

$$H^{\mathrm{Large}}_{11,\mathrm{Sp}(2,\mathbb{Z})}=2^{-3}C_{11}(k,j),\qquad H^{\mathrm{Large}}_{12,\mathrm{Sp}(2,\mathbb{Z})}=2^{-2}3^{-1}C_{12}(k,j).$$

Theorem 4.2. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd and $l_2=k-2>2$. Then, we have*

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\mathrm{K}(p)\big)=\sum_{t=1}^{12}H^{\mathrm{Large}}_{t,\mathrm{K}(p)}+\begin{cases} m(\sigma_k^-,\mathrm{K}(p)) & \textit{if } j=0,\\ 0 & \textit{if } j>0,\end{cases}$$

$$H^{\mathrm{Large}}_{1,\mathrm{K}(p)}=2^{-7}3^{-3}5^{-1}C_1(k,j)\cdot\big(p^2+1\big)+2^{-5}3^{-2}C_1'(k,j)\cdot 2p$$
$$-2^{-3}3^{-1}(j+2k-3)\cdot(p+1)-2^{-4}3^{-1}(j+1)\cdot 2+2^{-2}\cdot 2,$$

$$H^{\mathrm{Large}}_{2,\mathrm{K}(p)}=2^{-7}3^{-2}C_2(k,j)\cdot B(14,11;p,2)-2^{-4}3^{-1}C_2'(k,j)\cdot 2+2^{-5}(-1)^k\cdot\big\{8+2\,(-1|p)\big\},$$

$$H^{\mathrm{Large}}_{3,\mathrm{K}(p)}=2^{-5}3^{-1}C_3(k,j)\cdot B(2,5;p,2)-2^{-3}C_3'(k,j)\cdot 2,$$

$$H^{\mathrm{Large}}_{4,\mathrm{K}(p)}=2^{-3}3^{-3}C_4(k,j)\cdot B(2,10;p,3)$$
$$-2^{-2}3^{-2}C_4'(k,j)\cdot\big\{6+4\,(-3|p)\big\}+3^{-2}C_4''(k,j)\cdot 2\,(-3|p),$$

$$H^{\mathrm{Large}}_{5,\mathrm{K}(p)}=2^{-3}3^{-2}C_5(k,j)\cdot 2-2^{-2}3^{-1}C_5'(k,j)\cdot 2,$$

$$H^{\mathrm{Large}}_{6,\mathrm{K}(p)} = -2^{-7}3^{-1}C_6(k,j)\cdot B\big(\{4-(-1|p)\}\{p+(-1|p)\},7;p,2\big) - 2^{-7}3^{-1}C'_6(k,j)\cdot B\big(\{4+(-1|p)\}\{p+(-1|p)\},9;p,2\big) + 2^{-3}(-1)^{j/2+k}\cdot\{1+(-1|p)\},$$

$$H^{\mathrm{Large}}_{7,\mathrm{K}(p)} = -2^{-2}3^{-3}C_7(k,j)\cdot B\big(2^{-1}\{3-(-3|p)\}\{p+(-3|p)\},4;p,3\big) - 2^{-2}3^{-3}C'_7(k,j)\cdot B\big(2^{-1}\{3+(-3|p)\}\{p+(-3|p)\},5;p,3\big) + 2^{-1}3^{-1}[0,1,-1;3]_{j+2k}\cdot\{1+(-3|p)\},$$

$$H^{\mathrm{Large}}_{8,\mathrm{K}(p)} = 2^{-2}3^{-1}C_8(k,j)\cdot 2, \qquad H^{\mathrm{Large}}_{9,\mathrm{K}(p)} = 3^{-2}C_2(k,j)\cdot B\big(2,2^{-1};p,2\big),$$

$$H^{\mathrm{Large}}_{10,\mathrm{K}(p)} = 5^{-1}C_{10}(k,j)\cdot\begin{cases}2 & \text{if } p\equiv\pm1 \bmod 5,\\ 0 & \text{if } p\equiv 2,3 \bmod 5,\\ 1 & \text{if } p=5,\end{cases}$$

$$H^{\mathrm{Large}}_{11,\mathrm{K}(p)} = 2^{-3}C_{11}(k,j)\times\begin{cases}2 & \text{if } p\equiv\pm1 \bmod 8,\\ 0 & \text{if } p\equiv 3,5 \bmod 8,\\ 1 & \text{if } p=2,\end{cases}$$

$$H^{\mathrm{Large}}_{12,\mathrm{K}(p)} = 2^{-2}3^{-1}C_{12}(k,j)\cdot\begin{cases}2 & \text{if } p\equiv 1 \bmod 12,\\ 0 & \text{otherwise},\end{cases} - 2^{-2}3^{-1}C'_{12}(k,j)\cdot\begin{cases}2 & \text{if } p\equiv 11 \bmod 12,\\ 1 & \text{if } p=2,3,\\ 0 & \text{otherwise}.\end{cases}$$

Theorem 4.3. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd and $l_2=k-2>2$. Then, we have*

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\mathrm{Kl}(p)\big) = \sum_{t=1}^{12} H^{\mathrm{Large}}_{t,\mathrm{Kl}(p)} + \begin{cases} m(\sigma_k^-,\mathrm{Kl}(p)) & \text{if } j=0,\\ 0 & \text{if } j>0,\end{cases}$$

$$H^{\mathrm{Large}}_{1,\mathrm{Kl}(p)} = 2^{-7}3^{-3}5^{-1}C_1(k,j)\cdot(p+1)\big(p^2+1\big) + 2^{-5}3^{-2}C'_1(k,j)\cdot(p+1)^2 - 2^{-3}3^{-1}(j+2k-3)\cdot(2p+2) - 2^{-4}3^{-1}(j+1)\cdot(p+3) + 2^{-2}\cdot 4,$$

$$H^{\mathrm{Large}}_{2,\mathrm{Kl}(p)} = 2^{-7}3^{-2}C_2(k,j)\cdot B(14p+14,33;p,2) - 2^{-4}3^{-1}C'_2(k,j)\cdot(p+3) + 2^{-5}(-1)^k\cdot B\big(18+2\,(-1|p),17;p,2\big),$$

$$H^{\mathrm{Large}}_{3,\mathrm{Kl}(p)} = 2^{-5}3^{-1}C_3(k,j)\cdot B\big(p+2+(-1|p),7;p,2\big) - 2^{-3}C'_3(k,j)\cdot\{3+(-1|p)\},$$

$$H^{\mathrm{Large}}_{4,\mathrm{Kl}(p)} = 2^{-3}3^{-3}C_4(k,j)\cdot B\big(p+2+(-3|p),13;p,3\big) - 2^{-2}3^{-2}C'_4(k,j)\cdot B\big(13+7\,(-3|p),11;p,3\big) + 3^{-2}C''_4(k,j)\cdot B\big(2+2\,(-3|p),1;p,3\big),$$

$$H^{\mathrm{Large}}_{5,\mathrm{Kl}(p)} = 2^{-3}3^{-2}C_5(k,j)\cdot\{p+2+(-3|p)\} - 2^{-2}3^{-1}C'_5(k,j)\cdot\{3+(-3|p)\},$$

$$H^{\mathrm{Large}}_{6,\mathrm{Kl}(p)} = -2^{-7}3^{-1}C_6(k,j)\cdot 3(p+1)\{1+(-1|p)\} - 2^{-7}3^{-1}C'_6(k,j)\cdot 5(p+1)\{1+(-1|p)\} + 2^{-3}(-1)^{j/2+k}\cdot\{2+2\,(-1|p)\},$$

$$H^{\mathrm{Large}}_{7,\mathrm{Kl}(p)} = -2^{-2}3^{-3}C_7(k,j)\cdot(p+1)\{1+(-3|p)\} - 2^{-2}3^{-3}C'_7(k,j)\cdot 2(p+1)\{1+(-3|p)\} + 2^{-1}3^{-1}[0,1,-1;3]_{j+2k}\cdot\{2+2(-3|p)\},$$

$$H^{\mathrm{Large}}_{8,\mathrm{Kl}(p)} = 2^{-2}3^{-1}C_8(k,j)\cdot\{2+(-1|p)+(-3|p)\},$$

$$H^{\mathrm{Large}}_{9,\mathrm{Kl}(p)} = 3^{-2}C_9(k,j)\cdot\{2+2(-3|p)\},$$

$$H^{\mathrm{Large}}_{10,\mathrm{Kl}(p)} = 5^{-1}C_{10}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 5,\\ 1 & \textit{if } p=5,\\ 0 & \textit{otherwise},\end{cases}$$

$$H^{\mathrm{Large}}_{11,\mathrm{Kl}(p)} = 2^{-3}C_{11}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 8,\\ 1 & \textit{if } p=2,\\ 0 & \textit{otherwise},\end{cases}$$

$$H^{\mathrm{Large}}_{12,\mathrm{Kl}(p)} = 2^{-2}3^{-1}C_{12}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 12,\\ 0 & \textit{otherwise}.\end{cases}$$

Theorem 4.4. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd and $l_2=k-2>2$. Then, we have*

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\mathrm{Si}(p)\big)=\sum_{t=1}^{12}H^{\mathrm{Large}}_{t,\mathrm{Si}(p)}+\begin{cases}m(\sigma_k^-,\mathrm{Si}(p)) & \textit{if } j=0,\\ 0 & \textit{if } j>0,\end{cases}$$

$$H^{\mathrm{Large}}_{1,\mathrm{Si}(p)} = 2^{-7}3^{-3}5^{-1}C_1(k,j)\cdot(p+1)\big(p^2+1\big)+2^{-5}3^{-2}C'_1(k,j)\cdot(2p+2) - 2^{-3}3^{-1}(j+2k-3)\cdot(p+3)-2^{-4}3^{-1}(j+1)\cdot(2p+2)+2^{-2}\cdot 4,$$

$$H^{\mathrm{Large}}_{2,\mathrm{Si}(p)} = 2^{-7}3^{-2}C_2(k,j)\cdot B\big(7(p+1)^2,57;p,2\big) - 2^{-4}3^{-1}C'_2(k,j)\cdot(2p+2)+2^{-5}(-1)^k\cdot\{18+2(-1|p)\},$$

$$H^{\mathrm{Large}}_{3,\mathrm{Si}(p)} = 2^{-5}3^{-1}C_3(k,j)\cdot(p+1)\{1+(-1|p)\}-2^{-3}C'_3(k,j)\cdot\{2+2(-1|p)\},$$

$$H^{\mathrm{Large}}_{4,\mathrm{Si}(p)} = 2^{-3}3^{-3}C_4(k,j)\cdot(p+1)\{1+(-3|p)\} - 2^{-2}3^{-2}C'_4(k,j)\cdot\{10+10(-3|p)\}+3^{-2}C''_4(k,j)\cdot\{2+2(-3|p)\},$$

$$H^{\mathrm{Large}}_{5,\mathrm{Si}(p)} = 2^{-3}3^{-2}C_5(k,j)\cdot(p+1)\{1+(-3|p)\}-2^{-2}3^{-1}C'_5(k,j)\cdot\{2+2(-3|p)\},$$

$$H^{\mathrm{Large}}_{6,\mathrm{Si}(p)} = -2^{-7}3^{-1}C_6(k,j)\cdot B\big(3\{p+2+(-1|p)\},9;p,2\big) - 2^{-7}3^{-1}C'_6(k,j)\cdot B\big(5\{p+2+(-1|p)\},23;p,2\big)+2^{-3}(-1)^{j/2+k}\{3+(-1|p)\},$$

$$H^{\mathrm{Large}}_{7,\mathrm{Si}(p)} = -2^{-2}3^{-3}C_7(k,j)\cdot B\big(p+2+(-3|p),1;p,3\big) - 2^{-2}3^{-3}C'_7(k,j)\cdot B\big(2\{p+2+(-3|p)\},14;p,3\big) + 2^{-1}3^{-1}[0,1,-1;3]_{j+2k}\cdot\{3+(-3|p)\},$$

$$H^{\mathrm{Large}}_{8,\mathrm{Si}(p)} = 2^{-2}3^{-1}C_8(k,j)\cdot\{1+(-1|p)\}\{1+(-3|p)\},$$

$$H^{\mathrm{Hol}}_{9,\mathrm{Si}(p)} = 3^{-2}C_9(k,j)\cdot B\big(\{1+(-3|p)\}^2, 3/2; p, 2\big),$$

$$H^{\mathrm{Large}}_{10,\mathrm{Si}(p)} = 5^{-1}C_{10}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 5,\\ 1 & \textit{if } p=5,\\ 0 & \textit{otherwise},\end{cases}$$

$$H^{\mathrm{Large}}_{11,\mathrm{Si}(p)} = 2^{-3}C_{11}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 8,\\ 2 & \textit{if } p\equiv 3,5 \bmod 8,\\ 0 & \textit{if } p\equiv 7 \bmod 8\\ 1 & \textit{if } p=2,\end{cases}$$

$$H^{\mathrm{Large}}_{12,\mathrm{Si}(p)} = 2^{-2}3^{-1}C_{12}(k,j)\cdot\begin{cases}4 & \textit{if } p\equiv 1 \bmod 12,\\ 2 & \textit{if } p\equiv 5,7 \bmod 12,\\ 0 & \textit{if } p\equiv 11 \bmod 12,\\ 1 & \textit{if } p=2,3.\end{cases}$$

Theorem 4.5. *Let $(l_1,l_2)=(j+k-1,k-2)\in\Xi_1$. Assume that $l_1-l_2=j+1$ is odd and $l_2=k-2>2$. Then, we have*

$$m\big(D^{\mathrm{Large}}_{l_1,l_2},\mathrm{I}(p)\big)=\sum_{t=1}^{12}H^{\mathrm{Large}}_{t,\mathrm{I}(p)}+\begin{cases}m(\sigma_k^-,\mathrm{I}(p)) & \textit{if } j=0,\\ 0 & \textit{if } j>0,\end{cases}$$

$$H^{\mathrm{Large}}_{1,\mathrm{I}(p)}=2^{-7}3^{-3}5^{-1}C_1(k,j)\cdot(p+1)^2\big(p^2+1\big)+2^{-5}3^{-2}C_1'(k,j)\cdot 2(p+1)^2$$
$$-2^{-3}3^{-1}(j+2k-3)\cdot(4p+4)-2^{-4}3^{-1}(j+1)\cdot(4p+4)+2^{-2}\cdot 8,$$

$$H^{\mathrm{Large}}_{2,\mathrm{I}(p)}=2^{-7}3^{-2}C_2(k,j)\cdot(p+1)^2B(14,11;p,2)$$
$$-2^{-4}3^{-1}C_2'(k,j)\cdot(4p+4)+2^{-5}(-1)^k\cdot B\big(36+4\,(-1|p),34;p,2\big),$$

$$H^{\mathrm{Large}}_{3,\mathrm{I}(p)}=2^{-5}3^{-1}C_3(k,j)\cdot B\big(2(p+1)\{1+(-1|p)\},9;p,2\big)-2^{-3}C_3'(k,j)\cdot\{4+4\,(-1|p)\},$$

$$H^{\mathrm{Large}}_{4,\mathrm{I}(p)}=2^{-3}3^{-3}C_4(k,j)\cdot B\big(2(p+1)\{1+(-3|p)\},16;p,3\big)$$
$$-2^{-2}3^{-2}C_4'(k,j)\cdot B\big(20+20\,(-3|p),16;p,3\big)$$
$$+3^{-2}C_4''(k,j)\cdot B\big(4+4\,(-3|p),2;p,3\big),$$

$$H^{\mathrm{Large}}_{5,\mathrm{I}(p)}=2^{-3}3^{-2}C_5(k,j)\cdot 2(p+1)\{1+(-3|p)\}-2^{-2}3^{-1}C_5'(k,j)\cdot\{4+4\,(-3|p)\},$$

$$H^{\mathrm{Large}}_{6,\mathrm{I}(p)}=-2^{-7}3^{-1}C_6(k,j)\cdot B\big(6(p+1)\{1+(-1|p)\},15;p,2\big)$$
$$-2^{-7}3^{-1}C_6'(k,j)\cdot B\big(10(p+1)\{1+(-1|p)\},33;p,2\big)$$
$$+2^{-3}(-1)^{j/2+k}\cdot\{4+4\,(-1|p)\},$$

$$H^{\mathrm{Large}}_{7,\mathrm{I}(p)}=-2^{-2}3^{-3}C_7(k,j)\cdot B\big(2(p+1)\{1+(-3|p)\},4;p,3\big)$$
$$-2^{-2}3^{-3}C_7'(k,j)\cdot B\big(4(p+1)\{1+(-3|p)\},20;p,3\big)$$
$$+2^{-1}3^{-1}[0,1,-1;3]_{j+2k}\cdot\{4+4\,(-3|p)\},$$

$$H_{8,\mathrm{I}(p)}^{\mathrm{Large}} = 2^{-2}3^{-1}C_8(k,j)\cdot 2\big\{1+(-1|p)\big\}\big\{1+(-3|p)\big\},$$

$$H_{9,\mathrm{I}(p)}^{\mathrm{Large}} = 3^{-2}C_9(k,j)\cdot B\big(4+4\,(-3|p),2;p,3\big),$$

$$H_{10,\mathrm{I}(p)}^{\mathrm{Large}} = 5^{-1}C_{10}(k,j)\cdot \begin{cases} 8 & \textit{if } p\equiv 1 \bmod 5,\\ 1 & \textit{if } p=5,\\ 0 & \textit{otherwise},\end{cases}$$

$$H_{11,\mathrm{I}(p)}^{\mathrm{Large}} = 2^{-3}C_{11}(k,j)\cdot \begin{cases} 8 & \textit{if } p\equiv 1 \bmod 8,\\ 1 & \textit{if } p=2,\\ 0 & \textit{otherwise},\end{cases}$$

$$H_{12,\mathrm{I}(p)}^{\mathrm{Large}} = 2^{-2}3^{-1}C_{12}(k,j)\cdot \begin{cases} 8 & \textit{if } p\equiv 1 \bmod 12,\\ 0 & \textit{otherwise}.\end{cases}$$

Numerical examples of $m(D^{\mathrm{Large}}_{j+k-1,k-2},\mathrm{Sp}(2,\mathbb{Z}))$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	0	0	0	0	0	0	0	0	0	1	0	2	0	2	0	3	1	5
4	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	6	3	8
6	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4	9	6	12

Numerical examples of $m(D^{\mathrm{Large}}_{j+k-1,k-2},\mathrm{K}(2))$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	0	0	0	0	0	1	1	1	1	4	4	6	4	9	7	12	10	19
4	0	0	1	1	1	3	3	5	4	8	9	13	11	17	18	26	23	33
6	0	1	2	3	3	6	7	9	9	15	16	22	21	31	31	40	40	55

Numerical examples of $m(D^{\mathrm{Large}}_{j+k-1,k-2},\mathrm{Kl}(2))$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	0	0	0	1	1	4	3	6	4	13	11	20	14	29	22	40	32
4	0	1	2	4	4	9	9	16	14	26	26	40	35	55	53	79	71
6	2	4	6	10	11	19	21	30	30	47	49	68	67	95	94	125	124

Numerical examples of $m(D^{\mathrm{Large}}_{j+k-1,k-2},\mathrm{Si}(2))$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	0	0	0	2	0	5	1	9	3	16	6	24	10	35	17	48	24
4	0	2	1	5	3	11	7	19	12	30	21	45	31	63	46	86	63
6	1	4	4	10	8	19	16	32	26	49	41	71	59	99	84	132	112

Numerical examples of $m(D^{\mathrm{Large}}_{j+k-1,k-2},\mathrm{I}(2))$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	1	1	5	3	13	9	24	16	42	30	63	45	94	70	128
4	1	5	6	14	13	29	28	52	46	82	76	124	110	174	160	240
6	6	13	17	30	32	56	59	92	92	142	142	204	200	285	280	380

5. Applications

5.1. Relations between $m(D^{\mathrm{Hol}}_{l_1,l_2},\Gamma)$ and $m(D^{\mathrm{Large}}_{l_1,l_2},\Gamma)$

In this subsection, we will study relations between $m(D^{\mathrm{Hol}}_{l_1,l_2},\Gamma)$ and $m(D^{\mathrm{Large}}_{l_1,l_2},\Gamma)$ from the point of view of Arthur's conjecture. We refer to Arthur [2,4], Flicker [8], and Weissauer [44] for the whole conjecture for PGSp(2), to Roberts [28] for the Yoshida type, and to Piatetski-Shapiro [27] and Schmidt [31] for the Saito–Kurokawa type.

The discrete series D_k^+ of $\mathrm{SL}(2,\mathbb{R})$ was defined in Section 2.5. Some multiplicity formulas for $m(D_k^+,\mathrm{SL}(2,\mathbb{Z}))$ are well known (see, e.g., [24,42]). Therefore, we can deduce the following equalities from Theorems 3.1 and 4.1.

Theorem 5.1. *If $l_1-l_2=j+1>1$ and $l_2=k-2>2$, then we have*

$$m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{Sp}(2,\mathbb{Z})\big)=m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{Sp}(2,\mathbb{Z})\big)+m\big(D_{l_1-l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big)\times m\big(D_{l_1+l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big).$$

If $l_1-l_2=j+1=1$ and $l_2=k-2>2$, then we have

$$m\big(D_{k-1,k-2}^{\mathrm{Large}},\mathrm{Sp}(2,\mathbb{Z})\big)=m\big(D_{k-1,k-2}^{\mathrm{Hol}},\mathrm{Sp}(2,\mathbb{Z})\big)-m\big(D_{2k-3}^{+},\mathrm{SL}(2,\mathbb{Z})\big)+m\big(\sigma_k^{-},\mathrm{Sp}(2,\mathbb{Z})\big).$$

We easily see that these relations provide supporting evidence of Arthur's conjecture. The term $m(D_{l_1-l_2}^{+},\mathrm{SL}(2,\mathbb{Z}))\times m(D_{l_1+l_2}^{+},\mathrm{SL}(2,\mathbb{Z}))$ should mean the number of cuspidal automorphic representations $\pi=\bigotimes_v\pi_v$ of the Yoshida type, which satisfy that $\pi_\infty\cong D_{l_1,l_2}^{\mathrm{Large}}$ and π_v is unramified for any $v<\infty$. The term $m(D_{2k-3}^{+},\mathrm{SL}(2,\mathbb{Z}))-m(\sigma_k^{-},\mathrm{Sp}(2,\mathbb{Z}))$ should mean the number of cuspidal automorphic representations $\pi=\bigotimes_v\pi_v$ of the Saito–Kurokawa type, which satisfy that $\pi_\infty\cong D_{l_1,l_2}^{\mathrm{Hol}}$ and π_v is unramified for any $v<\infty$. Of course, this statement agrees with some known results for Siegel modular forms. It is well known that there exists the Saito–Kurokawa lifting from $S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ to $S_{k,0}(\mathrm{Sp}(2,\mathbb{Z}))$ if k is even (cf. [6]). Evdokimov and Oda independently proved that a Hecke-eigen Siegel cusp form belongs to the Maass space if and only if its Andrianov L-function has poles (cf. [7,26]). Miyazaki constructed the lifting from $S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ to a space of non-holomorphic Siegel modular forms related to σ_k^- if k is odd (cf. [25]). Hence, the number $m(D_{2k-3}^{+},\mathrm{SL}(2,\mathbb{Z}))-m(\sigma_k^{-},\mathrm{Sp}(2,\mathbb{Z}))$ should be the dimension of the Maass space of $S_{k,0}(\mathrm{Sp}(2,\mathbb{Z}))$.

By Theorem 5.1 we find that

$$m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{Sp}(2,\mathbb{Z})\big)=-\frac{1}{4}\mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{Sp}(2,\mathbb{Z})})-\frac{1}{2}m\big(D_{l_1-l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big)\times m\big(D_{l_1+l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big)$$

and

$$m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{Sp}(2,\mathbb{Z})\big)=-\frac{1}{4}\mathscr{L}_{\mu(l_1,l_2)}(h_{\mathrm{Sp}(2,\mathbb{Z})})+\frac{1}{2}m\big(D_{l_1-l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big)\times m\big(D_{l_1+l_2}^{+},\mathrm{SL}(2,\mathbb{Z})\big)$$

when $l_1-l_2>1$ and $l_2>2$. If it is possible to compute the full stable trace formula for Sp(2), then the term $-\frac{1}{4}\mathscr{L}_\mu(h_{\mathrm{Sp}(2,\mathbb{Z})})$ should be the contribution of the stable terms and the term $\frac{1}{2}m(D_{l_1-l_2}^{+},\mathrm{SL}(2,\mathbb{Z}))\times m(D_{l_1+l_2}^{+},\mathrm{SL}(2,\mathbb{Z}))$ should be the contribution of the endoscopic terms. Hence, the above equalities look like stablizations of $m(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{Sp}(2,\mathbb{Z}))$ and $m(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{Sp}(2,\mathbb{Z}))$. Spallone has studied unipotent terms of a stable version of Arthur's L^2-Lefschetz trace formula for GSp(2) in [36]. His calculation agrees with our conjecture.

We can also get analogues of Theorem 5.1 for $\mathrm{K}(p)$, $\mathrm{Kl}(p)$, $\mathrm{Si}(p)$, and $\mathrm{I}(p)$ by using the formulas of Sections 3 and 4 and Schmidt's results [32,33] for local newforms (cf. Table B.2). As an example, we will write it for $\mathrm{K}(p)$. As for $\mathrm{Kl}(p)$ and $\mathrm{Si}(p)$, we do not write them in this paper. Since $\mathrm{I}(p)$-newforms are related to the Steinberg representation of $\mathrm{GSp}(2,\mathbb{Q}_p)$, we will mention in the next subsection. An arithmetic subgroup $\Gamma_0(p)$ of $\mathrm{SL}(2,\mathbb{Q})$ is defined by

$$\Gamma_0(p)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})\;\middle|\; c\equiv 0 \bmod p\right\}.$$

We already know a multiplicity formula for $m(D_k^+,\Gamma_0(p))$ (see e.g., [24,42]). We set

$$\begin{aligned} m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{K}(p)\big)^{\mathrm{new}} &= m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{K}(p)\big)-2m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{Sp}(2,\mathbb{Z})\big) \\ &\quad + \begin{cases} m(D_{2k-3}^+,\mathrm{SL}(2,\mathbb{Z}))-m(\sigma_k^-,\mathrm{Sp}(2,\mathbb{Z})) & \text{if } j=0,\\ 0 & \text{otherwise,}\end{cases} \\ m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{K}(p)\big)^{\mathrm{new}} &= m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{K}(p)\big)-2m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{Sp}(2,\mathbb{Z})\big),\end{aligned}$$

and

$$m\big(\sigma_k^-,\mathrm{K}(p)\big)^{\mathrm{new}}=m\big(\sigma_k^-,\mathrm{K}(p)\big)-m\big(\sigma_k^-,\mathrm{Sp}(2,\mathbb{Z})\big).$$

These numbers should be the dimensions of the spaces of $\mathrm{K}(p)$-newforms. The $\mathrm{K}(p)$-newforms were studied by Ibukiyama [19–21] and Schmidt [32]. If $l_1-l_2=j+1>1$ and $l_2=k-2>2$, then we have

$$\begin{aligned} m\big(D_{l_1,l_2}^{\mathrm{Large}},\mathrm{K}(p)\big)^{\mathrm{new}} &= m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{K}(p)\big)^{\mathrm{new}} \\ &\quad + m\big(D_{l_1-l_2}^+,\Gamma_0^{(1)}(p)\big)^{\mathrm{new}}\times m\big(D_{l_1+l_2}^+,\mathrm{SL}(2,\mathbb{Z})\big) \\ &\quad + m\big(D_{l_1-l_2}^+,\mathrm{SL}(2,\mathbb{Z})\big)\times m\big(D_{l_1+l_2}^+,\Gamma_0^{(1)}(p)\big)^{\mathrm{new}}.\end{aligned}$$

If $l_1-l_2=j+1=1$ and $l_2=k-2>2$, then we have

$$\begin{aligned} m\big(D_{k-1,k-2}^{\mathrm{Large}},\mathrm{K}(p)\big)^{\mathrm{new}} &= m\big(D_{k-1,k-2}^{\mathrm{Hol}},\mathrm{K}(p)\big)^{\mathrm{new}} \\ &\quad + m\big(D_1^+,\Gamma_0^{(1)}(p)\big)^{\mathrm{new}}\times m\big(D_{2k-3}^+,\mathrm{SL}(2,\mathbb{Z})\big) \\ &\quad - m\big(D_{2k-3}^+,\Gamma_0^{(1)}(p)\big)^{\mathrm{new}}+m\big(\sigma_k^-,\mathrm{K}(p)\big)^{\mathrm{new}}.\end{aligned}$$

These equalities can be interpreted similarly to those of $\mathrm{Sp}(2,\mathbb{Z})$.

5.2. Numbers of automorphic representations

We denote by $\mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_p)}$ the Steinberg representation of $\mathrm{GL}(2,\mathbb{Q}_p)$ and by $\xi\mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_p)}$ the twist of $\mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_p)}$ via the unramified quadratic character ξ on $\mathbb{Q}_p^\times$. Let $n(D_k,\mathrm{St},p,+)$ (resp. $n(D_k,\mathrm{St},p,-)$) be the number of cuspidal automorphic representations $\pi=\bigotimes_v\pi_v$ of $\mathrm{PGL}(2,\mathbb{A})$ satisfying that $\pi_\infty=D_k$, $\pi_p=\mathrm{St}$ (resp. $\pi_p=\xi\mathrm{St}$) if $(k+1)/2$ is odd, $\pi_p=\xi\mathrm{St}$ (resp. $\pi_p=\mathrm{St}$) if $(k+1)/2$ is even, and π_v is unramified for each $v\neq\infty,p$. The signatures $+$ and $-$ of $n(D_k,\mathrm{St},p,\pm)$ mean the ϵ-factors of the L-functions of such automorphic representations. Under the assumption of Arthur's conjecture for $\mathrm{PGSp}(2)$, by Table B.2, the number

$$m\big(D_{l_1,l_2}^{\mathrm{Hol}},\mathrm{K}(p)\big)^{\mathrm{new}}-\begin{cases}0 & \text{if } l_1-l_2>1,\\ n(D_{l_1+l_2},\mathrm{St},p,-) & \text{if } l_1-l_2=1,\end{cases} \tag{5.1}$$

is equal to the number of cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of PGSp(2) of the general type, $\pi \cong D(l_1, l_2) \oplus D(-l_2, -l_1)$, π_p belongs to the class IIa (see Appendix B), π_v is unramified for any $v \neq \infty, p$. Using dimension formulas (see, e.g., [24]), Yamauchi's trace formula [45], and Casselman's result [5], we can explicitly calculate the number $n(D_k, \mathrm{St}, p, \pm)$. Hence, the numerical values of (5.1) can be computed.

Numerical examples of (5.1) for $p = 2$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	2	3
2	0	0	0	0	0	1	1	1	1	2	4	2	4	5	7	6	8	9	13
4	0	0	1	1	1	1	3	3	4	4	7	7	9	9	14	14	17	17	24
6	0	0	1	0	2	2	4	3	5	7	10	9	13	14	20	19	25	27	34

We set

$$n(R, \mathrm{St}, p) = m\big(R, \mathrm{I}(p)\big) - m\big(R, \mathrm{Si}(p)\big) - 2 \cdot m\big(R, \mathrm{Kl}(p)\big) + m\big(R, \mathrm{K}(p)\big) + 2 \cdot m\big(R, \mathrm{Sp}(2, \mathbb{Z})\big)$$

where $R = D^{\mathrm{Hol}}_{l_1,l_2}$, $D^{\mathrm{Large}}_{l_1,l_2}$ or σ_k^-. Hashimoto and Ibukiyama studied $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{St}, p)$ as dimensions of spaces of $\mathrm{I}(p)$-newforms in [19,14]. Schmidt also studied local and global newforms of $\mathrm{I}(p)$ in [32]. It follows from Table B.2 that the number $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{St}, p)$ (resp. $n(D^{\mathrm{Large}}_{l_1,l_2}, \mathrm{St}, p)$) is equal to the number of cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ satisfying $\pi_\infty = D(l_1, l_2) \oplus D(-l_2, -l_1)$ (resp. $\pi_\infty = D(l_1, -l_2) \oplus D(l_2, -l_1)$), π_p is the Steinberg representation of $\mathrm{GSp}(2, \mathbb{Q}_p)$ or its quadratic twist, and π_v is unramified for any $v \neq \infty, p$. By the multiplicity formulas we obtain

$$n\big(D^{\mathrm{Large}}_{l_1,l_2}, \mathrm{St}, p\big) = n\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{St}, p\big) + \begin{cases} n(\sigma_k^-, \mathrm{St}, p) & \text{if } l_1 - l_2 = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $l_1 - l_2 = j + 1 > 0$ and $l_2 = k - 2 > 2$. Note that $n(\sigma_k^-, \mathrm{St}, p) = 0$ under the assumption of Arthur's conjecture. Gross and Pollack have given a formula for L^2-Euler characteristic related to Steinberg representations in [11]. Their formula for Sp(2) agrees with ours.

Numerical examples of $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{St}, 2)$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	1	0	1	1	2	1	2	2	3	4	6	5	7	6
2	0	1	1	1	1	1	3	4	6	6	6	9	11	14	16	18	20	23	27
4	1	1	2	2	3	5	6	8	10	12	14	18	22	26	30	34	39	47	52
6	1	2	3	5	5	7	10	13	17	20	23	29	34	41	47	56	62	71	81

5.3. *Square integrable representation* Va

We consider the number

$$\begin{aligned} E(l_1, l_2, p) = {} & m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Si}(p)\big) - m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{K}(p)\big) - 2 \cdot m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Sp}(2, \mathbb{Z})\big) \\ & - \begin{cases} \{n(D_{l_1-l_2}, \mathrm{St}, p, -) \cdot n(D_{l_1+l_2}, \mathrm{St}, p, -) \\ \quad + n(D_{l_1-l_2}, \mathrm{St}, p, +) \cdot n(D_{l_1+l_2}, \mathrm{St}, p, +)\} & \text{if } l_2 \text{ is even,} \\ \{n(D_{l_1-l_2}, \mathrm{St}, p, -) \cdot n(D_{l_1+l_2}, \mathrm{St}, p, +) \\ \quad + n(D_{l_1-l_2}, \mathrm{St}, p, +) \cdot n(D_{l_1+l_2}, \mathrm{St}, p, -)\} & \text{if } l_2 \text{ is odd,} \end{cases} \\ & - \begin{cases} 0 & \text{if } l_1 - l_2 > 1, \\ n(D_{l_1+l_2}, \mathrm{St}, p, +) & \text{if } l_1 - l_2 = 1 \text{ and } l_2 \text{ is even,} \\ -n(D_{l_1+l_2}, \mathrm{St}, p, -) & \text{if } l_1 - l_2 = 1 \text{ and } l_2 \text{ is odd.} \end{cases} \end{aligned}$$

The following theorem was proved by some dimension formulas, Yamauchi's trace formula [45], and a computer.

Theorem 5.2. *(See [42, Theorem 3].) The number $E(l_1, l_2, p)$ is even and non-negative for any prime p and any (l_1, l_2), $l_1 > l_2 > 2$.*

Under the assumption of Arthur's conjecture, it is proved that $E(l_1, l_2, p)$ is even and non-negative for any $l_1 > l_2 > 0$ (cf. [42, Section 5]). Note that Arthur's conjecture includes the generalized Ramanujan conjecture which is necessary to prove it. Using this property we can consider numbers of cuspidal automorphic representations related to certain square integrable representations of $\mathrm{GSp}(2, \mathbb{Q}_p)$. We set

$$n\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Va}, p\big) = m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Kl}(p)\big) - \frac{1}{2} \cdot E(l_1, l_2, p) - 2 \cdot m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{K}(p)\big) + \begin{cases} n(D_{l_1+l_2}, \mathrm{St}, p) & \text{if } l_1 - l_2 = 1 \text{ and } l_2 \text{ is even}, \\ 0 & \text{otherwise}, \end{cases}$$

where we set $n(D_{l_1+l_2}, \mathrm{St}, p) = n(D_{l_1+l_2}, \mathrm{St}, p, -) + n(D_{l_1+l_2}, \mathrm{St}, p, +)$. If we assume Arthur's conjecture, then the number $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Va}, p)$ is the number of cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of the general type of PGSp(2), $\pi \cong D^{\mathrm{Hol}}_{l_1,l_2} \oplus D(-l_2, -l_1)$, π_p belongs to the class Va (cf. Appendix B), π_v is unramified for any $v \neq \infty, p$.

Numerical examples of $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Va}, 2)$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2
2	0	0	0	0	1	0	1	0	1	0	2	1	3	0	3	1	6	2	6
4	0	0	0	0	1	0	1	0	2	1	3	1	4	2	5	3	8	4	9
6	0	0	0	0	1	1	1	1	3	1	4	3	6	4	7	5	11	8	13

Numerical examples of $n(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Va}, 3)$.

$j \diagdown k$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	0	1	0	1	0	3	1	4	0	6	2	7	2	11
2	0	0	1	0	2	1	4	1	5	3	9	5	12	7	17	10	23	16	30
4	0	0	2	1	3	2	6	4	10	7	15	12	20	16	30	24	38	32	50
6	0	0	2	2	4	5	9	7	14	13	22	20	31	28	43	39	56	54	75

For only $p = 2$, we obtain the following relation. We can prove it by a direct calculation.

Proposition 5.3.

$$n\big(D^{\mathrm{Hol}}_{k-1,k-2}, \mathrm{Va}, 2\big) = \begin{cases} m(D^{\mathrm{Hol}}_{k+4,k+3}, \mathrm{Sp}(2, \mathbb{Z})) & \textit{if } k > 5 \textit{ and } k \textit{ is even}, \\ m(D^{\mathrm{Hol}}_{k-6,k-7}, \mathrm{Sp}(2, \mathbb{Z})) & \textit{if } k > 8 \textit{ and } k \textit{ is odd}, \\ 0 & \textit{if } 4 < k < 8 \textit{ and } k \textit{ is odd}. \end{cases}$$

This formula makes us guess that there exists a strange correspondence between Siegel modular forms. However, we have not known how to interpret Proposition 5.3 yet. It seems interesting to find what Proposition 5.3 means.

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Appendix A. Dimension formulas for Kl(p) and I(p)

In this appendix, we write dimension formulas for $\mathrm{Kl}(p)$ and $\mathrm{I}(p)$, which were used in Sections 4, 5.2, and 5.3. Ibukiyama and Hashimoto gave dimension formulas for the spaces of Siegel cusp forms of weight $\det^k$ ($k > 4$, $j = 0$) for $\mathrm{Kl}(p)$ and $\mathrm{I}(p)$ ($p > 3$) in [14]. We can easily generalize their formulas to any (k, j) ($k \geqslant 5$) by the result of [41]. Furthermore, it is not difficult to calculate dimension formulas for the cases $p = 2$ or 3 by the same method as [14]. We assume that $l_1 - l_2 = j + 1$ is odd, since $m(D^{\mathrm{Hol}}_{l_1,l_2}, \Gamma) = 0$ if $l_1 - l_2$ is even. The notation $H^{\mathrm{Hol}}_{t,\Gamma}$ means the total contribution of elements of Γ with the characteristic polynomial $f_t(\pm x)$ to $m(D^{\mathrm{Hol}}_{l_1,l_2}, \Gamma)$.

Theorem A.1. *Let $(l_1, l_2) = (j + k - 1, k - 2) \in \Xi_1$. Assume that $l_1 - l_2 = j + 1$ is odd and $l_2 = k - 2 > 2$. Then, we have*

$$m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{Kl}(p)\big) = \sum_{t=1}^{12} H^{\mathrm{Hol}}_{t,\mathrm{Kl}(p)},$$

$$\begin{aligned} H^{\mathrm{Hol}}_{1,\mathrm{Kl}(p)} = {} & 2^{-7}3^{-3}5^{-1}C_1(k, j) \cdot (p + 1)\big(p^2 + 1\big) - 2^{-5}3^{-2}C_1'(k, j) \cdot (p + 1)^2 \\ & + 2^{-3}3^{-1}(j + 1) \cdot (2p + 2) - 2^{-4}3^{-1}(j + 1) \cdot (p + 3), \end{aligned}$$

$$\begin{aligned} H^{\mathrm{Hol}}_{2,\mathrm{Kl}(p)} = {} & 2^{-7}3^{-2}C_2(k, j) \cdot B(14p + 14, 33; p, 2) \\ & - 2^{-4}3^{-1}C_2'(k, j) \cdot (p + 3) + 2^{-5}(-1)^k \cdot B\big(14 - 2\,(-1|p), 15; p, 2\big), \end{aligned}$$

$$H^{\mathrm{Hol}}_{3,\mathrm{Kl}(p)} = 2^{-5}3^{-1}C_3(k, j) \cdot B\big(p + 2 + (-1|p), 7; p, 2\big) - 2^{-3}C_3'(k, j) \cdot \big\{3 + (-1|p)\big\},$$

$$\begin{aligned} H^{\mathrm{Hol}}_{4,\mathrm{Kl}(p)} = {} & 2^{-3}3^{-3}C_4(k, j) \cdot B\big(p + 2 + (-3|p), 13; p, 3\big) \\ & - 2^{-2}3^{-2}C_4'(k, j) \cdot B\big(5 - (-3|p), 7; p, 3\big) \\ & - 3^{-2}C_4''(k, j) \cdot B\big(2 + 2\,(-3|p), 1; p, 3\big), \end{aligned}$$

$$H^{\mathrm{Hol}}_{5,\mathrm{Kl}(p)} = 2^{-3}3^{-2}C_5(k, j) \cdot \big\{p + 2 + (-3|p)\big\} - 2^{-2}3^{-1}C_5'(k, j) \cdot \big\{3 + (-3|p)\big\},$$

$$\begin{aligned} H^{\mathrm{Hol}}_{6,\mathrm{Kl}(p)} = {} & 2^{-7}3^{-1}C_6(k, j) \cdot 5(p + 1)\big\{1 + (-1|p)\big\} \\ & + 2^{-7}3^{-1}C_6'(k, j) \cdot 3(p + 1)\big\{1 + (-1|p)\big\} - 2^{-3}(-1)^{j/2} \cdot \big\{2 + 2\,(-1|p)\big\}, \end{aligned}$$

$$\begin{aligned} H^{\mathrm{Hol}}_{7,\mathrm{Kl}(p)} = {} & 2^{-2}3^{-3}C_7(k, j) \cdot 2(p + 1)\big\{1 + (-3|p)\big\} \\ & + 2^{-2}3^{-3}C_7'(k, j) \cdot (p + 1)\big\{1 + (-3|p)\big\} \\ & - 2^{-1}3^{-1}[1, -1, 0; 3]_j \cdot \big\{2 + 2\,(-3|p)\big\}, \end{aligned}$$

$$H^{\mathrm{Hol}}_{8,\mathrm{Kl}(p)} = 2^{-2}3^{-1}C_8(k, j) \cdot \big\{2 + (-1|p) + (-3|p)\big\},$$

$$H^{\mathrm{Hol}}_{9,\mathrm{Kl}(p)} = 3^{-2}C_9(k, j) \cdot \big\{2 + 2\,(-3|p)\big\},$$

$$H^{\mathrm{Hol}}_{10,\mathrm{Kl}(p)} = 5^{-1}C_{10}(k, j) \cdot \begin{cases} 4 & \text{if } p \equiv 1 \bmod 5, \\ 1 & \text{if } p = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^{\mathrm{Hol}}_{11,\mathrm{Kl}(p)} = 2^{-3} C_{11}(k,j) \cdot \begin{cases} 4 & \textit{if } p \equiv 1 \bmod 8, \\ 1 & \textit{if } p = 2, \\ 0 & \textit{otherwise}, \end{cases}$$

$$H^{\mathrm{Hol}}_{12,\mathrm{Kl}(p)} = -2^{-2}3^{-1} C'_{12}(k,j) \cdot \begin{cases} 4 & \textit{if } p \equiv 1 \bmod 12, \\ 0 & \textit{otherwise}. \end{cases}$$

Theorem A.2. *Let $(l_1, l_2) = (j+k-1, k-2) \in \Xi_1$. Assume that $l_1 - l_2 = j+1$ is odd and $l_2 = k-2 > 2$. Then, we have*

$$m\big(D^{\mathrm{Hol}}_{l_1,l_2}, \mathrm{I}(p)\big) = \sum_{t=1}^{12} H^{\mathrm{Hol}}_{t,\mathrm{I}(p)},$$

$$H^{\mathrm{Hol}}_{1,\mathrm{I}(p)} = 2^{-7}3^{-3}5^{-1} C_1(k,j) \cdot (p+1)^2\big(p^2+1\big) - 2^{-5}3^{-2} C'_1(k,j) \cdot 2(p+1)^2 + 2^{-3}3^{-1}(j+1) \cdot (4p+4) - 2^{-4}3^{-1}(j+1) \cdot (4p+4),$$

$$H^{\mathrm{Hol}}_{2,\mathrm{I}(p)} = 2^{-7}3^{-2} C_2(k,j) \cdot (p+1)^2 B(14, 11; p, 2) - 2^{-4}3^{-1} C'_2(k,j) \cdot (4p+4) + 2^{-5}(-1)^k \cdot B\big(28 - 4\,(-1|p), 30; p, 2\big),$$

$$H^{\mathrm{Hol}}_{3,\mathrm{I}(p)} = 2^{-5}3^{-1} C_3(k,j) \cdot B\big(2(p+1)\big\{1 + (-1|p)\big\}, 9; p, 2\big) - 2^{-3} C'_3(k,j) \cdot \big\{4 + 4\,(-1|p)\big\},$$

$$H^{\mathrm{Hol}}_{4,\mathrm{I}(p)} = 2^{-3}3^{-3} C_4(k,j) \cdot B\big(2(p+1)\big\{1 + (-3|p)\big\}, 16; p, 3\big) - 2^{-2}3^{-2} C'_4(k,j) \cdot B\big(4 + 4\,(-3|p), 8; p, 3\big) - 3^{-2} C''_4(k,j) \cdot B\big(4 + 4\,(-3|p), 2; p, 3\big),$$

$$H^{\mathrm{Hol}}_{5,\mathrm{I}(p)} = 2^{-3}3^{-2} C_5(k,j) \cdot 2(p+1)\big\{1 + (-3|p)\big\} - 2^{-2}3^{-1} C_5(k,j) \cdot \big\{4 + 4\,(-3|p)\big\},$$

$$H^{\mathrm{Hol}}_{6,\mathrm{I}(p)} = 2^{-7}3^{-1} C_6(k,j) \cdot B\big(10(p+1)\big\{1 + (-1|p)\big\}, 33; p, 2\big) + 2^{-7}3^{-1} C'_6(k,j) \cdot B\big(6(p+1)\big\{1 + (-1|p)\big\}, 15; p, 2\big) - 2^{-3}(-1)^{j/2} \cdot \big\{4 + 4\,(-1|p)\big\},$$

$$H^{\mathrm{Hol}}_{7,\mathrm{I}(p)} = 2^{-2}3^{-3} C_7(k,j) \cdot B\big(4(p+1)\big\{1 + (-3|p)\big\}, 20; p, 3\big) + 2^{-2}3^{-3} C'_7(k,j) \cdot B\big(2(p+1)\big\{1 + (-3|p)\big\}, 4; p, 3\big) - 2^{-1}3^{-1}[1, -1, 0; 3]_j \cdot \big\{4 + 4\,(-3|p)\big\},$$

$$H^{\mathrm{Hol}}_{8,\mathrm{I}(p)} = 2^{-2}3^{-1} C_8(k,j) \cdot 2\big\{1 + (-1|p)\big\}\big\{1 + (-3|p)\big\},$$

$$H^{\mathrm{Hol}}_{9,\mathrm{I}(p)} = 3^{-2} C_9(k,j) \cdot B\big(4 + 4\,(-3|p), 2; p, 3\big),$$

$$H^{\mathrm{Hol}}_{10,\mathrm{I}(p)} = 5^{-1} C_{10}(k,j) \cdot \begin{cases} 8 & \textit{if } p \equiv 1 \bmod 5, \\ 1 & \textit{if } p = 5, \\ 0 & \textit{otherwise}, \end{cases}$$

$$H^{\mathrm{Hol}}_{11,\mathrm{I}(p)} = 2^{-3} C_{11}(k,j) \cdot \begin{cases} 8 & \textit{if } p \equiv 1 \bmod 8, \\ 1 & \textit{if } p = 2, \\ 0 & \textit{otherwise}, \end{cases}$$

$$H^{\mathrm{Hol}}_{12,\mathrm{I}(p)} = -2^{-2}3^{-1} C'_{12}(k,j) \cdot \begin{cases} 8 & \textit{if } p \equiv 1 \bmod 12, \\ 0 & \textit{otherwise}. \end{cases}$$

Table B.1
Iwahori-spherical representations of $\mathrm{GSp}(2, F)$.

		Constituent of	Representation	Tempered	L^2	g	SK
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		χ_i, σ unit.		•	
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	χ, σ unit.		•	
	b	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$				•
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(1)}$	π, σ unit.		•	
	b	$(\chi \notin \{1, \nu^{\pm 2}\})$	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(1)}$				
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma \mathrm{St}_{\mathrm{GSp}(2)}$	σ unit.	•	•	
	b		$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(1)})$				
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$				
	d		$\sigma \mathbf{1}_{\mathrm{GSp}(2)}$				
V	a	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	σ unit.	•	•	
	b	$(\xi^2 = 1, \xi \neq 1)$	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$				•
	c		$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$				•
	d		$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$				
VI	a	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	σ unit.		•	
	b		$\tau(T, \nu^{-1/2}\sigma)$	σ unit.			•
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$				•
	d		$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$				

Table B.2
Dimensions of spaces of parahori-invariant vectors.

		Representation	$\mathbf{K}'_p$	$\mathbf{K}_p^{\mathrm{par}\,\prime}$	$\mathbf{K}_p^{\mathrm{kli}\,\prime}$	$\mathbf{K}_p^{\mathrm{sie}\,\prime}$	$\mathbf{K}_p^{\mathrm{iwa}\,\prime}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	1	2	4	4	8
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	2	1	4
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	1	1	2	3	4
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(1)}$	0	0	1	2	4
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(1)}$	1	2	3	2	4
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(2)}$	0	0	0	0	1
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(1)})$	0	0	1	2	3
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	0	1	2	1	3
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(2)}$	1	1	1	1	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	0	0	1	0	2
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	0	1	1	1	2
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	0	1	1	1	2
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	1	0	1	2	2
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	0	1	1	3
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0	0	1	1
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	0	1	1	0	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	1	1	2	2	3

Appendix B. Iwahori-spherical representations and dimensions of spaces of parahori-invariant vectors

In this appendix, we give Tables B.1 and B.2 for the convenience of the reader. Tables B.1 and B.2 are the same as [32, Tables 1 and 3] and [29, Tables A.1 and A.15]. I thank Prof. Ralf Schmidt for allowing me to write his tables.

We shall explain some notations. Note that our definition for GSp(2) is slightly different from that of [29,32]. If we replace J of [29,32] by $\begin{pmatrix} O_2 & I_2 \\ -I_2 & O_2 \end{pmatrix}$, then our definition agrees with that of [29,32]. We set

$$\mathcal{G} = \mathrm{GSp}(2).$$

We define the Borel subgroup B, the Siegel parabolic subgroup P, and the Klingen parabolic subgroup Q as

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \in \mathcal{G} \right\}, \qquad P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathcal{G} \right\}, \qquad Q = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in \mathcal{G} \right\}.$$

We set $F = \mathbb{Q}_p$. Let χ_1, χ_2, and σ be characters of $F^\times$. We denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $\mathcal{G}(F)$ obtained by normalized parabolic induction from the character of $B(F)$ given by

$$\begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & ca^{-1} & 0 \\ 0 & 0 & * & cb^{-1} \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c).$$

Let (π, V) be an admissible representation of $\mathrm{GL}(2, F)$. We also denote by $\pi \rtimes \sigma$ the representation of $\mathcal{G}(F)$ obtained by normalized parabolic induction from the representation of $P(F)$ on V given by

$$\begin{pmatrix} A & * \\ 0 & c\,{}^tA^{-1} \end{pmatrix} \mapsto \sigma(c)\pi(A).$$

Finally, we denote by $\sigma \rtimes \pi$ the representation of $\mathcal{G}(F)$ obtained by normalized parabolic induction from the representation of $Q(F)$ on V given by

$$\begin{pmatrix} t & * & * & * \\ 0 & a & * & b \\ 0 & 0 & t^{-1}(ad-bc) & 0 \\ 0 & c & * & d \end{pmatrix} \mapsto \sigma(t)\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

ν denotes the normalized absolute value on F, ξ denotes the unique non-trivial unramified quadratic character, St denotes the Steinberg representation, **1** denotes the trivial representation, and $L(\ )$ denotes the Langlands quotient. We set

$$\mathbf{K}'_p = \mathrm{GSp}(2, \mathbb{Z}_p), \qquad \mathbf{K}_p^{\mathrm{par}\,\prime} = x_2 M(4, \mathbb{Z}_p) x_2^{-1} \cap \mathrm{GSp}(2, \mathbb{Q}_p), \qquad \mathbf{K}_p^{\mathrm{iwa}\,\prime} = \mathbf{K}_p^{\mathrm{kli}\,\prime} \cap \mathbf{K}_p^{\mathrm{sie}\,\prime},$$
$$\mathbf{K}_p^{\mathrm{kli}\,\prime} = x_2 M(4, \mathbb{Z}_p) x_2^{-1} \cap \mathrm{GSp}(2, \mathbb{Z}_p), \qquad \mathbf{K}_p^{\mathrm{sie}\,\prime} = x_2 M(4, \mathbb{Z}_p) x_1^{-1} \cap \mathrm{GSp}(2, \mathbb{Z}_p)$$

where $x_1 = \mathrm{diag}(1, 1, p, p)$ and $x_2 = \mathrm{diag}(1, 1, p, 1)$.

Table B.1 lists all the irreducible admissible representations of $\mathcal{G}(F)$ supported in $B(F)$. Table B.1 is based on the result of Sally and Tadić [30]. Each Iwahori-spherical representations of $\mathcal{G}(F)$ is realized as a subrepresentation of an induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ with unramified characters χ_1, χ_2, and σ of $F^\times$. Representations in the same group I–VI are constituents of the same induced representation. The "tempered" column gives the precise condition for a representation to be tempered. The "L^2" column indicates which of the tempered representations are square-integrable. The "g" column indicates the generic representations. The "SK" column means that the representation appears as a local component in cusp forms that are CAP with respect to P, which are called Saito–Kurokawa representations. The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$, and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$. Neither IVb nor IVc are unitary. Any cuspidal automorphic representations of $\mathcal{G}$ do not have IVd for all the finite places.

We assume that the characters χ_1, χ_2, and σ are unramified in Table B.2. Table B.2 lists the dimension of the space of parahori-fixed vectors in each irreducible constituent of $\chi_1 \times \chi_2 \rtimes \sigma$. Table B.2 is based on the result of Schmidt [32,33].

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