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PARAMODULAR CUSP FORMS

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ABSTRACT. We classify Siegel modular cusp forms of weight two for the paramodular group $K(p)$ for primes $p < 600$. We find evidence that rational weight two Hecke eigenforms beyond the Gritsenko lifts correspond to certain abelian surfaces defined over $\mathbb{Q}$ of conductor p. The arithmetic classification is in the companion article by A. Brumer and K. Kramer, *Paramodular abelian varieties of odd conductor*. The Paramodular Conjecture, supported by these computations and consistent with the Langlands philosophy and the work of H. Yoshida, *Siegel's modular forms and the arithmetic of quadratic forms*, is a partial extension to degree two of the Shimura-Taniyama Conjecture. These nonlift Hecke eigenforms share Euler factors with the corresponding abelian variety A and satisfy congruences modulo ℓ with Gritsenko lifts, whenever A has rational ℓ-torsion.

1. INTRODUCTION

In 1980, H. Yoshida conjectured that for every abelian surface defined over $\mathbb{Q}$, there exists a discrete group $\Gamma \subseteq \mathrm{Sp}_2(\mathbb{Q})$ and a degree two Siegel modular form of weight two for Γ with the same L-function. He supported this conjecture by constructing lifts and giving specific examples. A broader context for this conjecture may be found in the recent article [37] of H. Yoshida.

Systematic computational evidence for nonlifts required specification of the discrete group of the putative Siegel modular form. The Paramodular Conjecture posits the paramodular group $K(N)$ as the group corresponding to certain abelian surfaces defined over $\mathbb{Q}$ of conductor N. Accordingly, this article studies spaces of Siegel paramodular cusp forms. We believe that the examples given here are the first nonlifts of weight two found. Although we have verified the equality of some Euler factors in our examples with those of abelian surfaces, we have not proven the equality of any L-functions. For a natural number N, the paramodular group $K(N)$ is defined by:

$$K(N) = \mathrm{Sp}_2(\mathbb{Q}) \cap \begin{pmatrix} * & * & */N & * \\ N* & * & * & * \\ N* & N* & * & N* \\ N* & * & * & * \end{pmatrix}, \quad \text{for } * \in \mathbb{Z}.$$

Let $S_2^k(K(N))$ denote the $\mathbb{C}$-vector space of Siegel modular cusp forms of weight k and degree two with respect to the group $K(N)$. A statement of the Paramodular Conjecture for general conductor N, a degree two version of the Shimura-Taniyama Conjecture, may be found in the companion article [5] by A. Brumer and K. Kramer. Here, however, we focus on the simplest case when the conductor is prime.

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1.1. **Paramodular Conjecture for abelian surfaces defined over $\mathbb{Q}$ stated only for prime conductor.** Let p be a prime. There is a bijection between lines of Hecke eigenforms $f \in S_2^2(K(p))$ that have rational eigenvalues and are not Gritsenko lifts and isogeny classes of abelian surfaces $\mathcal{A}$ defined over $\mathbb{Q}$ of conductor p. In this correspondence, we have

$$L(\mathcal{A}, s, \text{Hasse-Weil}) = L(f, s, \text{spin}).$$

In [5], the authors classify many odd numbers N according to whether or not an abelian surface of conductor N could exist; many examples of abelian surfaces are given as well. The first prime is $p = 277$ and the known surfaces of that conductor are all isogenous to the Jacobian $\mathcal{A}_{277}$ of the curve $y^2+y = x^5+5x^4+8x^3+6x^2+2x$. Our results cover primes $p < 600$ and are consistent with the Conjecture 1.1. In particular, there are no rational nonlift Hecke eigenforms where the Paramodular Conjecture indicates that there should be none.

1.2. **Theorem.** *For primes $p < 600$ and not in the set $\{277, 349, 353, 389, 461, 523, 587\}$, $S_2^2(K(p))$ is spanned by Gritsenko lifts.*

Each of the primes in the exceptional set of Theorem 1.2, has a known [5] abelian surface defined over $\mathbb{Q}$ with that prime conductor. For 587 there are actually two known isogeny classes. We prove the existence of a nonlift modular form for $p = 277$.

1.3. **Theorem.** *The subspace of Gritsenko lifts in $S_2^2(K(277))$ has dimension 10 whereas $S_2^2(K(277))$ has dimension 11. There is a rational Hecke eigenform f that is not a Gritsenko lift. The Euler factors of $L(f, s, \text{spin})$ for $q = 2$, 3, 5 and the linear coefficients of the Euler factors for $q = 7$, 11, 13, agree with those of $L(\mathcal{A}_{277}, s, \text{Hasse-Weil})$.*

The abelian surface $\mathcal{A}_{277}$ has rational 15-torsion. We have the following congruence on the modular side.

1.4. **Theorem.** *Let f be as in Theorem 1.3 and be chosen so that $f \in S_2^2(K(277))(\mathbb{Z})$ has Fourier coefficients of unit content. Let the first Fourier Jacobi coefficient of f be $\phi \in J_{2,277}$ and let $R = \text{Grit}(\phi) \in S_2^2(K(277))(\mathbb{Z})$. We have $f \equiv R \mod 15$.*

The above theorems answer, in the case of the paramodular group, the challenge posed in [4, pg. 16] to show that the first nonlift of weight two for prime level occurs at 277. Further examples, all currently conjectural, of weight two paramodular nonlifts and their Hecke eigenvalues and congruences may be found in Section 7; see Table 5 in particular. Please see our website [29] for thousands of Fourier coefficients and further details.

Computing Euler factors of the Hasse-Weil L-series of an abelian surface defined over $\mathbb{Q}$ is tractable because it reduces to counting points over finite fields. By contrast, directly computing Euler factors of the spin L-series of a Siegel modular eigenform requires too many Fourier coefficients to get very far. If the Paramodular Conjecture holds, then the L-series of the weight two nonlift f for $K(277)$ is the first genuine example of an L-series of a Siegel modular form that has ever been seen, in the sense that it is both computationally tractable and not of GL_2-type.

Computations in weight $k = 2$ pose a special challenge since no dimension formula is known. T. Ibukiyama used trace formula techniques [16] to give $\dim S_2^k(K(p))$ for $k \geq 5$ and any prime p. He has recently proven dimension formulae for $k = 3$ and 4; see [19]. The ring structure of $M_2(K(p))$, the graded ring of Siegel modular forms for $K(p)$, has been studied for $p = 2, 3$ and 5 in [17], [8] and [22], respectively. T. Ibukiyama has also proven [18] that $S_2^2(K(p)) = \{0\}$ for primes $p \le 23$. These were hitherto the only systematic computations concerning paramodular cusp forms of weight two.

For paramodular forms of weight two, the standard constructions of Siegel modular forms leave something to be desired. There are no paramodular Eisenstein series of weight two. The useful construction of tracing theta series from $M_2^k(\Gamma_0(p))$ to $M_2^k(K(p))$ always vanishes in weight two. See Theorem 3.5 for the proof of this fact. The Gritsenko lift, $\mathrm{Grit} : J_{2,p}^{\mathrm{cusp}} \to S_2^2(K(p))$, which constructs paramodular forms for $K(p)$ from Jacobi forms of index p, is a nontrivial construction; however, the subspace of lifts in weight two is precisely the uninteresting space in the context of arithmetic geometry. In order to construct nonlifts of weight two, we introduce a method of integral closure in degree two.

Given two linearly independent Gritsenko lifts $g_1, g_2 \in S_2^2(K(N))$, define the space $\mathcal{H}(g_1, g_2) = \{(H_1, H_2) \in S_2^4(K(N)) \times S_2^4(K(N)) : H_1 g_2 = H_2 g_1\}$. The map $\imath_{g_1,g_2} : S_2^2(K(N)) \to \mathcal{H}(g_1, g_2)$ given by $\imath_{g_1,g_2}(f) = (g_1 f, g_2 f)$ injects. If $\dim \mathcal{H}(g_1, g_2) \le \dim J_{2,N}^{\mathrm{cusp}}$ for some choice of g_1, g_2, then $S_2^2(K(N))$ consists entirely of lifts. Suppose, on the other hand, that all choices of g_1, g_2 yield $\dim \mathcal{H}(g_1, g_2) > \dim J_{2,N}^{\mathrm{cusp}}$. For $(H_1, H_2) \in \mathcal{H}(g_1, g_2)$ but not in $\imath_{g_1,g_2} \mathrm{Grit}(J_{2,N}^{\mathrm{cusp}})$, we might hope that the meromorphic $f = H_1/g_1 = H_2/g_2$ is actually holomorphic. In this case we use the initial Fourier expansion of f to search for an $F \in S_2^4(K(N))$ with $f^2 = F$, or $H_1^2 = g_1^2 F$. The validity of a weight 8 identity $H_1^2 = g_1^2 F$ is proof that H_1/g_1 is in the integral closure of $M_2(K(N))$ and is holomorphic.

Thus, to rule out nonlifts of weight two, one spans $S_2^4(K(p))$, while to construct nonlifts one must span $S_2^8(K(p))$ as well. For primes $p < 600$, $S_2^4(K(p))$ was spanned by tracing theta series, by multiplying weight two Gritsenko lifts and by smearing with Hecke operators. The ring structure of $M_2(K(p))$ plays a crucial role in spanning $S_2^4(K(p))$ because products of lifts do not in general span a Hecke stable subspace. This same multiplication of Fourier series, however, is computationally expensive. It was difficult to span $S_2^4(K(479))$ of dimension 440, for example, because the number of Gritsenko lifts in $S_2^2(K(479))$ is relatively small, just 8. Filling $S_2^4(K(479))^+$, which turned out to have dimension 341, required smearing products of Gritsenko lifts by Hecke operators nine times. This required computing 1.9 million Fourier coefficients for each of the 8 weight two Gritsenko lifts. The Fourier coefficients of the Gritsenko lifts were computed using the method of *theta blocks* due to V. Gritsenko, N. Skoruppa and D. Zagier [13]; see Section 4. Finding 99 linearly independent elements in $S_2^4(K(479))^-$ was achieved by theta tracing.

We hope this article will serve as a primer on computations with Siegel modular forms in degree two. Jacobi forms and Gritsenko lifts are computed on a large scale using theta blocks. Finite sets of Fourier coefficients are given that a priori determine both vanishing and congruence, including the troublesome primes two and three. Spaces of cusp forms are spanned, when the dimension is known, by theta tracing, multiplying lifts and smearing with Hecke operators. For small weights, when the dimension is not known, we introduce a method of integral closure in degree two. All potential nonlifts of weight two are found for prime paramodular levels less than 600. The results of this article are used in recent works by Dewar and Richter [9], Choi, Choie and O. Richter [7], Ryan and Tornaría [30], and Ash, Gunnells and McConnell [2], [3]. Some additional evidence for the Paramodular

Conjecture was given in [30], where Ryan and Tornaría developed a version of the Böcherer Conjecture for paramodular groups, proved it for Gritsenko lifts, and used the Paramodular Conjecture and the Fourier coefficients from our website [29] to numerically check their version on the rational nonlifts.

Finite sets of Fourier coefficients that determine vanishing and congruence in $S_2^k(K(p))$ for any $k \in \mathbb{Z}^+$ are given in Section 5. In particular, a generalization of Sturm's Theorem [33] to $n = 2$ may be of independent interest. For $f \in M_n^k$, denote the Fourier coefficients by $a(T; f)$. The Fourier coefficients of a level one elliptic modular form $f \in M_1^k$ are in the $\mathbb{Z}$-module spanned by the first $k/12$: $\forall n, a(n; f) \in \mathbb{Z}\langle a(j; f) : j \le k/12\rangle$. In Theorem 5.15 we prove that for level one Siegel forms $f \in M_2^k$, all the Fourier coefficients $a(T; f)$ are in the $\mathbb{Z}$-module spanned by those whose index T has dyadic trace less than or equal to $k/6$:

$$\forall f \in M_2^k,\, \forall T,\, a(T; f) \in \mathbb{Z}\langle a(S; f) : w(S) \le k/6\rangle.$$

A more natural proof by D. Choi, Y. Choie and T. Kikuta of another congruence criterion that works for primes $p > 3$ may be found in [6]. Section 8 contains examples of nonlifts of higher weight, in particular, weight 3 cusp forms whose construction was requested by A. Ash, P. Gunnells and M. McConnell [2]. We plan a sequel that studies $S_2^k(K(N))$ for composite N and makes more use of Fourier-Jacobi expansions.

2. Notation

For a commutative ring R, let $M_{m\times n}(R)$ denote the R-module of m-by-n matrices with coefficients in R. For $x \in M_{m\times n}(R)$, let $x' \in M_{n\times m}(R)$ denote the transpose. Let $V_n(R) = \{x \in M_{n\times n}(R) : x' = x\}$ be the symmetric n-by-n matrices over R; $V_n(\mathbb{R})$ is a euclidean vector space under the inner product $\langle x, y\rangle = \mathrm{tr}(xy)$. For $R \subseteq \mathbb{R}$, an element $x \in V_n(R)$ is called positive definite, written $x > 0$, when $v'xv > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$; we denote the set of these by $\mathcal{P}_n(R)$. When $x \in V_n(R)$ and $v'xv \ge 0$ for all $v \in \mathbb{R}^n$, we write $x \in \mathcal{P}_n^{\mathrm{semi}}(R)$. The half-integral matrices are $\mathcal{X}_n^{\mathrm{semi}} = \{T \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q}) : \forall\, v \in \mathbb{Z}^n,\, v'Tv \in \mathbb{Z}\}$ and $\mathcal{X}_n = \mathcal{X}_n^{\mathrm{semi}} \cap \mathcal{P}_n(\mathbb{Q})$.

Let $\mathrm{GL}_n(R) = \{x \in M_{n\times n}(R) : \det(x)$ is a unit in $R\}$ be the general linear group and $\mathrm{SL}_n(R) = \{x \in \mathrm{GL}_n(R) : \det(x) = 1\}$ the special linear group. For $x \in \mathrm{GL}_n(R)$ let x^* denote the inverse transpose. Let $I_n \in \mathrm{GL}_n(R)$ be the identity matrix and set $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) \in \mathrm{SL}_{2n}(R)$. The symplectic group is defined by $\mathrm{Sp}_n(R) = \{x \in \mathrm{GL}_{2n}(R) : x'J_nx = J_n\}$. For $T, u \in \mathrm{GL}_n(\mathbb{R})$, we define $T[u] = u'Tu$ and denote the $\mathrm{GL}_n(\mathbb{Z})$ equivalence class of T by $[T] = \bigcup_u T[u]$ for $u \in \mathrm{GL}_n(\mathbb{Z})$. We write $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ and for $R \subseteq \mathbb{R}$ define the group of positive R-similitudes by $\mathrm{GSp}_n^+(R) = \{x \in M_{2n\times 2n}(R) : \exists \nu \in \mathbb{R}^+ : g'J_ng = \nu J_n\}$. Each $\gamma \in \mathrm{GSp}_n^+(R)$ has a unique $\nu = \nu(\gamma) = \det(\gamma)^{1/n}$. For $S \in V_n(R)$, let $t(S) = \left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right)$ define a homomorphism $t : V_n(R) \to \mathrm{Sp}_n(R)$. For $U \in \mathrm{GL}_n(R)$, let $u(U) = \left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right)$ define a homomorphism $u : \mathrm{GL}_n(R) \to \mathrm{Sp}_n(R)$. We let $\Gamma_0(N) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z}) : C \equiv 0 \bmod N\}$, $G\Delta_n^+(R) = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(R)\}$ and $\Delta_n(R) = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(R)\}$. The group $\Gamma_0(N)$ is normalized by the Fricke involution $F_N = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & I_n \\ -N & 0 \end{smallmatrix}\right)$.

Define the Siegel upper half-space $\mathcal{H}_n = \{\Omega \in V_n(\mathbb{C}) : \mathrm{Im}\,\Omega \in \mathcal{P}_n(\mathbb{R})\}$. The group $\mathrm{GSp}_n^+(\mathbb{R})$ acts on $\mathcal{H}_n$ as $\gamma\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1}$ for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. For any function $f : \mathcal{H}_n \to \mathbb{C}$ and any $k \in \mathbb{Z}$, we follow Andrianov and, letting

TABLE 1. Dimensions for weight 4 paramodular cusp forms.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
$\dim S_2^4(K(p))$	0	0	0	1	1	2	2	3	3	4	6	8	7	9	8	10	11	16	17

$\langle n\rangle = n(n+1)/2$, define the group action for $\gamma \in \mathrm{GSp}_n^+(R)$ by

$$(f|_k\gamma)(\Omega) = \nu(\gamma)^{kn-\langle n\rangle}\det(C\Omega+D)^{-k} f(\gamma\langle\Omega\rangle).$$

Let Γ be a group commensurable with Γ_n. The complex vector space of Siegel modular forms of degree n and weight k automorphic with respect to Γ is denoted by $M_n^k(\Gamma)$ and is defined as the set of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and such that for all $Y_0 \in \mathcal{P}_n(\mathbb{R})$ and for all $\gamma \in \Gamma_n$, $f|\gamma$ is bounded on $\{\Omega \in \mathcal{H}_n : \operatorname{Im}\Omega > Y_0\}$. For $f \in M_n^k(\Gamma)$ the Siegel Φ-map is defined by $(\Phi f)(\Omega) = \lim_{\lambda\to+\infty} f(\left(\begin{smallmatrix} i\lambda & 0\\ 0 & \Omega\end{smallmatrix}\right))$ and the space of cusp forms is defined by $S_n^k(\Gamma) = \{f \in M_n^k(\Gamma) : \forall\gamma \in \Gamma_n, \Phi(f|\gamma) = 0\}$. The graded ring of Siegel modular forms is $M_n(\Gamma) = \bigoplus_k M_n^k(\Gamma)$ and the graded ideal of cusp forms is $S_n(\Gamma) = \bigoplus_k S_n^k(\Gamma)$.

Let $e(z) = e^{2\pi i z}$. By the Koecher principle, $f \in S_n^k(\Gamma)$ has a Fourier expansion

$$f(\Omega) = \sum a(T; f)e(\langle T, \Omega\rangle), \text{ also written } \mathrm{FS}_n(f) = \sum a(T; f)q^T,$$

where the summation is over $T \in \mathcal{P}_n(\mathbb{Q})$. The $a(T; f)$ satisfy

$$a(T[U]; f) = \det(U)^k a(T; f)$$

for all $U \in \mathrm{GL}_n(\mathbb{Q})$ such that $u(U) \in \Gamma$. We let $\mathrm{supp}(f) = \{T \in \mathcal{P}_n(\mathbb{Q}) : a(T; f) \neq 0\}$. The set $\mathrm{supp}(f)$ is contained in the lattice dual to $\{S \in V_n(\mathbb{Q}) : t(S) \in \Gamma\}$. The integrality properties of $\mathrm{supp}(f)$ are sometimes important. For $f \in S_n^k(\Gamma_0(N))$, we have $\mathrm{supp}(f) \subseteq \mathcal{X}_n$ and $a(T[U]; f) = \det(U)^k a(T; f)$ for all $U \in \mathrm{GL}_n(\mathbb{Z})$. Let ${}^N\mathcal{X}_2 = \{\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right) \in \mathcal{X}_2 : N|a\}$ and ${}^N\mathcal{X}_2^{\mathrm{semi}} = \{\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right) \in \mathcal{X}_2^{\mathrm{semi}} : N|a\}$. For $f \in S_2^k(K(N))$, we have $\mathrm{supp}(f) \subseteq {}^N\mathcal{X}_2$ and $a(T[U]; f) = \det(U)^k a(T; f)$ for all $U \in \hat{\Gamma}_0(N)$, where $\hat{\Gamma}_0(N) = \langle \Gamma_0(N), \left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right)\rangle$. The paramodular groups satisfy $\mathrm{Sp}_2(\mathbb{Q}) = K(p)\Delta_2(\mathbb{Q})$ so that there is essentially only one Fourier expansion to consider. For primes p, we have $\mathrm{Sp}_2(\mathbb{Q}) = \Gamma_0(p)\Delta_2(\mathbb{Q}) \cup \Gamma_0(p)E_1\Delta_2(\mathbb{Q}) \cup \Gamma_0(p)J_2\Delta_2(\mathbb{Q})$ with $E_1 = I_2 \boxplus J_1$, so that there are 3 basic Fourier expansions to consider for $\Gamma_0(p)$. Here, the pseudo-direct sum of two-by-two matrices, $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \boxplus \left(\begin{smallmatrix} \alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, means the four-by-four matrix $\left(\begin{smallmatrix} a & 0 & b & 0\\ 0 & \alpha & 0 & \beta\\ c & 0 & d & 0\\ 0 & \gamma & 0 & \delta\end{smallmatrix}\right)$. For square matrices A and B, we write $A \oplus B = \left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right)$.

3. Paramodular forms

For weights $k \geq 5$, the dimensions $\dim S_2^k(K(p))$ have been given in [16]; T. Ibukiyama [19] has recently proven the dimensions for weights $k = 3$ and 4. There are many ways to construct modular forms. The difficulty is knowing when to stop and thus these dimension formulae are powerful.

3.1. **Theorem** (T. Ibukiyama)**.** *Let $p \geq 5$ be a prime number:*

$$\dim S_2^4(K(p)) = \frac{p^2}{576} + \frac{p}{8} - \frac{143}{576} + \left(\frac{p}{96} - \frac{1}{8}\right)\left(\frac{-1}{p}\right) + \frac{1}{8}\left(\frac{2}{p}\right) + \frac{1}{12}\left(\frac{3}{p}\right) + \frac{p}{36}\left(\frac{-3}{p}\right).$$

Since weight one paramodular forms are trivial, compare Satz 1 in [32], only the dimensions for weight two cusp forms remain unknown for prime level.

For a group G, let G_1 and G_2 be subgroups that satisfy the following finiteness condition: $\forall g \in G, |G_1 \backslash G_1 g G_2| < +\infty$. Let $L(G_1, G)$ be the $\mathbb{C}$-vector space with a basis given by the left cosets $\bigcup_{g \in G}\{G_1 g\}$. The subgroup G_2 has a right action $L(G_1, G) \times G_2 \to L(G_1, G)$ given on basis elements by $(G_1 g, g_2) \mapsto G_1 g g_2$ and extended linearly. Denote the fixed subspace of G_2 by $H(G_1, G, G_2) = \{x \in L(G_1, G) : \forall g_2 \in G_2, x g_2 = x\}$, this is the space of Hecke operators for the triple (G_1, G, G_2). For a disjoint union $G_1 g G_2 = \bigcup_i G_1 g_i$, set $[G_1 g G_2] = \sum_i G_1 g_i \in H(G_1, G, G_2)$; then $H(G_1, G, G_2)$ is generated by these double cosets. We may check that the multiplication of Hecke operators $H(G_1, G, G_2) \times H(G_2, G, G_3) \to H(G_1, G, G_3)$ given by $(\sum_i G_1 g_i, \sum_j G_2 h_j) \mapsto \sum_{i,j} G_1 g_i h_j$ is well-defined. For $G = \mathrm{GSp}_n^+(\mathbb{Q})$ and subgroups G_i commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, the necessary finiteness condition is satisfied. We have an action of the Hecke operators on Siegel modular forms $M_n^k(G_1) \times H(G_1, G, G_2) \to M_n^k(G_2)$ given by mapping $(f, \sum_i G_1 g_i) \mapsto \sum_i f|_k g_i$. Hecke operators send cusp forms to cusp forms because of the factorization $\mathrm{GSp}_n^+(\mathbb{Q}) = \mathrm{Sp}_n(\mathbb{Z})\, G \Delta_n^+(\mathbb{Q})$. For $G_1 = G_2$, we have the Hecke algebra $H(G_1, G) = H(G_1, G, G_1)$ acting on $M_n^k(G_1)$. If $w \in G$ normalizes G_1, then the single coset $G_1 w = [G_1 w G_1]$ is a useful Hecke operator that is often just abbreviated by w. We use $B(N)$ to denote the Iwahori subgroup of $K(N)$,

$$B(N) = \mathrm{Sp}_2(\mathbb{Z}) \cap \begin{pmatrix} * & * & * & * \\ N* & * & * & * \\ N* & N* & * & N* \\ N* & N* & * & * \end{pmatrix}, \quad \text{for } * \in \mathbb{Z}.$$

For a description of the Hecke operators for the group $B(p)$, we refer to [15] and list the results here for the reader's convenience. The Hecke operator T_m is defined by the double coset $\{\gamma \in \mathrm{GSp}_n^+(\mathbb{Z}) : \nu(\gamma) = m\}$. For $f \in M_2^k(B(p))$ and $\left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) \in \mathcal{X}_2$ we have:

$$a\left(\left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) f | T_{q^\delta}\right) = \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{Z}: \alpha+\beta+\gamma=\delta; \\ \alpha,\beta,\gamma \geq 0}} \sum_{\substack{u \in R(q^\beta): a_u \equiv 0 \bmod q^{\beta+\gamma}; \\ b_u \equiv c_u \equiv 0 \bmod q^\gamma}} a\left(q^\alpha \left(\begin{smallmatrix} a_u q^{-\beta-\gamma} & b_u q^{-\gamma}/2 \\ b_u q^{-\gamma}/2 & c_u q^{\beta-\gamma} \end{smallmatrix}\right); f\right),$$

where $\left(\begin{smallmatrix} a_u & b_u/2 \\ b_u/2 & c_u \end{smallmatrix}\right) = u' \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) u$ and $R(q^\beta) \subseteq \Gamma_0(p)$ is any lift of $\mathbb{P}(\mathbb{Z}/q^\beta\mathbb{Z})$ under the map $\left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right) \mapsto (u_1, u_2)$. The operator T_q, for $(q, p) = 1$, is then given by $a\,(T; f|T_q) =$

$$a\,(qT; f) + q^{2k-3}\, a\left(\tfrac{1}{q}T; f\right) + q^{k-2} \sum_{j \bmod q} a\left(\tfrac{1}{q} T \left[\begin{smallmatrix} 1 & 0 \\ jp & q \end{smallmatrix}\right]; f\right) + q^{k-2}\, a\left(\tfrac{1}{q} T \left[\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right]; f\right).$$

The same formulas apply to $f \in M_2^k(K(p))$ because the number of single cosets in the double coset is the same. For an eigenform f, with eigenvalues $f|T(q^\delta) = \lambda_{q^\delta} f$, we use the spin Euler factor $Q_q(f; x)$ given for a q prime to the level by (this is the palindrome of the factor in [15]):

$$Q_q(f, x) = 1 - \lambda_q x + (\lambda_q^2 - \lambda_{q^2} - q^{2k-4}) x^2 - \lambda_q\, q^{2k-3} x^3 + q^{4k-6} x^4.$$

Following the work of Andrianov [1], the spin L-function is given, for $\mathrm{Re}(s) \gg 0$, by

$$L(f,s,\mathrm{spin}) = \prod_{\text{primes } q} Q_q(f, q^{-s})^{-1}.$$

Theta series give us modular forms on $\Gamma_0(p)$ and we can use the Hecke operator Tr, given below, to obtain modular forms on $K(p)$. In weight two these theta series trace to zero but for weights $k > 2$ we can use this method to construct paramodular forms. For even weights, this is the only method we have that constructs paramodular forms in the minus space. The plus and minus spaces of $S_2^k(K(p))$ are defined as follows. Define elements μ and $\tilde{\mu}$ as below. We note that μ normalizes $K(p)$; since $\mu^2 = -I_4$ the space $S_2^k(K(p))$ decomposes into μ-eigenspaces with eigenvalues ± 1 and we set $S_2^k(K(p))^{\pm} = \{f \in S_2^k(K(p)) : f|\mu = \pm f\}$.

$$\mu = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad \tilde{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{p} & 0 \end{pmatrix}.$$

Note that $\Gamma_0(N)$ is not a subgroup of $K(N)$; we let $\Gamma_0'(N) = K(N) \cap \mathrm{Sp}_2(\mathbb{Z})$.

3.2. **Theorem.** *Let the following double cosets define Hecke operators:*

$$[B(p)\Gamma_0'(p)] : M_2^k\left(B(p)\right) \to M_2^k\left(\Gamma_0'(p)\right) \text{ and}$$
$$[\Gamma_0'(p)K(p)] : M_2^k\left(\Gamma_0'(p)\right) \to M_2^k\left(K(p)\right).$$

We define $\mathrm{Tr} : M_2^k\left(\Gamma_0(p)\right) \to M_2^k\left(K(p)\right)$ *by* $\mathrm{Tr} = |_{M_2^k(\Gamma_0(p))}\,[B(p)\Gamma_0'(p)]\,[\Gamma_0'(p)K(p)]$. *For all* $f \in M_2^k\left(\Gamma_0(p)\right)$, *the Hecke operator* Tr *satisfies*

$$f|\,\mathrm{Tr} = \sum_{\beta \bmod p} f|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} + (f|E_1)\,|\tilde{\mu} + \sum_{\alpha,\beta \bmod p} (f|E_1)\,|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix}$$
$$+ \sum_{\alpha \bmod p} (f|F_p)\,|\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right).$$

Furthermore, we have $F_p\,\mathrm{Tr} = \mathrm{Tr}\,\mu$ *as Hecke operators in* $H\left(\Gamma_0(p), \mathrm{GSp}_n^+(\mathbb{Q}), K(p)\right)$.

Proof. From [15] we have $\Gamma_0'(p) = B(p) \cup B(p)s_2B(p)$ where

$$s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

A calculation shows that

$$s_2^{-1}B(p)s_2 \cap B(p) = \mathrm{Sp}_2(\mathbb{Z}) \cap \begin{pmatrix} * & * & * & * \\ p* & * & * & p* \\ p* & p* & * & p* \\ p* & p* & * & * \end{pmatrix}, \quad \text{for } * \in \mathbb{Z}.$$

Thus we have

$$B(p) = \bigcup_{\alpha \bmod p} \left(s_2^{-1}B(p)s_2 \cap B(p)\right) t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right) \text{ and hence}$$
$$B(p)s_2B(p) = \bigcup_{\alpha \bmod p} B(p)s_2\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right).$$

Thus $[B(p)\Gamma_0'(p)] = B(p) + \sum_{\alpha \mod p} B(p) s_2\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ as a Hecke operator. For the coset $\Gamma_0'(p)\backslash K(p)$, it follows close upon the definitions that

$$K(p) = \Gamma_0'(p)J(p) \cup \bigcup_{\beta \mod p} \Gamma_0'(p)t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} \text{ with the notation } J(p) = \begin{pmatrix} 0 & 0 & \frac{1}{p} & 0 \\ 0 & 0 & 0 & 1 \\ -p & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and hence that $[\Gamma_0'(p)K(p)] = \Gamma_0'(p)J(p) + \sum_{\beta \mod p} \Gamma_0'(p)t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix}$ as a Hecke operator. By the definition of multiplication of Hecke operators we have

$$\begin{aligned} \mathrm{Tr} &= \Gamma_0(p)\left(I + \sum_{\alpha \mod p} s_2\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right)\right)\left(J(p) + \sum_{\beta \mod p} t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \Gamma_0(p)\left(J(p) + \sum_{\beta \mod p} t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\alpha \mod p} s_2\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right) J(p)\right. \\ &\qquad\qquad \left. + \sum_{\alpha,\beta \mod p} s_2\, t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix}\right). \end{aligned}$$

$\Gamma_0(p)$ has three cusps, which we represent by I, $E_1 = s_2^{-1}$ and F_p. After determining the cusp, we select a simple upper triangular element for the $\Gamma_0(p)$-coset. We have

$$J(p) = u\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) F_p\mu \in \Gamma_0(p)\, F_p\mu.$$

For $\alpha \not\equiv 0 \mod p$, there is a $\tau \in \mathbb{Z}$ such that $\alpha\tau + 1 \equiv 0 \mod p$ and we have

$$s_2\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right) J(p) \in \Gamma_0(p)\, F_p\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \tau \end{smallmatrix}\right) \text{ for } \alpha\tau + 1 \equiv 0 \mod p.$$

For $\alpha \equiv 0 \mod p$, we have $s_2 J(p) \in \Gamma_0(p)\, E_1\tilde{\mu}$. If we note $s_2 \in \Gamma_0(p)E_1$, then the formula given for Tr follows.

Now we show $F_p\,\mathrm{Tr} = \mathrm{Tr}\,\mu$. Using the identity $\mu\, t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} \alpha = t\begin{pmatrix} \frac{\alpha}{p} & 0 \\ 0 & \beta \end{pmatrix} \mu$ we can see that the I_4 and F_p cusps swap:

$$\begin{aligned} F_p\left(\Gamma_0(p) \sum_{\beta \mod p} t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix}\right) &= \Gamma_0(p)F_p\mu \sum_{\beta \mod p} \mu t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left(\Gamma_0(p)F_p\mu \sum_{\beta \mod p} t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta \end{smallmatrix}\right)\right)\mu. \end{aligned}$$

To see that the E_1 cusp is stabilized, note that $F_pE_1 = -u\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) E_1\mu \in \Gamma_0(p)\, E_1\mu$ and therefore we have $F_p\,\Gamma_0(p)E_1 t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix} = \Gamma_0(p)E_1\mu t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix} = \Gamma_0(p)E_1 t\begin{pmatrix} \frac{\alpha}{p} & 0 \\ 0 & \beta \end{pmatrix}\mu$. Furthermore, we have $F_p\,\Gamma_0(p)E_1\tilde{\mu} = \Gamma_0(p)\, E_1\mu\tilde{\mu} = \Gamma_0(p)\, E_1\tilde{\mu}\,\mu$. □

In order to use the formula for Tr in Theorem 3.2 to provide Fourier expansions of paramodular forms, we must be able to expand a theta series at each of the three cusps: I_4, E_1 and J_2. General formulas from Andrianov [1, Prop 3.14] may be simplified to give the following results.

3.3. **Theorem.** *Let $k, q \in \mathbb{N}$. Let Q be a $2k$-by-$2k$ even quadratic form with qQ^{-1} even. Let $N(Q) = \{h \in \mathbb{Z}^{2k}/q\mathbb{Z}^{2k} : Qh = 0 \mod q\}$ be the $\mathbb{Z}/q\mathbb{Z}$-nullspace of Q.*

Define $\vartheta^Q[T] : \mathcal{H}_g \to \mathbb{C}$ *for* $T \in \mathbb{Z}^{2k\times g}/q\mathbb{Z}^{2k\times g}$ *by*

$$\vartheta^Q[T](\Omega) = \sum_{N\in\mathbb{Z}^{2k\times g}} e\left(\frac{1}{2}\langle Q[N+\tfrac{1}{q}T], \Omega\rangle\right).$$

For $g = 2$*, we have the following expansions at the other two cusps,*

$$\vartheta^Q|E_1 = i^k \det(Q)^{-1/2} \sum_{h\in N(Q)} \vartheta^Q[0,h],$$

$$\vartheta^Q|F_q = (-1)^k \det(Q)^{-1} q^k\, \vartheta^{qQ^*}.$$

Alternatively, we may use

$$\vartheta^Q|J_2 = \det(Q)^{-1} \sum_{a,b\in N(Q)} \vartheta^Q[a,b].$$

For small weights, in order to find the linear combinations of the $\vartheta^Q|\operatorname{Tr}$ that are cusp forms, it suffices to cancel the constant term. To explain this we need the following lemma.

3.4. **Lemma.** *The Witt map* $W : M_2(K(N)) \to M_1|\left(\begin{smallmatrix} N & 0\\ 0 & 1\end{smallmatrix}\right) \otimes M_1$*, given by*

$$(Wf)(\tau_1,\tau_2) = f\left(\begin{smallmatrix} \tau_1 & 0\\ 0 & \tau_2\end{smallmatrix}\right),$$

is a weight preserving homomorphism of graded rings.

Proof. This follows from the fact that $\left(\left(\begin{smallmatrix} N & 0\\ 0 & 1\end{smallmatrix}\right)^{-1} \mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} N & 0\\ 0 & 1\end{smallmatrix}\right)\right) \boxplus \mathrm{SL}_2(\mathbb{Z}) \subseteq K(N)$. □

Note that for $f \in M_2^k(K(N))$, the vanishing of Wf immediately implies the vanishing of Φf. Furthermore, for $k < 12$, the nontrivial M_1^k are spanned by a single Eisenstein series and so Wf vanishes precisely when the Fourier expansion of f has a constant term of zero. We will use this fact to construct paramodular cusp forms of weight four.

The trace Tr from $M_2(\Gamma_0(p))$ to $M_2(K(p))$ cannot be used on theta series to construct weight two forms. Indeed, Tr is identically zero on theta series in $M_2^2(\Gamma_0(p))$.

3.5. **Theorem.** *Let* p *be an odd prime. Let* Q *be a* 4*-by-*4 *even quadratic form of level* p *and square determinant. In degree two we have* $\vartheta^Q|\operatorname{Tr} = 0$*.*

Since $M_2^2(K(2)) = \{0\}$, the above theorem also holds for $p = 2$; see [17]. The remainder of this section is devoted to proving Theorem 3.5 and its consequences.

3.6. **Lemma.** *Let* p *be an odd prime. Let* Q *be a* 4*-by-*4 *even quadratic form of level* p *and square determinant. Then* $\det(Q) = p^2$*, the Hasse invariant of* Q *is* $(-1,-1)$ *and* Q *is equivalent over* $\mathbb{Q}$ *to* $\mathrm{Diag}(1, \mathrm{u}, p, \mathrm{u}\,p)$*, where the resdiue class of* u *in* $\mathbb{F}_p$ *is a nonsquare unit. Furthermore,* Q *has the property*

$$\forall b \in \mathbb{Z}^4,\quad b'Qb \equiv 0 \mod p^2 \iff b \equiv 0 \mod p. \tag{3.7}$$

Proof. This is elementary, compare page 152 of [24]. □

The notion of twinning is used in the proof of Theorem 3.5. Recall the Fricke involution in degree one: $F_p = \frac{1}{\sqrt{p}}\left(\begin{smallmatrix} 0 & 1\\ -p & 0\end{smallmatrix}\right)$.

3.8. **Definition.** For $T = \left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right) \in \mathcal{P}_2(\mathbb{Q})$, define $\mathrm{Twin}(T) = F_p' T F_p = \left(\begin{smallmatrix} pc & -b\\ -b & \frac{a}{p}\end{smallmatrix}\right) \in \mathcal{P}_2(\mathbb{Q})$.

Notice that twinning stabilizes ${}^p\mathcal{X}_2$ and respects the $\Gamma_0(p)$-equivalence class of T. Twins do have the same determinant but the $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes may differ. For example, $T = \left(\begin{smallmatrix} 10 & 5 \\ 5 & 4 \end{smallmatrix}\right) \in {}^5\mathcal{X}_2$ has $\mathrm{Twin}(T) = \left(\begin{smallmatrix} 20 & 5 \\ 5 & 2 \end{smallmatrix}\right) \in {}^5\mathcal{X}_2$ but $T = \left(\begin{smallmatrix} 10 & 5 \\ 5 & 4 \end{smallmatrix}\right) \in [\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)]$ whereas $\mathrm{Twin}(T) = \left(\begin{smallmatrix} 20 & 5 \\ 5 & 2 \end{smallmatrix}\right) \in [\left(\begin{smallmatrix} 2 & 1 \\ 1 & 8 \end{smallmatrix}\right)]$. The following elements twin and rescale the Fourier coefficients of a Siegel modular form:

$$\mu = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad \tilde{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{p} & 0 \end{pmatrix}; \quad \kappa = \begin{pmatrix} 0 & \frac{1}{p} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

3.9. **Lemma.** *Let $f \in M_2^k(\Gamma)$ have Fourier coefficients $a(T; f)$:*

$$\begin{aligned} a(T; f|\mu) &= a(\mathrm{Twin}(T); f), & \mathrm{supp}(f|\mu) &= \mathrm{Twin}(\mathrm{supp}(f)), \\ a(T; f|\tilde{\mu}) &= p^k a\left(\frac{1}{p}\mathrm{Twin}(T); f\right), & \mathrm{supp}(f|\tilde{\mu}) &= p\,\mathrm{Twin}(\mathrm{supp}(f)), \\ a(T; f|\kappa) &= p^{-k} a(p\,\mathrm{Twin}(T); f), & \mathrm{supp}(f|\kappa) &= \frac{1}{p}\mathrm{Twin}(\mathrm{supp}(f)). \end{aligned}$$

If f has μ-eigenvalue ± 1, then $a(\mathrm{Twin}(T); f) = \pm a(T; f)$, so that the μ-eigenspace is determined by the twins. It is useful to define a type of projection to ${}^p\mathcal{X}_2$.

3.10. **Definition.** Define $\pi : M_2(\Gamma(p)) \to \mathcal{O}(\mathcal{H}_2)$ by: If

$$f(\Omega) = \sum_{T \in \frac{1}{p}\mathcal{X}_2^{\mathrm{semi}}} a(T) e(\langle \Omega, T \rangle),$$

then $(\pi f)(\Omega) = \sum_{T \in {}^p\mathcal{X}_2^{\mathrm{semi}}} a(T) e(\langle \Omega, T \rangle)$.

For $f \in M_2(\Gamma_0(p))$, the four terms in $f|\,\mathrm{Tr}$,

$$\sum_{\beta \bmod p} f|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} + (f|E_1)|\tilde{\mu} + \sum_{\alpha,\beta \bmod p} (f|E_1)|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix} + \sum_{\alpha \bmod p} (f|F_p)|\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right),$$

have the following images under π:

$$\pi\left(\sum_{\beta \bmod p} f|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix}\right) = p\,\pi(f), \qquad \pi(f|E_1\tilde{\mu}) = f|E_1\tilde{\mu},$$

$$\pi\left(\sum_{\alpha,\beta \bmod p} (f|E_1)|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix}\right) = p^2\pi(f|E_1),$$

$$\pi\left(\sum_{\alpha \bmod p} (f|F_p)|\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right)\right) = p\,\pi(f|F_p\mu).$$

We will see that, when applied to weight two theta series, the sum of the first and third terms, and the the sum of the second and fourth terms, cancel separately.

3.11. **Lemma.** *Let p be an odd prime. Let Q be a 4-by-4 even quadratic form of level p and square determinant. For $b \in N(Q)$, $\pi\left(\vartheta^Q[0, b]\right) = 0$ unless $b \equiv 0 \bmod p$. We have*

$$\pi\left(\vartheta^Q|E_1\right) = -\frac{1}{p}\pi\left(\vartheta^Q\right) \text{ and } \sum_{\beta \bmod p} \vartheta^Q|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\alpha,\beta \bmod p} \left(\vartheta^Q|E_1\right)|t\begin{pmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{pmatrix} = 0.$$

Proof. We have $\vartheta^Q[0,b](\Omega) = \sum_{c,d\in\mathbb{Z}^2} e\left(\frac{1}{2}\langle \left(\begin{smallmatrix} c \\ d+b/p \end{smallmatrix}\right)' Q \left(\begin{smallmatrix} c \\ d+b/p \end{smallmatrix}\right), \Omega\rangle\right)$. We may write this as $\vartheta^Q[0,b](\Omega) = \sum_{c,d\in\mathbb{Z}^2} e\left(\frac{1}{2}\langle T,\Omega\rangle\right)$ for

$$T = \begin{pmatrix} c'Qc & c'Qd + \frac{1}{p}b'Qc \\ c'Qd + \frac{1}{p}b'Qc & d'Qd + \frac{2}{p}b'Qd + \frac{1}{p^2}b'Qb \end{pmatrix}.$$

If $b \in N(Q)$ and $T \in {}^p\mathcal{X}_2$, then $b'Qb \in p^2\mathbb{Z}$ and so by property (3.7) we have $b \equiv 0 \mod p$.

For the second part, notice $\vartheta^Q|E_1 = -\frac{1}{p}\sum_{b\in N(Q)} \vartheta^Q[0,b]$ by Theorem 3.3 so that we have $\pi\left(\vartheta^Q|E_1\right) = -\frac{1}{p}\pi\left(\vartheta^Q\right)$. To prove the third part, notice that

$$\sum_{\beta \mod p} \vartheta^Q|t\left(\begin{smallmatrix} \frac{\beta}{p} & 0 \\ 0 & 0 \end{smallmatrix}\right) + \sum_{\alpha,\beta \mod p} \left(\vartheta^Q|E_1\right)|t\left(\begin{smallmatrix} \frac{\beta}{p} & 0 \\ 0 & \alpha \end{smallmatrix}\right)$$

is already supported on ${}^p\mathcal{X}_2$. Therefore, it is equal to its projection, which is $p\,\pi\left(\vartheta^Q\right) + p^2\pi\left(\vartheta^Q|E_1\right) = p\,\pi\left(\vartheta^Q\right) + p^2\left(-\frac{1}{p}\pi\left(\vartheta^Q\right)\right) = 0.$ □

3.12. **Lemma.** *Let p be an odd prime. Let Q be a 4-by-4 even quadratic form of level p and square determinant. For $a,b \in N(Q)$, $\pi\left(\vartheta^Q[a,b]|\tilde{\mu}\right) = 0$ unless $a \equiv 0 \mod p$. We have*

$$\pi\left(\vartheta^Q|F_p\mu\right) = -\frac{1}{p}\pi\left(\vartheta^Q|E_1\tilde{\mu}\right) \text{ and } \left(\vartheta^Q|E_1\right)|\tilde{\mu} + \sum_{\alpha \mod p}\left(\vartheta^Q|F_p\right)|\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right) = 0.$$

Proof. We have $\vartheta^Q[a,b](\Omega) = \sum_{c,d\in\mathbb{Z}^2} e\left(\frac{1}{2}\langle \left(\begin{smallmatrix} c+a/p \\ d+b/p \end{smallmatrix}\right)' Q \left(\begin{smallmatrix} c+a/p \\ d+b/p \end{smallmatrix}\right), \Omega\rangle\right)$. We may write this as $\vartheta^Q[a,b](\Omega) = \sum_{c,d\in\mathbb{Z}^2} e\left(\frac{1}{2}\langle T,\Omega\rangle\right)$ for

$$T = \begin{pmatrix} c'Qc + \frac{2}{p}a'Qc + \frac{1}{p^2}a'Qa & * \\ * & * \end{pmatrix}.$$

We have $\pi\left(\vartheta^Q[a,b]|\tilde{\mu}\right) = 0$ unless $\operatorname{supp}(\vartheta^Q[a,b]|\tilde{\mu}) \cap {}^p\mathcal{X}_2$ is nonempty. However, since $\operatorname{supp}(\vartheta^Q[a,b]|\tilde{\mu}) = p\operatorname{Twin}(\operatorname{supp}(\vartheta^Q[a,b]))$, this is equivalent to $T \in \operatorname{supp}(\vartheta^Q[a,b]) \cap \frac{1}{p}({}^p\mathcal{X}_2)$. From $a \in N(Q)$ and $T \in \frac{1}{p}({}^p\mathcal{X}_2)$ we derive $a'Qa \equiv 0 \mod p^2$. By property (3.7) we have $a \equiv 0 \mod p$.

For the second part, note that $\vartheta^Q|F_p\mu = \vartheta^Q|J_2\tilde{\mu} = \frac{1}{p^2}\sum_{a,b\in N(Q)} \vartheta^Q[a,b]|\tilde{\mu}$ by Theorem 3.3. We have

$$\pi\left(\vartheta^Q|F_p\mu\right) = \frac{1}{p^2}\pi\left(\sum_{a,b\in N(Q)} \vartheta^Q[a,b]|\tilde{\mu}\right) = \frac{1}{p^2}\pi\left(\sum_{b\in N(Q)} \vartheta^Q[0,b]|\tilde{\mu}\right)$$

by the first part. Recognizing the formula for $\vartheta^Q|E_1$ from Theorem 3.3, we may conclude that $\pi\left(\vartheta^Q|F_p\mu\right) = \frac{1}{p^2}\pi\left(-p\vartheta^Q|E_1|\tilde{\mu}\right) = -\frac{1}{p}\pi\left(\vartheta^Q|E_1\tilde{\mu}\right)$.

To prove the third part, notice that $\left(\vartheta^Q|E_1\right)|\tilde{\mu} + \sum_{\alpha \mod p}\left(\vartheta^Q|F_p\right)|\mu\, t\left(\begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ is already supported on ${}^p\mathcal{X}_2$. Therefore, it is equal to its projection, which is $\pi\left(\vartheta^Q|E_1\tilde{\mu}\right) + p\,\pi\left(\vartheta^Q|F_p\mu\right) = \pi\left(\vartheta^Q|E_1\tilde{\mu}\right) + p\left(-\frac{1}{p}\pi\left(\vartheta^Q|E_1\tilde{\mu}\right)\right) = 0.$ □

Proof of Theorem 3.5. We add the final conclusions from Lemmas 3.11 and 3.12. □

Here is a consequence of Theorem 3.5 that is useful for proving the integrality of modular forms constructed by theta tracing.

3.13. **Theorem.** *Let p be an odd prime. Let $A, B \in \mathcal{P}_4(\mathbb{Z})$ be even with determinant p^2 and pA^{-1}, pB^{-1} both even. The Fourier coefficients of* $\mathrm{Tr}(\vartheta^{A\oplus B})$ *are multiples of* 4.

Proof. Let $Q = A \oplus B$, an 8×8 matrix. First, by the formulas in Theorems 3.2, 3.3 and Lemma 3.9, the constant term of $\mathrm{Tr}(\vartheta^Q)$ is $p + p^4/p^2 + p^2/p^2 + p = (p+1)^2$ which is a multiple of 4. We now show that $a(T; \mathrm{Tr}(\vartheta^Q)) \equiv 0 \mod 4$ for the nonzero $T \in {}^p\mathcal{X}_2$. By Theorem 3.5, we have that $\mathrm{Tr}(\vartheta^A) = 0 = \mathrm{Tr}(\vartheta^B)$. For $f = \vartheta^Q - \vartheta^A - \vartheta^B$ and $g = \mathrm{Tr}(f)$, it suffices to prove that $a(T; g) \equiv 0 \mod 4$. We have

$$a(T; g) = a(T; \mathrm{Tr}(f)) = p\, a(T; f) + a(T; f|E_1\tilde{\mu}) + p^2\, a(T; f|E_1) + p\, a(T; f|F_p\mu).$$

We will show that the Fourier coefficients of each of the four terms above are, individually, multiples of 4. When using the slash operator, it must be remembered that $f \in M_2^2(K(p)) \oplus M_2^4(K(p))$ is a sum of forms of different weights. Considering the first term, we wish to show that

$$p\, a(T; \vartheta^Q) - p\, a(T; \vartheta^A) - p\, a(T; \vartheta^B) \equiv 0 \mod 4.$$

We have

$$\begin{aligned}
a(T; \vartheta^Q) &= \#\{w \in \mathbb{Z}^{8\times 2} : w'Qw = 2T\} \\
&= \#\{(u, v) : u, v \in \mathbb{Z}^{4\times 2}, u'Au + v'Bv = 2T\}, \\
a(T; \vartheta^A) &= \#\{u \in \mathbb{Z}^{4\times 2} : u'Au = 2T\}, \\
a(T; \vartheta^B) &= \#\{v \in \mathbb{Z}^{4\times 2} : v'Bv = 2T\}.
\end{aligned}$$

Then, using that T is nonzero,

$$\begin{aligned}
&a(T; \vartheta^Q) - a(T; \vartheta^A) - a(T; \vartheta^B) \\
&\qquad = \#\{(u, v) : u, v \in \mathbb{Z}^{4\times 2}, u \neq 0, v \neq 0, u'Au + v'Bv = 2T\}.
\end{aligned}$$

This set can be partitioned into subsets of 4 elements each of the form $\{\pm u, \pm v\}$ because the u, v are nonzero. This proves that the above number is a multiple of 4. Now consider the second term. We wish to show that

$$a(T; \vartheta^Q|E_1\tilde{\mu}) - a(T; \vartheta^A|E_1\tilde{\mu}) - a(T; \vartheta^B|E_1\tilde{\mu}) \equiv 0 \mod 4.$$

Denoting $S = \frac{1}{p}\mathrm{Twin}(T)$, the left-hand side is

$$p^4\, a(S; \vartheta^Q|E_1) - p^2\, a(S; \vartheta^A|E_1) - p^2\, a(S; \vartheta^B|E_1).$$

Applying Theorem 3.3 for slashing with E_1, this becomes

$$\frac{p^4}{p^2}\, a(S; \sum_{\beta \in N(Q)} \vartheta^Q[0, \beta]) + \frac{p^2}{p}\, a(S; \sum_{h \in N(A)} \vartheta^A[0, h]) + \frac{p^2}{p}\, a(S; \sum_{\ell \in N(B)} \vartheta^B[0, \ell]).$$

Consider $\beta = \binom{h}{\ell} \in N(Q) = N(A \oplus B)$. We break the above sum into four parts:

(a) $\sum\{p^2\, a(S; \vartheta^Q[0, \binom{h}{\ell}]) : \binom{h}{\ell} \in N(Q), h \neq 0, \ell \neq 0\}$

(b) $+\sum\{p^2\, a(S; \vartheta^Q[0, \binom{h}{0}]) + p\, a(S; \vartheta^A[0, h]) : h \in N(A), h \neq 0\}$

(c) $+\sum\{p^2\, a(S; \vartheta^Q[0, \binom{0}{\ell}]) + p\, a(S; \vartheta^B[0, \ell]) : \ell \in N(B), \ell \neq 0\}$

(d) $+p^2\, a(S; \vartheta^Q[0, \binom{0}{0}]) + p\, a(S; \vartheta^A[0, 0]) + p\, a(S; \vartheta^B[0, 0])$.

We show parts (a) to (d) individually are 0 modulo 4. Part (a) is

$$p^2\,\#\{(u,v,h,\ell) : u,v \in \mathbb{Z}^4, \left(\begin{smallmatrix} h \\ \ell \end{smallmatrix}\right) \in N(Q), h \neq 0, \ell \neq 0,$$
$$(u+h/p)'A(u+h/p) + (v+\ell/p)'B(v+\ell/p) = 2S\}.$$

We can partition these (u,v,h,ℓ) into subsets of the form

$$\{(u,v,h,\ell),(u,-v,h,-\ell),(-u,v,-h,\ell),(-u,-v,-h,-\ell)\}.$$

Because p is odd, we have $h \not\equiv -h \mod p$ and $\ell \not\equiv -\ell \mod p$ for nonzero h,ℓ. Thus these subsets always have 4 distinct elements and this proves that Part (a) is a multiple of 4. To analyze Part (b), note that $\left(\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right) \in N(Q)$ if and only if $h \in N(A)$, so that

$$\sum \{a(S;\vartheta^Q[0,\left(\begin{smallmatrix} h \\ 0 \end{smallmatrix}\right)]) : h \in N(A), h \neq 0\}$$
$$= \#\{(u,v,h) : u,v \in \mathbb{Z}^4, h \in N(A), h \neq 0, (u+h/p)'A(u+h/p) + v'Bv = 2S\}.$$

We can put these (u,v,h) into equivalence classes of the form

$$\{(u,v,h),(u,-v,h),(-u,v,-h),(-u,-v,-h)\}.$$

These classes will have four elements unless $v=0$. So modulo 4, we can ignore all but the case where $v=0$, and thus modulo 4, Part (b) is equivalent to

$$\sum p^2\,\#\{(u,h) : u \in \mathbb{Z}^4, (u+h/p)'A(u+h/p) = 2S\}$$
$$+ \sum \{p\,a(S;\vartheta^A[0,h]) : h \in N(A), h \neq 0\}$$

But this is equal to

$$\sum \{(p^2+p)\,\#\{(u,h) : u \in \mathbb{Z}^4, (u+h/p)'A(u+h/p) = 2S\} : h \in N(A), h \neq 0\}.$$

Because we can pair $\pm(u,h)$, $\#\{(u,h) : u \in \mathbb{Z}^4, (u+h/p)'A(u+h/p) = 2S\}$ is an even number; since p^2+p is also even, the above is a multiple of 4. Thus Part (b) is a multiple of 4. Similarly, Part (c) is a multiple of 4. Finally, Part (d) can be rewritten as

$$p^2\,\left(a(S;\vartheta^Q[0,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)]) - a(S;\vartheta^A[0,0]) - a(S;\vartheta^B[0,0])\right)$$
$$+(p^2+p)\,a(S;\vartheta^A[0,0]) + (p^2+p)\,a(S;\vartheta^B[0,0]).$$

Here, the first line is a multiple of 4 by the exact argument as for the first term. The second line is a multiple of 4 because $a(S;\vartheta^A[0,0])$ and $a(S;\vartheta^B[0,0])$ are even because S is nonzero. This proves the second term is 0 modulo 4. The third term is

$$p^2\,a(T;\vartheta^Q|E_1) - p^2\,a(T;\vartheta^A|E_1) - p^2\,a(T;\vartheta^B|E_1),$$

and the same argument used for the second term will show that this is also 0 modulo 4. The fourth term can be rewritten as

$$p\,a(T;\vartheta^{pQ^*}|\mu) - p\,a(T;\vartheta^{pA^*}|\mu) - p\,a(T;\vartheta^{pB^*}|\mu).$$

Noting that $pQ^* = (pA^*) \oplus (pB^*)$, this is

$$p\,a(S;\vartheta^{(pA^*)\oplus(pB^*)}) - p\,a(S;\vartheta^{pA^*}) - p\,a(S;\vartheta^{pB^*}).$$

for $S = \mathrm{Twin}(T)$. The same argument used for the first term shows that this is a multiple of 4. Having shown all four terms are 0 modulo 4, we conclude that $a(T;g) \equiv 0 \mod 4$. $\square$

In the following theorem, $S_2^k(K(N))(\mathbb{Z})$ and $J_{k,N}^{\mathrm{cusp}}(\mathbb{Z})$ indicate $\mathbb{Z}$-modules of modular forms whose Fourier coefficients are integral. See Section 4 for Jacobi forms and Section 5, just before Theorem 5.9, for a discussion of the general notation $M_n^k(\Gamma)(R)$.

3.14. **Theorem.** *Let $g \in S_2^k(K(N))(\mathbb{Z})$ for $k \geq 2$. Let T_q be the Hecke operator for a fixed prime q. Then the following congruence holds:*

$$T_q(g^q) \equiv g \mod q.$$

Furthermore, if $\phi \in J_{qk,N}^{cusp}$ is the first Fourier-Jacobi coefficient of $T_q(g^q)$ and $\psi \in J_{k,N}^{cusp}(\mathbb{Z})$ that of g, then $\operatorname{Grit}(\psi) \equiv \operatorname{Grit}(\phi) \mod q$.

Proof. Take any $T \in {}^N\mathcal{X}_2$. Then we have

$$a(qT; g^q) = \sum_{s_i \in {}^N\mathcal{X}_2 :\, s_1 + \cdots + s_q = qT} a(s_1; g) \cdots a(s_q; g).$$

Since q is prime, unless $s_1 = \cdots = s_q$, then there are a multiple of q nontrivial ways to permute the $s_1, \ldots, s_q$. Thus we have $a(qT; g^q) \equiv a(T; g)^q \mod q$. From Fermat's congruence $x^q \equiv x \mod q$ for any integer x, we have $a(qT; g^q) \equiv a(T; g)$ mod q. Next, in the formula for any cusp form given in this section

$$a(T; T_q(f)) = a(qT; f) + \text{terms with coefficients that are positive powers of } q,$$

as long as f has weight at least 3. Here, the weight of g^q is at least 4, so we can conclude $a(T; T_q(g^q)) \equiv a(qT; g^q) \mod q$. Thus $a(T; T_q(g^q)) \equiv a(T; g) \mod q$ for all $T \in {}^N\mathcal{X}_2$. Hence we have the first assertion $T_q(g^q) \equiv g \mod q$.

Letting $\phi \in J_{qk,N}^{\mathrm{cusp}}$ be the first Fourier-Jacobi coefficient of $T_q(g^q)$, then ϕ is congruent modulo q to the first Fourier-Jacobi coefficient of g; that is, $\phi \equiv \psi$ mod q. Now, if $T = \begin{pmatrix} mN & r/2 \\ r/2 & n \end{pmatrix} \in {}^N\mathcal{X}_2$, then

$$a(T; \operatorname{Grit}(\phi)) = \sum_{\delta | (n,r,m)} \delta^{qk-1} c(\frac{mn}{\delta^2}, \frac{r}{\delta}; \phi),$$
$$a(T; \operatorname{Grit}(\psi)) = \sum_{\delta | (n,r,m)} \delta^{k-1} c(\frac{mn}{\delta^2}, \frac{r}{\delta}; \psi).$$

Since $\delta^{qk} \equiv \delta^k \mod q$ and since $c(\frac{mn}{\delta^2}, \frac{r}{\delta}; \phi) \equiv c(\frac{mn}{\delta^2}, \frac{r}{\delta}; \psi) \mod q$, we may conclude $a(T; \operatorname{Grit}(\phi)) \equiv a(T; \operatorname{Grit}(\psi)) \mod q$; that is, $\operatorname{Grit}(\phi) \equiv \operatorname{Grit}(\psi) \mod q$. □

4. Jacobi forms

The Gritsenko lift constructs paramodular forms from Jacobi forms and so we need to compute spaces of Jacobi forms. The basic reference for Jacobi forms is the book of Eichler and Zagier [10]. The weight 2 Jacobi forms in this article were originally computed from weight 3/2 modular forms on $\Gamma_0(4p)$. A more appealing technique, however, is the method of theta blocks, which seems to work very well for low weight. We thank N.-P. Skoruppa for explaining to us his joint work with V. Gritsenko and D. Zagier on theta blocks [13].

The following subgroup $\Gamma_\infty(\mathbb{Z})$ of $\mathrm{Sp}_2(\mathbb{Z})$ stabilizes the Fourier-Jacobi expansion of a level one Siegel modular form term by term and this gives some motivation for

TABLE 2. Dimensions for Jacobi cusp forms of weight 2 and index p.

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
$\dim J_{2,p}^{\mathrm{cusp}}$	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	2

the definition of Jacobi forms:

$$\Gamma_\infty(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z}) \cap \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad \text{for } * \in \mathbb{Z}.$$

To admit half-integral weights in the following definition, we require that $\mu(\gamma, \Omega) = \chi(\gamma)\det(C\Omega + D)^k$ be a factor of automorphy on $\Gamma_\infty(\mathbb{Z}) \times \mathcal{H}_2$, compare [12]. We write $q = e(\tau)$ and $\zeta = e(z)$.

4.1. **Definition.** A level one Jacobi form of weight $k \in \frac{1}{2}\mathbb{Z}$, index $m \in \mathbb{Q}$ and multiplier $\chi : \Gamma_\infty(\mathbb{Z}) \to \mathbb{C}$, denoted $\phi \in J_{k,m}(\chi)$, is a holomorphic map $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ given by $(\tau, z) \mapsto \phi(\tau, z)$ such that, if $\tilde{\phi} : \mathcal{H}_2 \to \mathbb{C}$ is defined by $\tilde{\phi}(\Omega) = \phi(\tau, z)e(m\omega)$ for $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$, then we have
1) $\forall \gamma \in \Gamma_\infty(\mathbb{Z})$, $\tilde{\phi}|_k\gamma = \chi(\gamma)\tilde{\phi}$, and
2) $\phi(\tau, z) = \sum_{n\geq 0, r\in\mathbb{Z}} c(n,r)q^n\zeta^r$, where $c(n,r) = 0$ unless $4mn - r^2 \geq 0$.

If $c(n,r) = 0$ unless $4mn - r^2 > 0$, then ϕ is called a *cusp* form and we write $\phi \in J_{k,m}^{\mathrm{cusp}}(\chi)$. In [10], pp. 121, 131-132 we can find dimension formulae for Jacobi forms. We thank N. Skoruppa for rewriting these for us. Let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

4.2. **Theorem.** *For $k \in \mathbb{Z}_{\geq 0}$, let $\{\{k\}\} = \dim S_1^k$. For $m \in \mathbb{N}$, let $\sigma_0(m)$ be the number of positive divisors of m. Let $\delta(k,m)$ be zero unless $k = 2$ and let $\delta(2,m) = \frac{1}{2}\sigma_0(m) - 1$ for nonsquare m and $\delta(2,m) = \frac{1}{2}\sigma_0(m) - \frac{1}{2}$ for square m.*

$$\text{For even } k \geq 2, \quad \dim J_{k,m}^{cusp} = \delta(k,m) + \sum_{j=0}^{m}\left(\{\{k+2j\}\} - \left\lfloor \frac{j^2}{4m} \right\rfloor\right).$$

$$\text{For odd } k \geq 3, \quad \dim J_{k,m}^{cusp} = \sum_{j=1}^{m-1}\left(\{\{k+2j-1\}\} - \left\lfloor \frac{j^2}{4m} \right\rfloor\right).$$

The Fourier expansions of theta blocks can be computed from the Dedekind eta function $\eta(\tau) = q^{\frac{1}{24}}\prod_{n\in\mathbb{N}}(1-q^n)$ and the odd Jacobi theta function:

$$\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{(2n+1)^2}{8}}\zeta^{\frac{2n+1}{2}} = q^{\frac{1}{8}}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\sum_{n\in\mathbb{N}}(-1)^{n+1}q^{\binom{n}{2}}\sum_{j\in\mathbb{Z}:|j|\leq n-1}\zeta^j.$$

A modular form may be viewed as a Jacobi form of index zero. For example, $\eta \in J_{\frac{1}{2},0}(\epsilon)$ with the multiplier $\epsilon : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ defined as in [21, p. 45]. Since $\eta^{24} = \Delta \in S_1^{12}$ we know that $\epsilon^{24} = 1$. Let $H(\mathbb{Z}) = \{u\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)t\left(\begin{smallmatrix} 0 & \mu \\ \mu & \kappa \end{smallmatrix}\right) \in \Gamma_\infty(\mathbb{Z}) : \lambda, \mu, \kappa \in \mathbb{Z}\}$ be the integral Heisenberg group. Define the character $v_H : H(\mathbb{Z}) \to \mathbb{C}$ by $v_H\left(u\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)t\left(\begin{smallmatrix} 0 & \mu \\ \mu & \kappa \end{smallmatrix}\right)\right) = (-1)^{\lambda+\mu+\kappa}$. Embed $i_\infty : \mathrm{SL}_2(\mathbb{Z}) \to \Gamma_\infty(\mathbb{Z})$ via $i_\infty(\sigma) = \sigma \boxplus I_2$. The isomorphism $\Gamma_\infty(\mathbb{Z})\backslash\{\pm I_4\} \cong i_\infty(\mathrm{SL}_2(\mathbb{Z})) \ltimes H(\mathbb{Z})$ allows us to view $\epsilon^a\, v_H^b$ as a map $\epsilon^a\, v_H^b : \Gamma_\infty(\mathbb{Z}) \to \mathbb{C}$ trivial on $\pm I_4$ for any integers $a, b \in \mathbb{Z}$ and we use this shorthand. For the theta function we have:

$$\vartheta \in J_{\frac{1}{2},\frac{1}{2}}(\epsilon^3 v_H).$$

For ϑ_d defined by $\vartheta_d(\tau, z) = \vartheta(\tau, dz)$ we have $\vartheta_d \in J_{\frac{1}{2},\frac{d^2}{2}}(\epsilon^3 v_H^d)$; see [12].

4.3. **Theorem** (Gritsenko, Skoruppa, Zagier). *Let $\ell \in \mathbb{N}$ and $t \in \mathbb{Z}$. Let $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$. Let $n = \sum_{i=1}^{\ell} n_i$ for brevity. Define a meromorphic function* $\mathrm{THBK}(t, \mathbf{n}, \mathbf{d}) : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ *by*

$$\mathrm{THBK}(t, \mathbf{n}, \mathbf{d})(\tau, z) = \eta(\tau)^t \prod_{i=1}^{\ell} \vartheta(\tau, d_i z)^{n_i}.$$

We have $\mathrm{THBK}(t, \mathbf{n}, \mathbf{d}) \in J_{k,m}^{cusp}$ *if and only if*

(1) $2k = t + n$,
(2) $2m = \sum_{i=1}^{\ell} n_i d_i^2$,
(3) $t + 3n \equiv 0 \mod 24$,
(4) $\forall d \in \mathbb{N}$, $\sum_{i:d|d_i} n_i \geq 0$,
(5) *The function* $\frac{k}{12} + \sum_{i=1}^{\ell} n_i \bar{B}_2(d_i x)$ *has a positive minimum on* $[0, 1]$. *Here* $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ *and* $\bar{B}_2(x) = B_2(x - [[x]])$ *is the periodic extension of its restriction to* $[0, 1]$.

We will only avail ourselves of the simplest cases. For $k = 2$, we always take $t = -6$ and all $n_i = 1$ so that $n = 10$. With this in mind, let

$$\mathrm{THBK}_2(d_1, d_2, \ldots, d_{10})(\tau, z) = \eta(\tau)^{-6} \prod_{i=1}^{10} \vartheta(\tau, d_i z).$$

For $k = 4$, a case used only incidentally, we always take $t = 0$ and all $n_i = 1$ so that $n = 8$. With this in mind, let

$$\mathrm{THBK}_4(d_1, d_2, \ldots, d_8)(\tau, z) = \prod_{i=1}^{8} \vartheta(\tau, d_i z).$$

In the articles of Gritsenko (see [11]) his lift is proved for the group $\Gamma[N] = UK(N)U$ where $U = u\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We restate his results for cusp forms on the paramodular group $K(N)$.

4.4. **Theorem** (Gritsenko). *Let $\phi \in J_{k,N}^{cusp}$ and let $\phi(\tau, z) = \sum_{n>0, r\in\mathbb{Z}} c(n, r) q^n \zeta^r$ be the Fourier expansion. There is a form* $\mathrm{Grit}(\phi) \in S_2^k(K(N))$ *given by*

$$\mathrm{Grit}(\phi)\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_{n,r,m} \left(\sum_{\delta|(n,r,m)} \delta^{k-1} c\left(\frac{mn}{\delta^2}, \frac{r}{\delta}\right) \right) q^{mN} \zeta^r e(n\omega).$$

For k even, $\mathrm{Grit}(\phi)$ *is in the μ-plus space; for k odd,* $\mathrm{Grit}(\phi)$ *is in the μ-minus space.*

Thus, the Fourier coefficients of $\mathrm{Grit}(\phi)$ are:

$$a\begin{pmatrix} mN & r/2 \\ r/2 & n \end{pmatrix} = \sum_{\delta|(n,r,m)} \delta^{k-1} c\left(\frac{mn}{\delta^2}, \frac{r}{\delta}\right).$$

Additionally, V. Gritsenko and K. Hulek [12] have developed a lift for Jacobi forms with certain characters. The case of a quadratic character will sometimes be useful here. For odd, squarefree $N \in \mathbb{N}$, consider a representation of N as the sum of four squares: $N = a^2 + b^2 + c^2 + d^2$, for a, b, c, $d \in \mathbb{N}$. We have $\vartheta_a\vartheta_b\vartheta_c\vartheta_d \in J^{\rm cusp}_{2,\frac{1}{2}N}(\epsilon^{12})$. The following theorem shows how to lift such forms to $K(N)$.

4.5. **Theorem** (Gritsenko and Hulek)**.** *Let $N \in \mathbb{N}$. There is a character $\chi_2^{(N)} : K(N) \to \{\pm 1\}$. Let $\phi \in J^{cusp}_{k,\frac{1}{2}N}(\epsilon^{12})$ and let $\phi(\tau,z) = \sum_{n,r\in\frac{1}{2}+\mathbb{Z}:2Nn>r^2,n>0} c(n,r)q^n\zeta^r$ be the Fourier expansion. There is a form $\operatorname{Grit}(\phi) \in S_2^k\left(K(N),\chi_2^{(N)}\right)$ given by*

$$\operatorname{Grit}(\phi)\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix} = \sum_{n,r,m\ odd\ \in\mathbb{N}}\left(\sum_{\delta|(n,r,m)}\delta^{k-1}c\left(\frac{mn}{2\delta^2},\frac{r}{2\delta}\right)\right)q^{\frac{mN}{2}}\zeta^{\frac{r}{2}}e(\frac{n}{2}\omega).$$

The point is that for $i \in \{1,2\}$ and $\phi_i \in J^{\rm cusp}_{k_i,\frac{1}{2}N}(\epsilon^{12})$, we have a paramodular form with trivial character $\operatorname{Grit}(\phi_1)\operatorname{Grit}(\phi_2) \in S_2^{k_1+k_2}(K(N))$.

5. Vanishing theorems and congruences

Computations often require a priori sets of Fourier coefficients that determine linear dependence among Siegel modular forms. We discuss such sets in degree two and prove that some also determine congruences among Fourier coefficients. The forms that index Fourier coefficients are a partially ordered set with no natural linear order. Indeed, the intrinsic measure of the vanishing order of a Fourier series is the closure in $\mathcal{P}_n^{\rm semi}(\mathbb{R})$ of the convex ray hull of its support. For computational purposes, however, ordering the support with a convex function ϕ is a versatile expedient. We review the results of [26] and [27].

5.1. **Definition.** A function $\phi : \mathcal{P}_n^{\rm semi}(\mathbb{R}) \to \mathbb{R}_{\ge 0}$ is called *type one* if

(1) For all $s \in \mathcal{P}_n(\mathbb{R})$, $\phi(s) > 0$,
(2) for all $\lambda \in \mathbb{R}_{\ge 0}$ and $s \in \mathcal{P}_n^{\rm semi}(\mathbb{R})$, $\phi(\lambda s) = \lambda\phi(s)$,
(3) for all $s_1, s_2 \in \mathcal{P}_n^{\rm semi}(\mathbb{R})$, $\phi(s_1 + s_2) \ge \phi(s_1) + \phi(s_2)$.

Type one functions are continuous on $\mathcal{P}_n(\mathbb{R})$ and respect the partial order on $\mathcal{P}_n^{\rm semi}(\mathbb{R})$. Basic examples are: For $s \in \mathcal{P}_n^{\rm semi}(\mathbb{R})$, define

(1) $m(s) = \inf_{u\in\mathbb{Z}_n\setminus\{0\}} u'su$, the minimum function,
(2) $\tilde{tr}(s) = \inf_{u\in\mathrm{GL}_n(\mathbb{Z})} \operatorname{tr}(u'su)$, the reduced trace,
(3) $\delta(s) = \det(s)^{1/n}$, the reduced determinant,
(4) $w(s) = \inf_{u\in\mathcal{P}_n(\mathbb{R})} \frac{\langle u,s\rangle}{m(u)}$, the dyadic trace.

For $n = 2$, the dyadic trace of a Minkowski reduced $s = \left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right) \in \mathcal{P}_2(\mathbb{R})$ is given by $w(s) = a + c - |b|$; see [25]. For $n = 2$, Minkowski reduced means $2|b| \le a \le c$.

5.2. **Vanishing Theorem.** *Let ϕ be type one. For all $n \in \mathbb{Z}^+$ there exists a $c_n(\phi) \in \mathbb{R}_{>0}$ such that: For any subgroup $\Gamma \subseteq \Gamma_n$ with finite index I and coset decomposition $\Gamma_n = \bigcup_{i=1}^I \Gamma M_i$, we have*

$$(5.3)\qquad \forall\, k \in \mathbb{Z}^+, \forall\, f \in S_n^k(\Gamma),\quad \frac{1}{I}\sum_{i=1}^{I}\inf\phi\left(\operatorname{supp}(f|M_i)\right) > c_n(\phi)\,k \implies f \equiv 0.$$

For $n = 2$, we may take $c_n(\phi) = \inf\phi\left(\left[\frac{1}{30}\left(\begin{smallmatrix} 3 & 1\\ 1 & 3\end{smallmatrix}\right)\right]\right)$.

Proof. This is Theorem 2.5 from [28], except for the last comment, which is Corollary 5.8 from [26]. □

We obtain our Vanishing Theorem for automorphic forms on $K(p)$ by viewing them as automorphic with respect to $\Gamma_0'(p) = K(p) \cap \Gamma_2$. To use Theorem 5.2 we need coset representatives $\Gamma_2 = \bigcup_{i=1}^{(1+p)(1+p^2)} \Gamma_0'(p) Y_i$; these may be found in [14, p. 71].

5.4. **Theorem** (Hashimoto, Ibukiyama). *As a complete set of* $1+p+p^2+p^3$ *representatives of the coset space* $\mathrm{Sp}_2(\mathbb{Z})/\Gamma_0'(p)$ *we may take*

$$X_1(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & -a \\ c & 0 & 0 & 1 \end{pmatrix}; \quad X_2(a,b) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 \\ b & a & 1 & 0 \end{pmatrix};$$

$$X_3(a) = \begin{pmatrix} 0 & -a & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 \end{pmatrix}; \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where a, b, c *run over the integers modulo* p.

It is helpful to abbreviate

$$\kappa = u\begin{pmatrix} 0 & \frac{1}{p} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{p} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

5.5. **Corollary.** *A complete set of* $1+p+p^2+p^3$ *coset representatives* Y *for* $\Gamma_0'(p)Y \in \Gamma_0'(p)\backslash \mathrm{Sp}_2(\mathbb{Z})$ *and a representative from* $\Delta_2(\mathbb{Q})$ *for* $K(p)Y$ *and a representative,* I_4 *or* κ*, for* $K(p)Y\Delta_2(\mathbb{Z})$ *is given by*

$$X_4^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in K(p)\begin{pmatrix} 0 & \frac{1}{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z});$$

$$X_3(a)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -a & 0 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix} \in K(p)\begin{pmatrix} -\frac{1}{p} & -\frac{a}{p} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & a & -1 \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z}),$$

where a *runs over the integers modulo* p*;*

$$X_2(a,b)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -a & -b & 0 & 1 \\ 0 & -a & 1 & 0 \end{pmatrix} \in K(p)\begin{pmatrix} -\hat{b}a & -\frac{1}{p} & 0 & \frac{\hat{b}}{p} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -1 & -\hat{b}a \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z}),$$

where a, b, $\hat{b}$ *run over the integers modulo* p *with* $b\hat{b} \equiv 1 \mod p$*;*

$$X_2(a,0)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -a & 0 & 0 & 1 \\ 0 & -a & 1 & 0 \end{pmatrix} \in K(p)\begin{pmatrix} -\frac{1}{p} & \hat{a} & 0 & \frac{\hat{a}}{p} \\ 0 & 1 & -\hat{a} & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & -\hat{a}p & 1 \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z}),$$

where a, $\hat{a}$ run over the integers modulo p with $a\hat{a} \equiv 1 \mod p$;

$$X_2(0,0)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \Delta_2(\mathbb{Z});$$

$$X_1(a,b,c)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ -b & -c & 1 & a \\ -c & 0 & 0 & 1 \end{pmatrix} \in K(p) \begin{pmatrix} -\frac{1}{p} & -\frac{\hat{b}c}{p} & \frac{\hat{b}}{p} & \frac{\hat{b}a}{p} \\ 0 & 1 & -a\hat{b} & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & -\hat{b}c & 1 \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z}),$$

where a, b, $\hat{b}$, c run over the integers modulo p with $b\hat{b} \equiv 1 \mod p$;

$$X_1(a,0,c)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & -c & 1 & a \\ -c & 0 & 0 & 1 \end{pmatrix} \in K(p) \begin{pmatrix} 0 & -\frac{1}{p} & \frac{\hat{c}}{p} & \frac{\hat{c}a}{p} \\ 1 & 0 & 0 & -\hat{c} \\ 0 & 0 & 0 & -p \\ 0 & 0 & 1 & 0 \end{pmatrix} \subseteq K(p)\kappa\,\Delta_2(\mathbb{Z}),$$

where a, c, $\hat{c}$ run over the integers modulo p with $c\hat{c} \equiv 1 \mod p$;

$$X_1(a,0,0)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Delta_2(\mathbb{Z}),$$

where a runs over the integers modulo p.

Proof. First, it is the $Y = X_i^{-1}$ from Theorem 5.4 that give left coset representatives. Finally, one can check these assertions directly by taking inverses and multiplying. In the case of $X_2(a,b)^{-1}$, for example, the following element is in $K(p)$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -a & -b & 0 & 1 \\ 0 & -a & 1 & 0 \end{pmatrix} \begin{pmatrix} -\hat{b}a & -\frac{1}{p} & 0 & \frac{\hat{b}}{p} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -1 & -\hat{b}a \end{pmatrix}^{-1} = \begin{pmatrix} -p & a\hat{b} & -\frac{\hat{b}}{p} & 0 \\ 0 & -1 & 0 & 0 \\ bp & a(1-b\hat{b}) & \frac{b\hat{b}-1}{p} & 0 \\ ap & -a^2\hat{b} & 0 & -1 \end{pmatrix}.$$

Furthermore, the following element is in $\Delta_2(\mathbb{Z})$:

$$\begin{pmatrix} 0 & \frac{1}{p} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\hat{b}a & -\frac{1}{p} & 0 & \frac{\hat{b}}{p} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -1 & -\hat{b}a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a\hat{b} & -1 & 0 & \hat{b} \\ 0 & 0 & 1 & -a\hat{b} \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

□

With these coset representatives, we may use Theorem 5.2 to prove:

5.6. **Theorem.** *Let $f \in S_2^k(K(p))$. Let ϕ be a type one $\mathrm{GL}_2(\mathbb{Z})$-class function. Unless $f \equiv 0$, we have*

$$\inf \phi(\mathrm{supp}(f)) + p \inf \phi(\mathrm{supp}(f|\mu)) \le \phi\left(\tfrac{1}{30}\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)\right) k(1+p^2).$$

5.7. **Corollary.** *For nontrivial $f \in S_2^k(K(p))$ we have*

$$\min \delta(\operatorname{supp}(f)) \le \frac{\sqrt{2}}{15} k \, \frac{1+p^2}{1+p}.$$

If f is additionally a μ-eigenform then we have

$$\min w(\operatorname{supp}(f)) \le \frac{k}{6} \frac{1+p^2}{1+p},$$

$$\min \tilde{tr}(\operatorname{supp}(f)) \le \frac{k}{5} \frac{1+p^2}{1+p},$$

$$\min m(\operatorname{supp}(f)) \le \frac{k}{10} \frac{1+p^2}{1+p}.$$

Proof of Theorem 5.6 *and Corollary* 5.7. Let $\Gamma_2 = \bigcup_{i=1}^{I} \Gamma_0'(p) Y_i$ with $I = (1+p)(1+p^2)$. For nontrivial f, Theorem 5.2 gives

$$\frac{1}{I} \sum_{i=1}^{I} \inf \phi(\operatorname{supp}(f|Y_i)) \le c_2(\phi)\, k. \tag{5.8}$$

For each i we have $Y_i \in K(p)\kappa^{\epsilon_i} \left(\begin{smallmatrix} u_i & * \\ 0 & u_i^* \end{smallmatrix}\right)$ with $u_i \in \mathrm{GL}_2(\mathbb{Z})$, with $\epsilon_i = 0$ in $1+p$ cases and with $\epsilon_i = 1$ in p^2+p^3 cases by Corollary 5.5. Since ϕ is a class function and $\operatorname{supp}(f|Y_i) = u_i' \kappa'^{\epsilon_i} \operatorname{supp}(f) \kappa^{\epsilon_i} u_i$, we have $\phi(\operatorname{supp}(f|Y_i)) = \phi(\operatorname{supp}(f|\kappa^{\epsilon_i}))$. When $\epsilon_i = 0$ this is $\phi(\operatorname{supp}(f))$ and when $\epsilon_i = 1$ this is

$$\phi(\operatorname{supp}(f|\kappa)) = \phi\left(\frac{1}{p} \operatorname{supp}(f|\mu)\right)$$

by Lemma 3.9. In view of these two cases, condition (5.8) becomes

$$\frac{1}{I}\left((1+p) \inf \phi(\operatorname{supp}(f)) + (p^2+p^3) \inf \phi\left(\frac{1}{p} \operatorname{supp}(f|\mu)\right)\right) \le c_2(\phi)\, k,$$

which is equivalent to the conclusion of Theorem 5.6.

In the corollary, we have replaced inf by min since the infimum of $\phi(\operatorname{supp}(f|Y_i))$ is attained whenever ϕ has finite shells or takes values in a lattice. If f is a μ-eigenform then $\operatorname{supp}(f|\mu) = \operatorname{supp}(f)$ and the conclusions for w, m and $\tilde{tr}$ follow upon evaluating $w(1/30\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)) = 1/6$, $\tilde{tr}(1/30\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)) = 1/5$ and $m(1/30\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)) = 1/10$. In general, twinning does not change the determinant, so $\delta(\operatorname{supp}(f|\mu)) = \delta(\operatorname{supp}(f))$ and we note that $\delta(1/30\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)) = \sqrt{2}/15$. □

The goal of this section is to give similar sets of Fourier coefficients that determine congruences. The essential difficulty is at the primes two and three. Results of this type depend upon the work of J. I. Igusa [20]. Let R be a ring. For $A \subseteq \mathbb{C}$, let

$$M_n^k(\Gamma)(A) = \{f \in M_n^k(\Gamma) : \forall T \in \mathcal{P}_n^{\mathrm{semi}}(\mathbb{Q}), a(T; f) \in A\},$$

with a similar meaning for cusp forms $S_n^k(\Gamma)(A)$ and Jacobi forms $J_{k,m}(A)$. We see that $M_n^k(\Gamma)(A)$ is an R-module, or $M_n(\Gamma)(A)$ a ring, whenever A is. Further, for a prime ℓ, let $M_n^k(\Gamma)(\mathbb{F}_\ell)$ denote the reduction modulo ℓ of the coefficients of the Fourier series from $M_n^k(\Gamma)(\mathbb{Z})$. Let $R_\ell : M_n^k(\Gamma)(\mathbb{Z}) \to M_n^k(\Gamma)(\mathbb{F}_\ell)$ be the natural reduction map.

Using Igusa's work we can prove the following:

5.9. **Theorem.** *Let $K \subseteq \mathbb{C}$ be a number field, $\mathcal{O}$ its ring of integers and $\mathbf{p}$ a prime ideal in $\mathcal{O}$. Let $f \in S_2^k(K(p))(\mathcal{O})$ be a μ-eigenform with $a(T;f) \in \mathbf{p}$ for all $T \in {}^p\mathcal{X}_2$ satisfying $w(T) \le \frac{k}{6}\frac{p^2+1}{p+1}$. Then we have $a(T;f) \in \mathbf{p}$ for all $T \in {}^p\mathcal{X}_2$.*

The proof we will give is valid only for primes p because we rely on the coset decomposition of Corollary 5.5. The proof of Theorem 5.9 is at the end of this section. These results partially generalize Sturm's Theorem [33] on elliptic modular forms for $\Gamma_0(N)$, which we state for level one in the next theorem. For a ring $R \subseteq \mathbb{C}$ and a set $S \subseteq \mathbb{C}$, let $R\langle S\rangle$ denote the R-module generated by finite R-linear combinations of elements from S. Recall the notation $\{\{k\}\} = \dim S_1^k$.

5.10. **Theorem.** *If $f \in M_1^k(\mathbb{C})$, then $f \in M_1^k(\mathbb{Z}\langle a(j;f) : j \le \{\{k\}\}\rangle)$.*

Theorem 5.9 is proven by reduction to a congruence criterion for integral forms of level one which is of independent interest. For $g = 2$, we will show in Theorem 5.15 that

$$\text{If } f \in M_2^k(\mathbb{C}), \text{ then } f \in M_2^k\left(\mathbb{Z}\langle a(T;f) : w(T) \le \frac{k}{6}\rangle\right). \tag{5.11}$$

For $k \ne 2$, each $M_1^k(\mathbb{Z})$ has a basis $\{h_i\}$, where each $h_i(\tau)$ has a Fourier expansion $q^i + 0(q^{i+1})$ for $0 \le i \le \dim S_1^k$. The following theorem is an immediate consequence. For an R-module V, we let $\mathrm{Sym}(V \otimes V)$ be the kernel of the map $V \otimes V \to V \wedge V$.

5.12. **Theorem.** *If $f \in \mathrm{Sym}\,(M_1 \otimes M_1)^k(\mathbb{C})$, then we have*

$$f \in \mathrm{Sym}\,(M_1 \otimes M_1)^k\,(\mathbb{Z}\langle a(i,j;f) : i,j \le \{\{k\}\}\rangle)\,.$$

For a ring $R \subseteq \mathbb{C}$, this shows that $\mathrm{Sym}\,(M_1 \otimes M_1)^k(R) = \mathrm{Sym}\left(M_1^k(R) \otimes M_1^k(R)\right)$.

Interest in the intermediate ring $\mathrm{Sym}\,(M_1 \otimes M_1)^k(R)$ stems from the Witt map [35]

$$\begin{aligned} W : M_2^k(R) &\to \mathrm{Sym}\,(M_1 \otimes M_1)^k(R) \\ (\Omega \mapsto f(\Omega)) &\mapsto \left((\tau_1,\tau_2) \mapsto f\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}\right) \end{aligned}$$

for which the Fourier coefficients obey

$$a(i,j;W(f)) = \sum_b a\left(\begin{pmatrix} i & b\\ b & j\end{pmatrix}; f\right).$$

The following exact sequence is often the basis of computing in the ring $M_2(\mathbb{C})$.

$$0 \to M_2^{k-10}(\mathbb{C}) \xrightarrow{\cdot X_{10}} M_2^k(\mathbb{C}) \xrightarrow{W} \mathrm{Sym}\,(M_1 \otimes M_1)^k(\mathbb{C}) \to 0.$$

Igusa showed that the following sequence is exact [20, Lemma 7, p. 163].

$$0 \to M_2^{k-10}(\mathbb{Z}) \xrightarrow{\cdot X_{10}} M_2^k(\mathbb{Z}) \xrightarrow{W} \mathrm{Sym}\,(M_1 \otimes M_1)^k(\mathbb{Z}),$$

but that the Witt map from $M_2^k(\mathbb{Z})$ to $\mathrm{Sym}\,(M_1 \otimes M_1)^k(\mathbb{Z})$ does *not* surject when twelve divides k. The following lemma has greater applicability if we note that $M_2^k(R) = M_2^k(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for $\mathbb{Z}$-modules R; cf. [20, p. 150].

5.13. **Theorem** ([20], Lemma 13, p. 171))**.** *We have*

$$W\left(M_2^k(\mathbb{Z})\right) = \mathrm{Sym}\,(M_1 \otimes M_1)^k(\mathbb{Z})$$

if and only if $k \not\equiv 0 \mod 12$.

We need a lemma that records the reach of each Fourier expansion. Some familiarity with the geometry of numbers, as in [26], is required for the proof of the next lemma. In particular, for $A \in \mathcal{P}_2(\mathbb{R})$, let $\varpi(A) = \mathbb{R}_{\geq 0}\langle xx'; x \in \mathbb{Z}^2 : x'Ax = m(A)\rangle$ be the cone in $\mathcal{P}_2^{\mathrm{semi}}(\mathbb{R})$ generated by the outer products of the minimal vectors of A. Also, for $X \subseteq \mathcal{P}_2^{\mathrm{semi}}(\mathbb{R})$, Semihull$(X)$ denotes the closure in $\mathcal{P}_2^{\mathrm{semi}}(\mathbb{R})$ of the convex ray hull of X. Finally, we need to mention the technique of restriction to modular curves [27] in the simplest case. For $s \in \mathcal{P}_n(\mathbb{Z})$, define $\phi_s : \mathcal{H}_1 \to \mathcal{H}_n$ by $\tau \mapsto s\tau$ and note that, for any $\ell \in \mathbb{N}$ with $\ell s^{-1} \in \mathcal{P}_n(\mathbb{Z})$, the pullback is a ring homomorphism $\phi_s^* : M_n \to M_1(\Gamma_0(\ell))$ that multiplies weights by n. If $f \in M_n^k$ has a Fourier series $\mathrm{FS}_n(f) = \sum_T a(T)q^T$, then $\mathrm{FS}_1(\phi_s^* f) = \sum_j \left(\sum_{T:\langle s,T\rangle = j} a(T)\right) q^j$.

5.14. **Lemma.** *Let $R \subseteq \mathbb{C}$ be a $\mathbb{Z}$-module. Let $b > 0$. Let $f \in M_2^k(\mathbb{C})$ and $f' \in M_2^{k-10}(\mathbb{C})$ with $f = X_{10}f'$. Then we have*

$$\left(\forall T \in \mathcal{X}_2^{semi} : w(T) < b,\ a(T;f) \in R\right)$$
$$\implies \forall T \in \mathcal{X}_2^{semi} : w(T) < b - \frac{3}{2},\ a(T;f') \in R.$$

Proof. Let $\mathrm{supp}_R(f') = \{T \in \mathcal{X}_2^{\mathrm{semi}} : a(T;f') \notin R\}$. Our goal is to prove the inequality $\min w(\mathrm{supp}_R(f')) \geq b - 3/2$. Letting $K = \mathrm{Semihull}(\mathrm{supp}_R(f'))$, it is equivalent to show that $\min w(K) \geq b - 3/2$. Suppose, by way of contradiction, that $T \in \mathrm{supp}_R(f')$ is a vertex of K with minimal dyadic trace $w(T) < b - 3/2$. By changing representatives within the $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of T, we may assume $T \in \varpi(A)$ for $A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$.

In the dual $K^{\sqcup} = \{y \in \mathcal{P}_2^{\mathrm{semi}}(\mathbb{R}) : \forall x \in K, \langle x,y\rangle \geq 1\}$, the vertex T corresponds to the convex two-dimensional face $F = \{y \in K^{\sqcup} : \langle T,y\rangle = 1\}$. The point $Y = A/w(T)$ is in this face and also in the interior of the cone $\varpi(\hat{A})$ for $\hat{A} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Let N be a neighborhood of Y with $N \subseteq \varpi(\hat{A})$. For any $H \in N \cap F^\circ$, we have both (1) $\{x \in K : \langle x,H\rangle = 1\} = \{T\}$ because $H \in F^\circ$ and (2) $\{x \in [\hat{A}] : \langle x,H\rangle = w(H)\} = \{\hat{A}\}$ because $H \in \varpi(\hat{A})^\circ$. Consider the continuous function $\langle \hat{A} + T, \cdot\rangle/m(\cdot)$ evaluated at Y:

$$\frac{\langle \hat{A}+T,Y\rangle}{m(Y)} = \frac{\langle \hat{A}+T,A\rangle}{m(A)} = \frac{\langle \hat{A},A\rangle}{m(A)} + \frac{\langle T,A\rangle}{m(A)} = \frac{3}{2} + w(T) < \frac{3}{2} + \left(b - \frac{3}{2}\right) = b.$$

As Y is an accumulation point of $N \cap F^\circ$, there is $H \in N \cap F^\circ$ with $\langle \hat{A}+T,H\rangle/m(H) < b$.

The leading term of $\phi_H^* X_{10}$ is $a(\hat{A};X_{10})q^{\langle \hat{A},H\rangle} = q^{\langle \hat{A},H\rangle}$ by (2). The leading term of $\phi_H^* f'$ modulo $R[[q]]$ is $a(T;f')q^{\langle T,H\rangle}$ by (1). Thus $\phi_H^* f = (\phi_H^* X_{10})(\phi_H^* f')$ has a leading term modulo $R[[q]]$ given by $a(T;f')q^{\langle \hat{A}+T,H\rangle}$ and hence

$$\sum_{X \in \mathrm{supp}(f):\langle X,H\rangle = \langle \hat{A}+T,H\rangle} a(X;f) \equiv a(T;f') \quad \text{modulo } R.$$

For $X \in \mathrm{supp}(f)$ with $\langle X,H\rangle = \langle \hat{A}+T,H\rangle$ we have

$$w(X) \leq \frac{\langle X,H\rangle}{m(H)} = \frac{\langle \hat{A}+T,H\rangle}{m(H)} < b,$$

so that $a(X;f) \in R$ by hypothesis and we obtain $a(T;f') \in R$ and the contradiction $T \notin \mathrm{supp}_R(f')$. $\square$

TABLE 3. Functions from the proof of Theorem 5.15.

1	$k \mod 12$	0	10	8	6	4	2
2	$\nu(k)$	0	1	2	3	4	5
3	$\hat{k}$	$k+4$	k	k	k	k	k
4	k'	$k-6$	$k-10$	$k-10$	$k-10$	$k-10$	$k-10$
5	$\nu(k')$	3	0	1	2	3	4
6	$12\{\{k\}\}$	k	$k-10$	$k-8$	$k-6$	$k-4$	$k-14$
7	$k-\nu(k)$	k	$k-1$	$k-2$	$k-3$	$k-4$	$k-5$
8	$k'-\nu(k')$	$k-9$	$k-10$	$k-11$	$k-12$	$k-13$	$k-14$

5.15. **Theorem.** *Let $f \in M_2^k(\mathbb{C})$, then $f \in M_2^k\left(\mathbb{Z}\langle a(T;f) : w(T) \le \frac{k}{6}\rangle\right)$. Furthermore, $f \in M_2^k\left(\mathbb{Z}\langle a(T;f) : w(T) \le \frac{k-\nu(k)}{6}\rangle\right)$ for $\nu(k) = 0,1,2,3,4,5$ for $k \equiv 0,10,8,6,4,2 \mod 12$, respectively.*

Proof. It suffices to prove the second assertion and we do this by induction on k. Let $A = \mathbb{Z}\langle a(T;f) : w(T) \le (k-\nu(k))/6\rangle$. Let $f \in M_2^k(\mathbb{C})$ and consider $\hat{f} = f$ for $k \not\equiv 0 \mod 12$ and $\hat{f} = E_4 f$ for $k \equiv 0 \mod 12$. Thus, in all cases, the weight $\hat{k}$ of $\hat{f}$ is not divisible by 12. The Witt image $W(\hat{f}) \in \mathrm{Sym}\,(M_1 \otimes M_1)^{\hat{k}}(\mathbb{C})$ has Fourier coefficients

$$a(i,j;W(\hat{f})) = \sum_b a\left(\begin{pmatrix} i & b \\ b & j\end{pmatrix}; \hat{f}\right).$$

By Theorem 5.12, we have

$$W(\hat{f}) \in \mathrm{Sym}\,(M_1 \otimes M_1)^{\hat{k}}\left(\mathbb{Z}\langle a(i,j;W(\hat{f})) : i,j \le \{\{\hat{k}\}\}\rangle\right).$$

For $i,j \le \{\{\hat{k}\}\}$, we have $w\left(\begin{smallmatrix} i & b \\ b & j\end{smallmatrix}\right) \le \mathrm{tr}\left(\begin{smallmatrix} i & b \\ b & j\end{smallmatrix}\right) = i+j \le 2\{\{\hat{k}\}\}$, so that $a(i,j;W(\hat{f})) \in A$ and $W(\hat{f}) \in \mathrm{Sym}\,(M_1 \otimes M_1)^{\hat{k}}(A)$ if $2\{\{\hat{k}\}\} \le (k-\nu(k))/6$. This inequality holds in all six cases; see Table 3. By Lemma 5.13, there is an $F \in M_2^{\hat{k}}(A)$ such that $W(F) = W(\hat{f})$. Therefore, $F - \hat{f} = X_{10} f'$ for some $f' \in M_2^{k'}(\mathbb{C})$ with $k' = \hat{k} - 10$. For all T with $w(T) \le (k-\nu(k))/6$ we have $a(T; F-\hat{f}) \in A$. Since $(k-\nu(k))/6 \in \frac{1}{2}\mathbb{Z}$ and since $w\left(\mathcal{X}_2^{\mathrm{semi}}\right) \subseteq \frac{1}{2}\mathbb{Z}$, the strict inequality $w(T) < (k-\nu(k))/6 + 1/2$ is equivalent to the inequality $w(T) \le (k-\nu(k))/6$. By Lemma 5.14, we have $a(T;f') \in A$ for all T with $w(T) < (k-\nu(k))/6 - 1$, or equivalently, $w(T) \le (k-\nu(k))/6 - 3/2$.

By the induction hypothesis we have

$$f' \in M_2^{k'}\left(\mathbb{Z}\langle a(T;f') : w(T) \le (k'-\nu(k'))/6\rangle\right).$$

Thus we have $f' \in M_2^{k'}(A)$ since $(k'-\nu(k'))/6 \le (k-\nu(k))/6 - 3/2$, an inequality that holds with equality in all six cases. Since $X_{10} \in S_2^{10}(\mathbb{Z})$, we have $F - \hat{f} = X_{10} f' \in S_2^{\hat{k}}(A)$ and we also have $\hat{f} \in M_2^{\hat{k}}(A)$. This is either $f \in M_2^k(A)$ or $E_4 f \in M_2^{k+4}(A)$, from which $f \in M_2^k(A)$ follows. It remains to check the base case of the induction. It suffices to note that for $k < 10$, nontrivial $M_2^k(\mathbb{C})$ are spanned by one Eisenstein series. $\square$

Table 3 is an aid to checking the proof of Theorem 5.15.

Theorem 5.15 is a module criterion for integral forms of level one. To prove a congruence criterion for $K(p)$, we use the explicit coset representatives from Corollary 5.5.

Proof of Theorem 5.9. Write the Fourier expansion of f as $\mathrm{FS}(f)=\sum_T a(T)q^T$. For each coset $\Gamma_0'(p)Y\in\Gamma_0'(p)\backslash\Gamma_2$ as in Corollary 5.5, we consider the Fourier expansion of $f|Y$. There are $I=1+p+p^2+p^3$ of these cosets. These Y break down into $p+1$ cases of one type where $K(p)Y\Delta_2(\mathbb{Z})=K(p)\Delta_2(\mathbb{Z})$ and p^2+p^3 cases of another type where $K(p)Y\Delta_2(\mathbb{Z})=K(p)\kappa\Delta_2(\mathbb{Z})$. For the first type we have

$$Y=\delta=\begin{pmatrix} U & XU^* \\ 0 & U^* \end{pmatrix}\in\Delta_2(\mathbb{Z})$$

for $U\in\mathrm{GL}_2(\mathbb{Z})$ and $X\in M_{2\times 2}^{\mathrm{sym}}(\mathbb{Z})$, so that for $f_Y=\det(U)^k f$ we have the Fourier expansion

$$\mathrm{FS}(f_Y)=\sum_{T\in{}^p\mathcal{X}_2} a(T)q^{T[U]}.$$

For the second type we have $Y=\kappa\delta$. Using $\kappa=\mu\frac{1}{\sqrt{p}}\left(\begin{smallmatrix} I & 0\\ 0 & pI\end{smallmatrix}\right)$ along with our assumption $f|\mu=\pm f$ and letting $f_Y=\pm\det(U)^k p^k f$ and $\zeta_p=e(\frac{1}{p})$, we have the Fourier expansion

$$\mathrm{FS}(f_Y)=\sum_{T\in{}^p\mathcal{X}_2} a(T)\zeta_p^{\langle X,T\rangle}q^{\frac{1}{p}T[U]}.$$

Each of these Fourier expansions has coefficients in $\mathcal{O}[\zeta_p]$ and furthermore in $\mathbf{p}[\zeta_p]$ for T such that $w(T)\le\frac{k}{6}\frac{p^2+1}{p+1}$. We consider the product of these series:

$$F=\left(\prod_{\Gamma_0'(p)Y\in\Gamma_0'(p)\backslash\Gamma_2} f_Y\right)\in M_2^{kI}\left(\mathcal{O}[\zeta_p]\right).$$

From the product for F we see that $\mathrm{supp}_{\mathbf{p}[\zeta_p]}(F)\subseteq\sum_Y\mathrm{supp}_{\mathbf{p}[\zeta_p]}(f_Y)$ and so

$$\begin{aligned}\min\ w\left(\mathrm{supp}_{\mathbf{p}[\zeta_p]}(F)\right)&\ge\min\ w\left(\sum_Y\mathrm{supp}_{\mathbf{p}[\zeta_p]}(f_Y)\right)\\ &\ge\min\left(\sum_Y w\left(\mathrm{supp}_{\mathbf{p}[\zeta_p]}(f_Y)\right)\right)\\ &\ge\sum_Y\min\ w\left(\mathrm{supp}_{\mathbf{p}[\zeta_p]}(f_Y)\right)>(p+1)\frac{k}{6}\frac{p^2+1}{p+1}+\frac{1}{p}(p^2+p^3)\frac{k}{6}\frac{p^2+1}{p+1}=\frac{k}{6}.\end{aligned}$$

By Theorem (5.11), we have $F\in S_2^{kI}\left(\mathbf{p}[\zeta_p]\right)$.

Now we will show that $\mathrm{FS}(f)=\sum_T a(T)q^T$ has $a(T)\in\mathbf{p}$. We proceed by contradiction; if not, there is a $T_0\in{}^p\mathcal{X}_2$ with $a(T_0)\notin\mathbf{p}$. Pick a prime ideal L containing $\mathbf{p}\mathcal{O}_{K(\zeta_p)}$ in $\mathcal{O}_{K(\zeta_p)}$. Denote by

$$R_L: M_2\left(\mathcal{O}_{K(\zeta_p)}\right)\to M_2\left(\mathcal{O}_{K(\zeta_p)}/L\right)$$

the reduction of Fourier coefficients modulo L. Then $R_L(F)=0$ but we can show each $R_L(f_Y)$ is nonzero at q^{T_0} or at $q^{\frac{1}{p}T_0[U]}$: for if $a(T_0)\zeta_p^j\in L$ for some $j\in\mathbb{Z}$ and $a(T_0)\in\mathcal{O}$, then $a(T_0)\in L\cap\mathcal{O}=\mathbf{p}$. We now have a contradiction using the fact that power series over the domain $\mathcal{O}_{K(\zeta_p)}/L$ cannot be zero divisors. $\square$

We conclude this section by presenting vanishing and congruence conditions for Jacobi forms. We thank O. Richter for suggesting this.

5.16. **Corollary.** *Let $p \in \mathbb{N}$ be prime or one. Let $\phi \in J_{k,p}^{cusp}$ have the Fourier expansion*

$$\phi(\tau, z) = \sum_{n\in\mathbb{N}, r\in\mathbb{Z}:\, 4np>r^2} c(n,r)q^n\zeta^r = \sum_{D\in\mathbb{N}, r\in\mathbb{Z}:\, -D\equiv r^2 \mod 4p} c(D)q^{\frac{D+r^2}{4p}}\zeta^r.$$

The form ϕ is trivial if and only if $c(D)=0$ whenever $D \le \frac{8}{225}\left(k\frac{1+p^2}{1+p}\right)^2$.

Let K be a number field, $\mathcal{O}$ its integers and $\mathbf{p}$ a prime ideal in $\mathcal{O}$. Let $\phi \in J_{k,p}^{cusp}$ have all $c(D) \in \mathcal{O}$. We have all $c(D) \in \mathbf{p}$ if and only if $c(D) \in \mathbf{p}$ whenever $D \le \frac{1}{27}\left(k\frac{1+p^2}{1+p}\right)^2$.

Proof. By Theorem 2.2 of [10, p. 23], for index p prime or 1, the Fourier coefficients $c(n,r)$ depend only upon $4np-r^2$ so that we may write $c(n,r)=c(4np-r^2)$. First examine the vanishing condition. The Fourier coefficients of $\mathrm{Grit}(\phi) \in S_2^k(K(p))$ are

$$a\begin{pmatrix} mp & r/2 \\ r/2 & n \end{pmatrix} = \sum_{\delta|(n,r,m)} \delta^{k-1} c\left(\frac{mn}{\delta^2}, \frac{r}{\delta}\right) = \sum_{\delta} \delta^{k-1} c\left(\frac{4nm-r^2}{\delta}\right);$$

compare Theorem 4.4. From $c(D)=0$ whenever $D \le \frac{8}{225}\left(k\frac{1+p^2}{1+p}\right)^2$, we see that $a(T)=0$ whenever $4\det(T) \le \frac{8}{225}\left(k\frac{1+p^2}{1+p}\right)^2$. Equivalently, $a(T)=0$ for $\delta(T) \le \frac{\sqrt{2}}{15}k\frac{1+p^2}{1+p}$, which proves the vanishing of $\mathrm{Grit}(\phi)$ by Corollary 5.7. Hence ϕ also vanishes.

Now examine the congruence condition for $\mathrm{Grit}(\phi) \in S_2^k(K(p))^{\epsilon}(\mathcal{O})$ for $\epsilon = (-1)^k$. If $a(T; \mathrm{Grit}(\phi)) \in \mathbf{p}$ for $T = \left(\begin{smallmatrix} mp & r/2 \\ r/2 & n \end{smallmatrix}\right) \in {}^p\mathcal{X}_2$ satisfying $w(T) \le \frac{k}{6}\frac{1+p^2}{1+p}$, then by Theorem 5.9 we have $a(T; \mathrm{Grit}(\phi)) \in \mathbf{p}$ for all T. It would follow that $c(D) = c(n,r;\phi) = a\left(\left(\begin{smallmatrix} p & r/2 \\ r/2 & n \end{smallmatrix}\right); \mathrm{Grit}(\phi)\right) \in \mathbf{p}$ for all D. So suppose T satisfies $w(T) \le \frac{k}{6}\frac{1+p^2}{1+p}$; we will show that $a(T; \mathrm{Grit}(\phi)) \in \mathbf{p}$. From $w(T) \ge \frac{2}{\mu_2}\delta(T)$, we see $\delta(T) \le \frac{1}{\sqrt{3}}w(T) \le \frac{k}{6\sqrt{3}}\frac{1+p^2}{1+p}$; thus $D = 4mnp - r^2 = 4\det(T) \le \frac{1}{27}\left(k\frac{1+p^2}{1+p}\right)^2$ and $c(D) \in \mathbf{p}$. Therefore, by the above formula for the Fourier coefficients of $\mathrm{Grit}(\phi)$, we have $a(T; \mathrm{Grit}(\phi)) \in \mathbf{p}$. $\square$

6. Integral closure

The constructions considered so far generate a large subring $R \subseteq M_2(K(p))$. Let R be the Hecke stable subring generated by Gritsenko lifts and traces of theta series. For weights $k \ge 3$, the dimension formulae of Ibukiyama reveal when R contains $S_2^k(K(p))$; indeed, we usually have containment in our examples for $k \ge 4$. We may use the following lemmas to construct nonlifts in $S_2^2(K(p))$ by studying the integral closure of $S_2(K(p))$. This technique, in the case of elliptic modular forms, was used by J. Tate [34] to construct "nonbanal" examples of weight one cusp forms for which the Artin Conjecture could be tested. The present article, in much the same spirit, aims at nonbanal examples of weight two paramodular cusp forms for which the Paramodular Conjecture can be tested.

Recall that $\mathcal{X}_n = \{T \in \mathcal{P}_n(\mathbb{Q}) : \forall\, v \in \mathbb{Z}^n,\ v'Tv \in \mathbb{Z}\}$ and ${}^N\mathcal{X}_2 = \{\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathcal{X}_2 : N|a\}$.

6.1. **Definition.** Set $\mathcal{H}_N(2) = \{H \in S_2^4(K(N)) : \operatorname{supp}(H) \subseteq {}^N\mathcal{X}_2 + {}^N\mathcal{X}_2\}$. Also, define $\mathcal{H}_N(2)^{\pm} = \{H \in S_2^4(K(N))^{\pm} : \operatorname{supp}(H) \subseteq {}^N\mathcal{X}_2 + {}^N\mathcal{X}_2\}$.

The next lemma is useful when $S_2^2(K(N))$ has linearly independent Gritsenko lifts.

6.2. **Lemma.** *Let $g_1, g_2 \in S_2^2(K(N))$ be nontrivial. Define a linear map*

$$\begin{aligned} \imath_{g_1,g_2} : S_2^2(K(N)) &\to \{(H_1, H_2) \in \mathcal{H}_N(2) \times \mathcal{H}_N(2) : H_1\, g_2 = H_2\, g_1\} \\ f &\mapsto (fg_1, fg_2). \end{aligned}$$

The map $\imath_{g_1,g_2}$ is injective.

Proof. It suffices to point out that the image of $\imath_{g_1,g_2}$ is contained in $\mathcal{H}_N(2) \times \mathcal{H}_N(2)$. □

6.3. **Corollary.** *Let $g_1, g_2 \in S_2^2(K(N))$ be nontrivial. For primes ℓ, we have the inequality* $\dim_{\mathbb{C}} S_2^2(K(N)) \le \dim_{\mathbb{F}_\ell}\{(H_1, H_2) \in \mathcal{H}_N(2)(\mathbb{F}_\ell) \times \mathcal{H}_N(2)(\mathbb{F}_\ell) : H_1\, R_\ell(g_2) = H_2\, R_\ell(g_1)\}$.

This corollary finds an upper bound on $\dim S_2^2(K(N))$ from a basis of $S_2^4(K(N))(\mathbb{Z})$ modulo ℓ. When this upper bound equals $\dim J_{2,N}^{\text{cusp}}$ then $S_2^2(K(N))$ is spanned by lifts. It is also useful to have versions of Lemma 6.2 that treat the plus and minus spaces separately. We define a set, $\mathcal{F}_N$, of indices that are $\hat{\Gamma}_0(N)$-equivalent to those that appear in the first Fourier-Jacobi coefficient of forms from $S_2(K(N))$.

6.4. **Definition.** Set $\mathcal{F}_N = \{T \in {}^N\mathcal{X}_2 : \exists \left(\begin{smallmatrix} N & b \\ b & c \end{smallmatrix}\right) \in {}^N\mathcal{X}_2, \exists U \in \hat{\Gamma}_0(N) : T[U] = \left(\begin{smallmatrix} N & b \\ b & c \end{smallmatrix}\right)\}$ for $N \in \mathbb{N}$. Set ${}^N\mathcal{X}_2' = {}^N\mathcal{X}_2 \setminus \mathcal{F}_N$. Define $\mathcal{H}_N'(2) = \{H \in \mathcal{H}_N(2) : \operatorname{supp}(H) \subseteq {}^N\mathcal{X}_2 + {}^N\mathcal{X}_2'\}$. Define $\mathcal{H}_N''(2) = \{H \in \mathcal{H}_N(2) : \operatorname{supp}(H) \subseteq {}^N\mathcal{X}_2' + {}^N\mathcal{X}_2'\}$. Set $\mathcal{H}_N'(2)^{\pm} = \mathcal{H}_N'(2) \cap S_2^4(K(N))^{\pm}$ and $\mathcal{H}_N''(2)^{\pm} = \mathcal{H}_N''(2) \cap S_2^4(K(N))^{\pm}$.

6.5. **Corollary.** *Let $N \in \mathbb{N}$. Let $g_1, g_2 \in S_2^2(K(N))^+$ be nontrivial. We have the inequalities* $\dim_{\mathbb{C}} S_2^2(K(N))^- \le \dim\{(H_1, H_2) \in \mathcal{H}_N(2)^- \times \mathcal{H}_N(2)^- : H_1\, g_2 = H_2\, g_1\}$ *and* $\dim_{\mathbb{C}} \left(S_2^2(K(N))^+ / \operatorname{Grit}(J_{2,N}^{cusp})\right) \le \dim\{(H_1, H_2) \in \mathcal{H}_N'(2)^+ \times \mathcal{H}_N'(2)^+ : H_1\, g_2 = H_2\, g_1\}$.

Proof. When g_1 and g_2 are plus forms, the restriction of $\imath_{g_1,g_2}$ to $S_2^2(K(N))^-$ has an image in $\mathcal{H}_N(2)^- \times \mathcal{H}_N(2)^-$. The injectivity of $\imath_{g_1,g_2}$ then proves the first inequality. For the second inequality, we note that each $f \in S_2^2(K(N))^+$ has a unique represenative $\hat{f} \in f + \operatorname{Grit}(J_{2,N}^{\text{cusp}})$ whose first Fourier Jacobi coefficient vanishes. Thus, we have $\operatorname{supp}(\hat{f}) \subseteq {}^N\mathcal{X}_2'$. The map $\hat{\imath}_{g_1,g_2}$ defined on $S_2^2(K(N))^+$ by $\hat{\imath}_{g_1,g_2}(f) = (g_1\hat{f}, g_2\hat{f})$ has kernel $\operatorname{Grit}(J_{2,N}^{\text{cusp}})$ and an image in $\mathcal{H}_N'(2)^+ \times \mathcal{H}_N'(2)^+$. □

This next lemma is useful when $S_2^2(K(N))$ has a nontrivial Gritsenko lift.

6.6. **Lemma.** *Let $g \in S_2^2(K(N))$ be nontrivial. Define a (nonlinear) map*

$$\begin{aligned} \jmath_g : S_2^2(K(N)) &\to \{(F, H) \in \mathcal{H}_N(2) \times \mathcal{H}_N(2) : H^2 = Fg^2\} \\ f &\mapsto (f^2, fg). \end{aligned}$$

The map $\jmath_g$ is bijective.

Proof. The map $\jmath_g$ is clearly injective. On the other hand, suppose we have F, $H \in \mathcal{H}_N(2) \subseteq S_2^4(K(N))$ with $H^2 = Fg^2$. The function $f = H/g$ is a weight 2 meromorphic form whose square, $f^2 = H^2/g^2 = F$, is holomorphic. Hence f is holomorphic and $\jmath_g(f) = (f^2, fg) = (F, H)$. $\square$

Our strategy to construct nonlifts is to use Lemma 6.2 to find a meromorphic function f with distinct representations $f = H_1/g_1 = H_2/g_2$. One then gains a pretty good idea of whether or not f is holomorphic by applying the formulae for the action of the Hecke operators on Fourier coefficients to the initial Fourier expansion of f. To prove f is holomorphic we use Lemma 6.6. This requires demonstrating the vanishing of $H^2 - Fg^2 \in S_2^8(K(p))^+$. One way to verify these identities in weight 8 is to span $S_2^8(K(p))^+$ but this is not always computationally feasible. Another path to proving holomorphicity would be to study the divisors of g_1 and g_2; note, however, that $K(p)\backslash K(p)(\mathcal{H}_1 \oplus \mathcal{H}_1)$ will always be a common divisor of any weight two paramodular cusp forms.

For small levels, when the dimension of the Gritsenko lifts is less than 2, nonlifts were eliminated by the Restriction Technique; compare [27]. In order to avoid a lengthy description of the Restriction Technique here, we provide the following lemmas. The first was used for levels 37, 43 and 53.

6.7. **Lemma.** *If* $\dim J_{2,N}^{cusp} \geq 1$ *and* $\dim S_2^4(K(N)) = 1 + \dim J_{4,N}^{cusp}$*, then* $\mathcal{H}_N''(2) = \{0\}$*. Also, if* $\dim \mathcal{H}_N(2) \leq 1 \leq \dim J_{2,N}^{cusp}$*, then* $\mathcal{H}_N''(2) = \{0\}$*.*

Proof. Let $f = \mathrm{Grit}(\phi) \in S_2(K(N))^+$ for a nontrivial $\phi \in J_{2,N}^{\mathrm{cusp}}$. Since the first Fourier-Jacobi coefficient of f^2 is zero, we have $\mathrm{Grit}(J_{4,N}^{\mathrm{cusp}}) \subsetneq \mathbb{C}f^2 + \mathrm{Grit}(J_{4,N}^{\mathrm{cusp}}) \subseteq S_2^4(K(N))^+$. Applying the hypothesis $\dim S_2^4(K(N)) = 1 + \dim J_{4,N}^{\mathrm{cusp}}$, we have $S_2^4(K(N)) = S_2^4(K(N))^+ = \mathbb{C}f^2 + \mathrm{Grit}(J_{4,N}^{\mathrm{cusp}})$. From this we can show $\mathcal{H}_N''(2) = \{0\}$.

Any $H \in \mathcal{H}_N''(2) \subseteq S_2^4(K(N))$ can be expressed as $H = \alpha f^2 + \mathrm{Grit}(\psi)$ for some $\alpha \in \mathbb{C}$ and $\psi \in J_{4,N}^{\mathrm{cusp}}$. The first Fourier-Jacobi coefficient of H is ψ and so $\psi = 0$ and $H = \alpha f^2$. The Fourier-Jacobi expansion of H begins $H\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \alpha\, \phi(w,z)^2 e(2N\omega) + \dots$ and because ϕ is nontrivial, $\mathrm{supp}(\phi^2 e(\cdot)^{2N})$ contains a definite index $T = \begin{pmatrix} 2N & r/2 \\ r/2 & m \end{pmatrix}$ with $r \in \mathbb{Z}$ and $m \in \mathbb{N}$. However, no element of this form is in ${}^N\mathcal{X}_2' + {}^N\mathcal{X}_2'$, so that $H \in \mathcal{H}_N''(2)$ implies that $\alpha = 0$ and $H = 0$. This proves the first assertion. The final assertion follows from the same argument because any $H \in \mathcal{H}_N''(2)$ can be written $H = \alpha f^2$ for f as above. $\square$

Items (4) and (5) of the next lemma are the most commonly used on the website [29] to prove that, for most primes $p < 600$, $S_2^2(K(p))$ consist entirely of lifts.

6.8. **Lemma.** *Let* $N \in \mathbb{N}$*.*

(1) *If* $S_2^4(K(N)) = \mathrm{Grit}(J_{4,N}^{cusp})$*, then* $\mathcal{H}_N(2) = \{0\}$*.*
(2) *If* $\mathcal{H}_N(2)^+ = \{0\}$*, then* $S_2^2(K(N)) = \{0\}$*.*
(3) *If* $\mathcal{H}_N''(2) = \{0\}$*, then* $S_2^2(K(N)) = \mathrm{Grit}(J_{2,N}^{cusp})$*.*
(4) *If* $\mathcal{H}_N''(2)^+ = \{0\}$*, then* $S_2^2(K(N))^+ = \mathrm{Grit}(J_{2,N}^{cusp})$*.*
(5) *If* $\dim \mathcal{H}_N(2)^- < \dim J_{2,N}^{cusp}$*, then* $S_2^2(K(N))^- = \{0\}$*.*
(6) *If* $\dim \mathcal{H}_N''(2) \leq 2$*, then* $\mathrm{Grit}(J_{2,N}^{cusp})$ *has codimension at most* 1 *in* $S_2^2(K(N))$*.*
(7) *If* $\dim \mathcal{H}_N''(2)^+ \leq 2$*, then* $\mathrm{Grit}(J_{2,N}^{cusp})$ *has codimension at most* 1 *in* $S_2^2(K(N))^+$*.*

TABLE 4. Dimensions of the μ-plus and μ-minus subspaces in $S_2^4\,(K(p))$

p	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163
$\dim S_2^4\,(K(p))^+$	18	23	32	27	32	27	38	33	44	38	45	51	51	59	65	65
$\dim S_2^4\,(K(p))^-$	1	0	0	1	1	2	0	1	2	3	2	2	3	2	3	4

p	167	173	179	181	191	193	197	199	211	223	227	229	233	239	241
$\dim S_2^4\,(K(p))^+$	55	62	65	83	73	92	78	91	100	106	91	121	106	105	133
$\dim S_2^4\,(K(p))^-$	8	8	6	3	7	4	10	6	7	12	18	7	13	15	7

p	251	257	263	269	271	277	281	283	293	307	311	313	317	331
$\dim S_2^4\,(K(p))^+$	113	124	120	134	149	161	149	155	149	177	163	200	174	211
$\dim S_2^4\,(K(p))^-$	18	18	23	20	17	17	18	24	31	30	32	21	34	26

p	337	347	349	353	359	367	373	379	383	389	397	401	409	419
$\dim S_2^4\,(K(p))^+$	227	192	239	212	210	241	263	264	226	256	289	274	318	272
$\dim S_2^4\,(K(p))^-$	25	47	29	42	45	45	39	39	62	48	49	48	39	69

p	421	431	433	439	443	449	457	461	463	467	479	487	491	499
$\dim S_2^4\,(K(p))^+$	333	287	343	333	297	333	378	335	362	321	341	393	363	422
$\dim S_2^4\,(K(p))^-$	43	73	53	64	82	65	59	83	76	98	99	88	98	81

p	503	509	521	523	541	547	557	563	569	571	577	587	593	599
$\dim S_2^4\,(K(p))^+$	363	400	426	437	506	478	460	443	502	530	558	480	518	519
$\dim S_2^4\,(K(p))^-$	120	104	101	112	90	119	138	156	121	117	114	169	156	156

Proof. For (1) it is enough to show that $\mathcal{H}_N(2) \cap \mathrm{Grit}(J_{4,N}^{\mathrm{cusp}}) = \{0\}$. For $f \in \mathcal{H}_N(2)$ the first Fourier-Jacobi coefficient is 0, hence $f \in \mathrm{Grit}(J_{4,N}^{\mathrm{cusp}})$ further implies that $f = \mathrm{Grit}(0) = 0$. For (2), if $S_2^2(K(N)) \neq \{0\}$, then there is a nontrivial f in $S_2^2(K(N))^+$ or $S_2^2(K(N))^-$. In either case, $f^2 \in \mathcal{H}_N(2)^+$ is nontrivial as well. For (3), suppose that $f \in S_2^2(K(N))$ is not a Gritsenko lift. Let $\phi \in J_{2,N}^{\mathrm{cusp}}$ be the first Fourier-Jacobi coefficient of f. Then $\hat{f} = f - \mathrm{Grit}(\phi)$ has a trivial first Fourier-Jacobi coefficient but is itself nontrivial. From $\mathrm{supp}(\hat{f}) \subseteq {}^N\mathcal{X}_2'$ we see that $\hat{f}^2 \in \mathcal{H}_N''(2)$ and $\mathcal{H}_N''(2) \neq \{0\}$. Item (4) follows from the same argument. For (5): when $f \in S_2^2(K(N))^-$ is nontrivial then $f\,\mathrm{Grit}(J_{2,N}^{\mathrm{cusp}}) \subseteq \mathcal{H}_N(2)^-$ so that $\dim J_{2,N}^{\mathrm{cusp}} \leq \dim \mathcal{H}_N(2)^-$. For (6), if $f, g \in S_2^2(K(N))$ are linearly independent modulo Gritsenko lifts, then $\hat{f}^2$, $\hat{f}\hat{g}$, $\hat{g}^2 \in \mathcal{H}_N''(2)$ are linearly independent, noting that any quadratic has linear factors over $\mathbb{C}$. Item (7) follows from the same argument. $\square$

7. EXAMPLES OF WEIGHT TWO

We explain how the theorems stated in the Introduction were proven. We first construct initial Fourier expansions of cusp forms in $S_2^4\,(K(p))$ by multiplying Gritsenko lifts from $S_2^2\,(K(p))$, by applying Hecke operators and by tracing theta series from $S_2^4\,(\Gamma_0(p))$. The dimension formula of Ibukiyama in Theorem 3.1 tells us if and when we have spanned $S_2^4\,(K(p))$. In this manner we were able to span $S_2^4\,(K(p))$ for $p < 600$. A nontrivial minus form of weight four first appears for $p = 83$.

For example, $\dim S_2^4\,(K(229)) = 128$ and products of the 7 weight two Gritsenko lifts give 28 linearly independent cusp forms of weight 4. Applying the Hecke operators T_2, T_3, T_2^2 and T_5, we span spaces of dimension 56, 84, 112 and at least 121, respectively. When the action of the Hecke operators seemed to stabilize, we used Theorems 3.2 and 3.3 to compute initial Fourier expansions of $\mathrm{Tr}\,(\vartheta_P \vartheta_Q) \in$

$M_2^4(K(229))$ for $P, Q \in \mathcal{A}$ where

$$\mathcal{A} = \left\{ \begin{pmatrix} 10 & 1 & -3 & -1 \\ 1 & 12 & 0 & 3 \\ -3 & 0 & 24 & 10 \\ -1 & 3 & 10 & 24 \end{pmatrix}, \begin{pmatrix} 10 & 1 & 2 & -1 \\ 1 & 12 & 1 & 4 \\ 2 & 1 & 12 & 2 \\ -1 & 4 & 2 & 40 \end{pmatrix}, \begin{pmatrix} 12 & 2 & -1 & -1 \\ 2 & 14 & 5 & 6 \\ -1 & 5 & 16 & 8 \\ -1 & 6 & 8 & 28 \end{pmatrix}, \begin{pmatrix} 12 & 3 & 5 & -1 \\ 3 & 18 & 8 & 0 \\ 5 & 8 & 18 & 3 \\ -1 & 0 & 3 & 20 \end{pmatrix} \right\}.$$

Any linear combination of these that cancels the constant term in the Fourier expansion gives an element of $S_2^4(K(229))$ as explained after Lemma 3.4. The addition of these linear combinations of theta traces increased the dimension of the constructed subspace to at least 128 and hence spanned $S_2^4(K(229))$. The involution μ twins the indices of the Fourier coefficients so that the dimensions of the plus and minus subspaces are easily computed to be 121 and 7. Notice that the theta traces were only necessary to fill the minus space. For the case $p = 229$ just described, the seven theta blocks $\mathrm{THBK}_2(\Sigma_i)$ are: $[2,2,3,4,5,7,7,9,10,11]$, $[2,2,3,3,5,5,7,8,10,13]$, $[2,2,2,3,4,5,6,8,10,14]$, $[1,3,4,4,5,6,7,8,11,11]$, $[1,3,3,4,6,6,7,9,10,11]$, $[1,3,3,4,5,7,8,8,10,11]$, $[1,3,3,4,4,5,7,8,10,13]$. See [29] for full comments on these computations.

Proof of Theorem 1.2. This is an application of Lemmas 6.2, 6.6, 6.7, 6.8 and Corollaries 6.3, and 6.5. Using initial Fourier expansions of a basis for $S_2^4(K(p))$ we compute initial Fourier expansions of linear combinations whose span contains $\mathcal{H}_p(2)$, $\mathcal{H}_p(2)^{\pm}$ and $\mathcal{H}_p''(2)^{\pm}$. If $\dim \mathcal{H}_p''(2)^+ = 0$ then $S_2^2(K(p))^+$ has no nonlifts by Lemma 6.8, item (4). If $\dim \mathcal{H}_p(2)^- < \dim J_{2,p}$ then $S_2^2(K(p))^-$ is trivial by Lemma 6.8, item (5). Otherwise, for various pairs of Gritsenko lifts $g_1, g_2 \in S_2^2(K(p))$, we compute upper bounds for $\dim\{(H_1,H_2) \in \mathcal{H}_p(2) \times \mathcal{H}_p(2) : H_1 g_2 = H_2 g_1\}$ by linear algebra. If any of these dimensions equal $\dim J_{2,p}$, then the injectivity of $\imath_{g_1,g_2}$ from Lemma 6.2 tells us that $S_2^2(K(p)) = \mathrm{Grit}(J_{2,p})$. These computations may also be performed modulo a prime ℓ, we usually take $\ell = 19$, and we use Corollary 6.3 to get the same conclusion: $S_2^2(K(p)) = \mathrm{Grit}(J_{2,p})$. When this last technique was used, the two Gritsenko lifts g_1 and g_2 that worked were recorded at [29].

Let us give an illustration of this last technique in the case of $p = 229$. We have $\dim J_{2,229} = 7$, $\dim \mathcal{H}_{229}(2) \le 31$ and for $g_1 = \mathrm{THBK}_2(2,2,3,4,5,7,7,9,10,11)$ and $g_2 = \mathrm{THBK}_2(2,2,3,3,5,5,7,8,10,13)$ we have $\dim_{\mathbb{F}_{19}}\{(H_1,H_2) \in (\mathcal{H}_{229}(2) \times \mathcal{H}_{229}(2))(\mathbb{F}_{19}) : H_1 R_{229}(g_2) = H_2 R_{229}(g_1)\} \le 7$. Therefore we have $S_2^2(K(229)) = \mathrm{Grit}(J_{2,229})$. We note that this process requires choosing 31 linearly independent elements from $S_2^4(\mathbb{Z})$ whose $\mathbb{C}$-span contains $\mathcal{H}_{229}(2)$ and whose reductions modulo 19 remain linearly independent over $\mathbb{F}_{19}$. For other primes, we simply note which case of Lemma 6.8 applies. □

Although the primes p for which the Gritsenko lifts span $S_2^2(K(p))$ produce no new paramodular cusp forms—they do give important evidence for the Paramodular Conjecture 1.1 because these p should also be, and so far are, primes for which there is no abelian surface defined over $\mathbb{Q}$ of conductor p. The more interesting primes p are those where $\dim \mathrm{Grit}(J_{2,p}) < \dim S_2^2(K(p))$ since, in order to test the Paramodular Conjecture, we must construct nonlift paramodular Hecke eigenforms. The first example is $p = 277$.

7.1. **Theorem.** *We have* $\dim S_2^2(K(277)) = 11$ *whereas the dimension of Gritsenko lifts in* $S_2^2(K(277))$ *is* $\dim J_{2,277} = 10$. *Let* $G_i = \mathrm{Grit}(\mathrm{THBK}_2(\Sigma_i))$ *for* $1 \le i \le 10$ *be the lifts of the* 10 *theta blocks given by* $\Sigma_i \in \{[2,4,4,4,5,6,8,9,10,14]$, $[2,3,4,5,5,7,7,9,10,14]$, $[2,3,4,4,5,7,8,9,11,13]$, $[2,3,3,5,6,6,8,9,11,13]$, [2, 3, 3, 5, 5, 8, 8, 8, 11, 13], [2, 3, 3, 5, 5, 7, 8, 10, 10, 13], [2, 3, 3, 4, 5, 6, 7, 9, 10, 15], [2, 2, 4, 5, 6, 7, 7, 9, 11, 13], [2, 2, 4, 4, 6, 7, 8, 10, 11, 12], [2, 2, 3, 5, 6, 7, 9, 9, 11, 12]}. *Let*

$$\begin{aligned}Q =& -14G_1^2 - 20G_8G_2 + 11G_9G_2 + 6G_2^2 - 30G_7G_{10} + 15G_9G_{10} + 15G_{10}G_1\\&-30G_{10}G_2 - 30G_{10}G_3 + 5G_4G_5 + 6G_4G_6 + 17G_4G_7 - 3G_4G_8 - 5G_4G_9\\&-5G_5G_6 + 20G_5G_7 - 5G_5G_8 - 10G_5G_9 - 3G_6^2 + 13G_6G_7 + 3G_6G_8 - 10G_6G_9\\&-22G_7^2 + G_7G_8+15G_7G_9 + 6G_8^2 - 4G_8G_9 - 2G_9^2 + 20G_1G_2 - 28G_3G_2+23G_4G_2\\&+7G_6G_2-31G_7G_2 + 15G_5G_2 + 45G_1G_3 - 10G_1G_5 - 2G_1G_4 - 13G_1G_6-7G_1G_8\\&+39G_1G_7 - 16G_1G_9 - 34G_3^2 + 8G_3G_4 + 20G_3G_5 + 22G_3G_6 + 10G_3G_8\\&+21G_3G_9 - 56G_3G_7 - 3G_4^2,\\L =& -G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1.\end{aligned}$$

Let $f = Q/L$ define a meromorphic paramodular form of weight 2. The form f is holomorphic and $S_2^2(K(277))(\mathbb{Q}) = \mathrm{Span}_{\mathbb{Q}}(f, G_1, \dots, G_{10})$. The form f is a Hecke eigenform with spin Euler factors

$$\begin{aligned}Q_2(f,x) &= 1 + 2x + 4x^2 + 4x^3 + 4x^4,\\Q_3(f,x) &= 1 + x + x^2 + 3x^3 + 9x^4,\\Q_5(f,x) &= 1 + x - 2x^2 + 5x^3 + 25x^4.\end{aligned}$$

Additionally, let

$$\begin{aligned}\hat{Q} =& -55G_5^2 - 13G_8G_2 - 148G_7G_8 + 17G_4G_9 + 123G_4G_8 + 160G_4G_7-12G_4G_6\\&+73G_4G_5 - 163G_{10}G_3 - 148G_{10}G_2 + 211G_{10}G_1 - 62G_9G_{10} - 94G_7G_{10}\\&+145G_9G_2 - 18G_7G + 58G_6G_9-138G_6G_8 - 10G_6G_7 + 52G_5G_9 - 109G_5G_8\\&-73G_5G_6 - 4G_6G_2 + 154G_4G_2 - 156G_3G_2+17G_1G_2 - 5G_8G_9 + 333G_3G_5\\&+37G_3G_4 + 26G_1G_9 + 235G_1G_7 - 71G_1G_8 + 49G_1G_6-34G_1G_4 - 151G_1G_5\\&+396G_1G_3 - 2G_5G_2 - 19G_7G_2 - 245G_3G_7 - 58G_3G_9 + 83G_3G_8 + 113G_3G_6\\&-54G_7^2 - 63G_8^2 + 12G_9^2 - 24G_4^2 - 404G_3^2 + 89G_5G_7 - 196G_1^2\\&+8G_2^2 - 24G_6^2 + 5G_{10}G_5 + 63G_{10}G_6 + 9G_8G_{10} - 3G_4G_{10},\\\hat{L} =& G_{10} + 15G_1 + 6G_2 - 5G_3 + 5G_4 - 6G_5 - 5G_6 - 13G_7 + 7G_8 - 8G_9.\end{aligned}$$

We have the following identity in $S_2^8(K(277))$:

$$Q^2 + \hat{L}QL + \hat{Q}L^2 = 0. \tag{7.2}$$

Proof. For the pair of Gritsenko lifts

$$\begin{aligned}v_1 &= L = -G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1,\\v_2 &= G_1 - 3G_2 + G_3 - 2G_4 + 2G_6 + G_7 + 2G_8 - 2G_9,\end{aligned}$$

we computed $\dim\{(H_1,H_2) \in \mathcal{H}_{277}(2) \times \mathcal{H}_{277}(2) : H_1v_2 = H_2v_1\} \le 11$. Therefore we have $\dim S_2^2(K(277)) \le 11$ by Lemma 6.2. There is a pair $(Q, H_2) \in \mathcal{H}_{277}(2) \times \mathcal{H}_{277}(2)$ that is not an image of $\imath_{v_1,v_2}(\mathrm{Grit}(J_{2,277}))$, that appears to satisfy $Qv_2 = H_2v_1$ and that produces an initial Fourier expansion for $f = Q/v_1$ that is consistent with being a Hecke eigenform. To prove that f is holomorphic, it suffices to prove the identity (7.2). For if we have $Q^2 + \hat{L}QL + \hat{Q}L^2 = 0$, then we have

$$f^2 + \hat{L}f + \hat{Q} = 0, \text{ for } f = \frac{Q}{L},$$

so that f is in the integral closure of $M_2(K(277))$ and hence is holomorphic. An application of the Siegel ϕ map to $f^2 + \hat{L}f + \hat{Q} = 0$ shows that $(\phi f)^2 = 0$ so that f is a cusp form. From $Q, L \in S_2(K(277))(\mathbb{Z})$ we see that $f \in S_2^2(K(277))(\mathbb{Q})$. The Fourier expansion of f is computed by the long division of L into Q. The action of the Hecke operators on $S_2^2(K(277))(\mathbb{Q})$ and the Euler factors of f are computed from these Fourier coefficients.

The identity (7.2) was proven by spanning $S_2^8(K(277))$. The space $S_2^8(K(277))$ was spanned in the same manner as $S_2^4(K(277))$ but at greater expense. Products of the 56 weight four Gritsenko lifts gave at least 1496 linearly independent elements, at which point the Hecke operators T_2 and T_3 were applied, resulting in at least 1760 linearly independent elements in $S_2^8(K(277))^+$. Theta traces $\mathrm{Tr}\,(\vartheta_P \vartheta_Q) \in M_2^4\,(K(277))$ for $P, Q \in \mathcal{A}$ were computed and multiplied by $S_2^4(K(277))^+$ where

$$\mathcal{A} = \{ \begin{pmatrix} 16 & 5 & 5 & -5 \\ 5 & 16 & 7 & 1 \\ 5 & 7 & 18 & 3 \\ -5 & 1 & 3 & 26 \end{pmatrix}, \begin{pmatrix} 14 & 1 & -7 & -6 \\ 1 & 20 & 3 & -6 \\ -7 & 3 & 20 & 5 \\ -6 & -6 & 5 & 22 \end{pmatrix}, \begin{pmatrix} 12 & 3 & 1 & -5 \\ 3 & 14 & 6 & -2 \\ 1 & 6 & 20 & 1 \\ -5 & -2 & 1 & 30 \end{pmatrix},$$
$$\begin{pmatrix} 14 & 4 & 1 & 4 \\ 4 & 14 & 6 & 5 \\ 1 & 6 & 18 & 2 \\ 4 & 5 & 2 & 30 \end{pmatrix}, \begin{pmatrix} 14 & 1 & -5 & 3 \\ 1 & 14 & 0 & -1 \\ -5 & 0 & 16 & 6 \\ 3 & -1 & 6 & 32 \end{pmatrix}, \begin{pmatrix} 16 & 5 & 5 & -3 \\ 5 & 16 & 7 & -8 \\ 5 & 7 & 18 & 0 \\ -3 & -8 & 0 & 28 \end{pmatrix} \}.$$

Symmetrization of these with respect to μ gave an additional 57 cusp forms, so that $\dim S_2^8(K(277))^+ \geq 1817$. Products from $S_2^4(K(277))^+ \, S_2^4(K(277))^-$ gave at least 595 linearly independent weight 8 minus forms. Finally, the Hecke operator T_2 was applied to the minus forms computed in this manner, for an estimate $\dim S_2^8(K(277))^- \geq 712$. These steps gave a subspace that spanned at least 2529 dimensions over $\mathbb{F}_{19}$. This is the correct dimension by Ibukiyama's formula. Therefore, the dimension of the plus space in $S_2^8(K(277))$ is 1817 and the dimension of the minus space is 712. This gave a determining set of 2529 Fourier coefficients for $S_2^8(K(277))$. The identity (7.2) was then checked to vanish on this determining set of 2529 Fourier coefficients. $\square$

In connection with Theorem 7.1 we mention that the hyperelliptic curve C of genus 2 defined by $y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$ has a Jacobi variety $\mathrm{Jac}(C)$ defined over $\mathbb{Q}$ with conductor 277. We refer to the companion article [5] for these arithmetic results. The Euler factors of the Hasse-Weil L-function of $\mathrm{Jac}(C)$ are identical to those of the spin L-function of f for the primes $q = 2, 3$ and 5. We know further equalities of eigenvalues but not nearly enough to prove that these L-functions are equal. The abelian surface $\mathrm{Jac}(C)$ has rational 15-torsion. This is of a piece with the congruence in the following theorem. We thank A. Brumer for suggesting the formula for Q/L below.

7.3. **Theorem.** *Let f be as in Theorem* 7.1. *We have $f \in S_2^2\,(K(277))\,(\mathbb{Z})$. The first Fourier Jacobi coefficient of f is*

$$\begin{aligned} \phi = & -5\,\mathrm{THBK}_2(\Sigma_5) + 3\,\mathrm{THBK}_2(\Sigma_4) - 3\,\mathrm{THBK}_2(\Sigma_6) + 4\,\mathrm{THBK}_2(\Sigma_7) \\ & + 6\,\mathrm{THBK}_2(\Sigma_8) + 2\,\mathrm{THBK}_2(\Sigma_9) - 2\,\mathrm{THBK}_2(\Sigma_3) - 2\,\mathrm{THBK}_2(\Sigma_2) \\ & - \mathrm{THBK}_2(\Sigma_1) \in J_{2,277}. \end{aligned}$$

Let $R = \mathrm{Grit}(\phi) = -5G_5 + 3G_4 - 3G_6 + 4G_7 + 6G_8 + 2G_9 - 2G_3 - 2G_2 - G_1 \in S_2^2\,(K(277))\,(\mathbb{Z})$. We have

$$\forall\, T \in {}^{277}\mathcal{X}_2, a(T; f) \equiv a(T; R) \mod 15.$$

Proof. By a result of Shimura [31], we have $Nf \in S_2^2(K(277))(\mathbb{Z})$ for some positive integer N. We choose N to be minimal with this property and show that $N = 1$. If not, suppose a prime ℓ divides N. From $f = Q/L$, we have $L(Nf) = NQ$. In $S_2^2(K(277))(\mathbb{F}_\ell)$ we have $R_\ell(L)R_\ell(Nf) = 0$ because Q has integral Fourier coefficients. The Fourier expansion of L has unit content:

$$L(\Omega) = e\left(\langle \Omega, \tfrac{1}{2}\left(\begin{smallmatrix} 98\cdot 277 & 233 \\ 233 & 2 \end{smallmatrix}\right)\rangle\right) + e\left(\langle \Omega, \tfrac{1}{2}\left(\begin{smallmatrix} 26\cdot 277 & 120 \\ 120 & 2 \end{smallmatrix}\right)\rangle\right) + 2\,e\left(\langle \Omega, \tfrac{1}{2}\left(\begin{smallmatrix} 326\cdot 277 & 601 \\ 601 & 4 \end{smallmatrix}\right)\rangle\right) + \dots$$

where the "$+\dots$" indicates $\hat{\Gamma}_0(277)$-equivalent terms and terms of higher dyadic trace. Therefore $R_\ell(L)$ is nontrivial and cannot be a zero divisor. This shows that $R_\ell(Nf) = 0$ and that $\frac{N}{\ell} f$ is integral, contradicting the minimality of N. The congruence now follows formally from the quotient $f = Q/L$. To see this note that

$$\frac{Q}{L} = R + 15\,\frac{J}{L}$$

holds identically in the $G_1, \dots, G_{10}$, considered as ten variables, if we set

$$\begin{aligned} J = {} & -G_7G_8 + G_4G_7 - 2G_{10}G_3 - 2G_{10}G_2 + G_{10}G_1 + G_9G_{10} - 2G_7G_{10} + G_9G_2 \\ & + G_7G_9 - G_6G_9 + G_6G_7 - G_5G_9 + 2G_4G_2 - 2G_3G_2 + G_1G_2 + 2G_3G_5 - G_1G_9 \\ & + 3G_1G_7 - G_1G_6 - G_1G_5 + 3G_1G_3 - G_7G_2 - 4G_3G_7 + G_3G_9 + 2G_3G_6 - 2G_7^2 \\ & - 2G_3^2 + 2G_5G_7 - G_1^2. \end{aligned}$$

Therefore we have $R_\ell(L)R_\ell(f - R) = 0$ for $\ell = 3$ and 5. The conclusion follows since $R_\ell(L)$ is not a zero divisor. $\square$

Let $\Gamma_0(p)^*$ be the modular group generated by $\Gamma_0(p)$ and the Fricke involution F_p. In $\mathrm{Grit}(J_{2,p}) \cong S_1^2(\Gamma_0(p)^*)$, the 10 eigenforms break into a nine-dimensional piece and a rational eigenform: $2G_1 + G_2 - 2G_3 + G_5 - 2G_7 - G_9$. This rational eigenform is visibly congruent to R modulo 3. Suitable multiples of the other eigenforms are each congruent to R modulo a prime above 5. We have given the computations for the case $p = 277$ in some detail. We now tabulate the results for the other exceptional primes given in Theorem 1.2.

Table 5 lists data for every prime $p < 600$ that possibly could have a Hecke eigenform in $S_2^2(K(p))$ not in the image of the Gritsenko lift from $J_{2,p}$. The existence of a nonlift has been proven in the first case but remains conjectural in others. The ϵ and the $\mathcal{O}$ columns indicate that there is a nonlift $f \in S_2^2(K(p))^\epsilon(\mathcal{O})$. It is defined by $f = Q/L$ for some $Q \in S_2^4(K(p))$ and $L \in S_2^2(K(p))$. There is an identity in $S_2^8(K(p))^+$, $Q^2 + LQ\hat{L} + L^2\hat{Q} = 0$ for some $\hat{Q} \in S_2^4(K(p))$ and $\hat{L} \in S_2^2(K(p))$, that would certify the holomorphicity of f. We give Q, $\hat{Q}$, L and $\hat{L}$ at [29] for each case. The verification of the weight 8 identity is expensive and has only been completed for $p = 277$. If the dimensions $\dim S_2^8(K(p))^+$ were only known, then it is very likely that the Fourier coefficients for elements in this space already posted at [29] would suffice to show the existence of the nonlifts in all cases. So there is at most one nonlift in each $S_2^2(K(p))^\epsilon$ and if a nonlift does not exist then the dimension of $S_2^2(K(p))^\epsilon$ would need to be reduced by one. A future publication with V. Gritsenko will show that there is a Borcherds product in $S_2^2(K(587))^-$ and so the existence of a second nonlift will be rigorously verified. Table 5 gives some Hecke eigenvalues of these nonlifts and Euler factors of the weight two form f for q prime to the level p may be computed as

$$Q_q(f, x) = 1 - \lambda_q x + (\lambda_q^2 - \lambda_{q^2} - 1)x^2 - q\,\lambda_q x^3 + q^2 x^4.$$

TABLE 5. Hecke eigenforms in $S_2^2(K(p))^{\epsilon}(\mathcal{O})$ but not in $\mathrm{Grit}(J_{2,p})$.

p	$\dim J_{2,p}$	$\dim S_2^2(K(p))$	$\mathcal{O}$	ϵ	ℓ	λ_2	λ_3	λ_4	λ_5	λ_7	λ_9	λ_{11}
277	10	11	$\mathbb{Z}$	$+$	$\{3,5\}$	-2	-1	-1	-1	1	-1	-2
349	11	12	$\mathbb{Z}$	$+$	$\{13\}$	-2	-1	1	-1	-2	1	1
353	11	12	$\mathbb{Z}$	$+$	$\{11\}$	-1	-2	-3	1	0	-1	2
389	11	12	$\mathbb{Z}$	$+$	$\{2,5\}$	-1	-2	-2	1	-3	1	4
461	12	13	$\mathbb{Z}$	$+$	$\{7\}$	0	-3	-3	1	0	2	2
523	17	18	$\mathbb{Z}$	$+$	$\{2,5\}$	-1	0	-2	-4	2	-1	2
587	18	20	$\mathbb{Z}$	$+$	$\{11\}$	-1	0	-3	0	-2	-2	
587	18	20	$\mathbb{Z}$	$-$	$\{\}$	-3	-4	3	-2	0	6	-1

For the identifications of these Euler factors with those of known rational abelian varieties see the companion article [5]. Additionally, each entry displays primes ℓ for which we conjecture the congruence $f \equiv \mathrm{Grit}(\phi) \mod \ell$, where ϕ is the first Fourier-Jacobi coefficient of f; the congruences have been proven contingent upon the existence of the nonlift f. We cannot compute enough Fourier coefficients to verify these congruences with Theorem 5.9 but congruences can be detected by the form of Q and L, as in the proof of Theorem 7.3. By examining minors of rational bases, we have also been able to prove is that these are the only possible such congruences; see [29].

Finally, we remark that for $p = 277$, there is a 4-dimensional space of weight 2 Gritsenko lifts whose product with the nonlift f lands in the span of the products of the weight 2 Gritsenko lifts. For $p = 353$, the analogous space is 3 dimensional. These curious subspaces of weight two Gritsenko lifts are instrinsic, as are the corresponding subspaces in $J_{2,p}$ and in Kohnen's plus space $S_1^{3/2}(\Gamma_0(4p))^+$; see [10]. There is no corresponding space in $S_1^2(\Gamma_0(p))$ because the Shimura correspondence is noncanonical. We do, however, have an intrinsic subspace of $S_1^2(\Gamma_0(p)) \otimes S_1^2(\Gamma_0(p))$ constructed by composing the Witt map with the Saito-Kurokawa lift. It would be interesting to give an alternative characterization of these spaces directly in terms of elliptic modular forms. We state the following Theorem as representative of the unproven cases in Table 5.

7.4. **Theorem.** *We have* $\dim S_2^2(K(353)) \le 12$ *whereas the dimension of Gritsenko lifts in* $S_2^2(K(353))$ *is* $\dim J_{2,353} = 11$. *Let* $G_i = \mathrm{Grit}(\mathrm{THBK}_2(\Sigma_i))$ *for* $1 \le i \le 11$ *be the lifts of the* 11 *theta blocks given by* $\Sigma_i \in \{[3,4,4,4,6,7,8,10,12,16]$, $[3,3,4,4,5,7,7,10,12,17]$, $[2,3,5,5,6,7,9,10,11,16]$, $[2,3,5,5,6,7,8,10,13,15]$, $[2,3,5,5,5,7,10,10,12,15]$, $[2,3,4,6,6,7,9,9,13,15]$, $[2,3,4,6,6,7,8,10,14,14]$, $[2,3,4,5,6,7,9,11,13,14]$, $[2,3,4,5,5,7,8,9,12,17]$, $[2,3,4,4,6,8,10,11,12,14]$, $[2,3,3,5,6,9,9,11,12,14]\}$. *Let*

$$
\begin{aligned}
Q = &- G_{10}G_{11} - 2G_{11}G_2 + G_{10}G_3 + 4G_{11}G_3 + 2G_2G_3 - 4G_3^2 - 5G_{11}G_4 + 5G_3G_4 \\
&+ G_{11}G_5 - G_3G_5 + 6G_{11}G_6 - 6G_3G_6 - 11G_1G_7 + 11G_{10}G_7 - 9G_{11}G_7 + 9G_3G_7 \\
&+ 11G_4G_7 - 11G_6G_7 + 11G_7^2 + 11G_1G_8 - 11G_{10}G_8 + 9G_{11}G_8 - 9G_3G_8 \\
&- 11G_4G_8 + 11G_6G_8 - 22G_7G_8 + 11G_8^2 - 2G_{11}G_9 + 2G_3G_9, \\
L = &- G_{11} + G_3.
\end{aligned}
$$

Let $f = Q/L$ *define a meromorphic form of weight* 2. *Let* $\{[1, 8, 12, 12], [2, 3, 4, 18], [2, 3, 12, 14], [2, 6, 12, 13], [3, 10, 10, 12], [4, 7, 12, 12], [5, 6, 6, 16], [8, 8, 9, 12]\}$ *be labeled* $\{\Xi_1, \ldots, \Xi_8\}$. *For* $\Xi_i = [a,b,c,d]$ *and* $\Xi_j = [\alpha,\beta,\gamma,\delta]$, *define*

$W(i,j) = \mathrm{Grit}(\vartheta_a\vartheta_b\vartheta_c\vartheta_d)\,\mathrm{Grit}(\vartheta_\alpha\vartheta_\beta\vartheta_\gamma\vartheta_\delta) - \mathrm{Grit}\left(\mathrm{THBK}_4(a,b,c,d,\alpha,\beta,\gamma,\delta)\right)$. *Set*

$$\begin{aligned}\hat{Q} =&22G_1^2 - 165G_1G_{10} + 133G_{10}^2 + 385G_1G_{11} - 646G_{10}G_{11} + 407G_{11}^2 + 224G_{10}G_2\\ &-214G_{11}G_2 + 125G_2^2 - 286G_1G_3 + 330G_{10}G_3 - 562G_{11}G_3 + 121G_2G_3 + 127G_3^2\\ &-165G_1G_4+230G_{10}G_4-491G_{11}G_4+119G_2G_4+330G_3G_4+113G_4^2-220G_1G_5\\ &+231G_{10}G_5 - 25G_{11}G_5 - 22G_2G_5 + 113G_3G_5 + 110G_4G_5 - 100G_5^2 + 121G_1G_6\\ &-177G_{10}G_6 + 103G_{11}G_6 - 123G_2G_6 - 165G_3G_6 - 60G_4G_6 + 110G_5G_6 - 30G_6^2\\ &-572G_1G_7+594G_{10}G_7-479G_{11}G_7+99G_2G_7+457G_3G_7+330G_4G_7-103G_5G_7\\ &-66G_6G_7 + 205G_7^2 + 473G_1G_8 - 495G_{10}G_8 + 644G_{11}G_8 - 220G_2G_8 - 501G_3G_8\\ &-352G_4G_8 - 18G_5G_8 + 66G_6G_8 - 432G_7G_8 + 227G_8^2 + 242G_1G_9 - 214G_{10}G_9\\ &+213G_{11}G_9-377G_2G_9-198G_3G_9+132G_5G_9+64G_6G_9-242G_7G_9+185G_8G_9\\ &+224G_9^2 + 264W(1,2) + 528W(2,3) - 143W(2,4) - 178W(3,4) + 264W(1,5)\\ &+528W(2,5)+528W(3,5)-143W(4,5)-242W(2,6)-242W(3,6)-242W(5,6)\\ &-143W(1,7) + 35W(3,7) - 35W(4,7) - 143W(1,8) - 264W(3,8) + 143W(4,8)\\ &-143W(7,8),\\ \hat{L} =&9G_{10} - 3G_{11} - 4G_2 + G_4 + G_6 + 7G_9.\end{aligned}$$

The following equation in $S_2^8(K(353))$ holds on all Fourier coefficients for $T \in {}^{277}\mathcal{X}_2$ satisfying $\det(2T) \le 5000$. If we indeed have

$$Q^2 + \hat{L}QL + \hat{Q}L^2 = 0, \tag{7.5}$$

then the form f is holomorphic and $S_2^2(K(353))(\mathbb{Q}) = \mathrm{Span}_{\mathbb{Q}}(f, G_1, \dots, G_{11})$. Furthermore, the form f is a Hecke eigenform with spin Euler factors

$$\begin{aligned}Q_2(f,x) &= 1 + x + 3x^2 + 2x^3 + 4x^4,\\ Q_3(f,x) &= 1 + 2x + 4x^2 + 6x^3 + 9x^4.\end{aligned}$$

8. Examples for weights $k > 2$

For weights greater than two, the constructions of paramodular cusp forms in this article become easier because the dimension formulae of Ibukiyama apply. Existence of a nonlift follows whenever we have $\dim S_2^k(K(p)) > \dim J_{k,p}^{\mathrm{cusp}}$. Furthermore, nonlifts of higher weight occur at lower prime levels and identities and congruences may sometimes be proven directly from Corollary 5.7 and Theorem 5.9. As these nonlifts are the first examples likely to arise in related work, we give some examples here. Already, there has been interest in the weight three case. A. Ash, P. Gunnells and M. McConnell in [2] studied $H^5(\Gamma_0(p), \mathbb{C})$ and found cusp forms and computed Euler 2 and 3 factors at levels $p = 61$, 73 and 79. They predicted the existence of corresponding Siegel modular cusp forms for $\Gamma_0'(p)$ and requested a construction. The paramodular cusp forms in $S_2^3(K(p))$ constructed here have macthing Euler factors at 2 and 3; we additionally computed the Euler 5-factor. We thank A. Brumer for bringing this topic to our attention. Recall our normalization for the q-Euler factor of a Hecke eigenform $f \in S_2^k(K(p))$:

$$Q_q(f,x) = 1 - \lambda_q x + (\lambda_q^2 - \lambda_{q^2} - q^{2k-4})x^2 - \lambda_q\, q^{2k-3}x^3 + q^{4k-6}x^4.$$

These Euler factors possess the symmetry $x^4 q^{4k-6} Q_q\left(q^{3-2k}/x\right) = Q_q(x)$. The first three examples use theta blocks of weight 3:

$$\mathrm{THBK}_3(d_1, d_2, \ldots, d_9)(\tau, z) = \eta(\tau)^{-3} \prod_{i=1}^{9} \vartheta(\tau, d_i z).$$

1. Example. We have $\dim S_2^3(K(61)) = 7$ and $\dim J_{3,61}^{\mathrm{cusp}} = 6$. There is a nonlift Hecke eigenform $f \in S_2^3(K(61))^-(\mathbb{Z})$ with Euler factors:

$$\begin{aligned} Q_2(f,x) &= 1 + 7x + 24x^2 + 56x^3 + 64x^4, \\ Q_3(f,x) &= 1 + 3x + 3x^2 + 81x^3 + 729x^4, \\ Q_5(f,x) &= 1 - 3x + 85x^2 - 375x^3 + 15625x^4. \end{aligned}$$

For $\ell = 43$, f is congruent to an element of $\mathrm{Grit}\left(J_{3,61}^{\mathrm{cusp}}\right)(\mathbb{Z})$ modulo ℓ and this is the only such congruence. We may define the nonlift f via

$$f = -9B[1] - 2B[2] + 22B[3] + 9B[4] - 10B[5] + 19B[6] - 43B[1]B[6]/B[2],$$

where, for $1 \le i \le 6$, the $B[i] = \mathrm{Grit}\left(\mathrm{THBK}_3(\Xi_i)\right)$ are Gritsenko lifts of the theta blocks given by $\Xi_i = [2, 2, 2, 3, 3, 3, 3, 5, 7]$, $[2, 2, 2, 2, 3, 4, 4, 4, 7]$, $[2, 2, 2, 2, 3, 3, 4, 6, 6]$, $[1, 2, 3, 3, 3, 3, 4, 4, 7]$, $[1, 2, 3, 3, 3, 3, 3, 6, 6]$, $[1, 2, 2, 2, 4, 4, 4, 5, 6]$. The integrality of f may be checked, using Theorem 5.9, by computing the $a(T; f)$ to be integers for $T \in {}^{61}\mathcal{X}_2$ with $w(T) \le \frac{3}{6}\frac{61^2+1}{61+1} < 30.02$; there are 1477 such $\hat{\Gamma}_0(61)$-classes. Alternatively, we may note, as in the proof of Theorem 7.3, that the Fourier expansion of $B[2]$ has unit content because $a(\frac{1}{2}\left(\begin{smallmatrix} 244 & 22 \\ 22 & 2 \end{smallmatrix}\right); B[2]) = -1$.

2. Example. We have $\dim S_2^3(K(73)) = 9$ and $\dim J_{3,73}^{\mathrm{cusp}} = 8$. There is a nonlift Hecke eigenform $f \in S_2^3(K(73))^-(\mathbb{Z})$ with Euler factors:

$$\begin{aligned} Q_2(f,x) &= 1 + 6x + 22x^2 + 48x^3 + 64x^4, \\ Q_3(f,x) &= 1 + 2x + 3x^2 + 54x^3 + 729x^4, \\ Q_5(f,x) &= 1 + 130x^2 + 15625x^4. \end{aligned}$$

For $\ell \in \{3, 13\}$, f is congruent to an element of $\mathrm{Grit}\left(J_{3,73}^{\mathrm{cusp}}\right)(\mathbb{Z})$ modulo ℓ and this is the only such congruence. We may define the nonlift

$$\begin{aligned} f = 9B[1] + 19B[2] + 2B[3] - 13B[4] + 34B[5] - 15B[6] - 12B[7] \\ - 10B[8] - 39B[2]B[6]/B[4], \end{aligned}$$

where the $B[i] = \mathrm{Grit}\left(\mathrm{THBK}_3(\Xi_i)\right)$ are the Gritsenko lifts of the theta blocks given, for $1 \le i \le 8$, by $\Xi_i = [2,3,3,3,3,4,4,5,7]$, $[2,3,3,3,3,3,5,6,6]$, $[2,2,3,4,4,4,4,4,7]$, $[2,2,3,3,4,4,4,6,6]$, $[2,2,3,3,3,5,5,5,6]$, $[2,2,2,4,4,4,5,5,6]$, $[2,2,2,2,3,4,4,5,8]$, $[2,2,2,2,2,4,5,6,7]$. From the Fourier coefficient $a(\frac{1}{2}\left(\begin{smallmatrix} 146 & 17 \\ 17 & 2 \end{smallmatrix}\right); B[4]) = 1$, we see that the Fourier expansion of $B[4]$ has unit content, so that f is integral.

3. Example. We have $\dim S_2^3(K(79)) = 8$ and $\dim J_{3,79}^{\mathrm{cusp}} = 7$. There is a nonlift Hecke eigenform $f \in S_2^3(K(79))^-(\mathbb{Z})$ with Euler factors:

$$\begin{aligned} Q_2(f,x) &= 1 + 5x + 14x^2 + 40x^3 + 64x^4, \\ Q_3(f,x) &= 1 + 5x + 42x^2 + 135x^3 + 729x^4, \\ Q_5(f,x) &= 1 - 3x + 80x^2 - 375x^3 + 15625x^4. \end{aligned}$$

For $\ell = 2$, f is congruent to an element of $\mathrm{Grit}\left(J_{3,79}^{\mathrm{cusp}}\right)(\mathbb{Z})$ modulo ℓ^5 and this is the only such congruence. We may define the nonlift f via

$$\begin{aligned} f = (&-32B[1]^2 + 32B[2]^2 + 32B[1]B[3] - 64B[2]B[3] + 32B[3]^2 \\ &+ 26B[1]B[4] - 38B[2]B[4] + 19B[3]B[4] + 3B[4]^2 + 32B[1]B[5] \\ &- 17B[4]B[5] - 32B[2]B[6] + 32B[3]B[6] + 27B[4]B[6] + 64B[1]B[7] \\ &+ 64B[2]B[7] - 96B[3]B[7] - 68B[4]B[7] - 32B[5]B[7] - 32B[6]B[7])/B[4], \end{aligned}$$

where the $B[i] = \mathrm{Grit}\left(\mathrm{THBK}_3(\Xi_i)\right)$ are the Gritsenko lifts of the theta blocks given, for $1 \le i \le 7$, by $\Xi_i = [2,2,3,3,3,3,5,5,8], [2,2,2,3,4,4,4,5,8], [2,2,2,2,4,4,5,6,7], [2,2,2,2,2,4,4,5,9], [1,3,3,3,3,4,4,5,8], [1,2,3,4,4,4,4,4,8], [1,2,3,3,3,4,5,6,7]$. From the Fourier coefficient $a(\frac{1}{2}\left(\begin{smallmatrix} 1106 & 47 \\ 47 & 2 \end{smallmatrix}\right); B[4]) = -1$, we see that the Fourier expansion of $B[4]$ has unit content and so f is integral.

4. Example. We have $\dim S_2^4\left(K(83)\right)^- = 1$. This is the lowest prime level for which a weight four minus form occurs. There is a nonlift Hecke eigenform $f \in S_2^4\left(K(83)\right)^-(\mathbb{Z})$ with Euler factors:

$$\begin{aligned} Q_2(f,x) &= 1 + 17x + 132x^2 + 544x^3 + 1024x^4, \\ Q_3(f,x) &= 1 + 23x + 270x^2 + 5589x^3 + 59049x^4. \end{aligned}$$

We may define the nonlift f via

$$f = \frac{1}{48}\left(\mathrm{Tr}\left(\vartheta_Q^2\right) - \mathrm{Tr}\left(\vartheta_Q^2\right)|\mu\right) \text{ for } Q = \left(\begin{smallmatrix} 4 & 1 & 1 & 0 \\ 1 & 6 & 2 & -2 \\ 1 & 2 & 8 & 3 \\ 0 & -2 & 3 & 44 \end{smallmatrix}\right).$$

To prove the integrality of f, we use forms of determinant 83^2 and level 83, see [23]:

$$B = \left(\begin{smallmatrix} 4 & 1 & 0 & 1 \\ 1 & 6 & 3 & 1 \\ 0 & 3 & 16 & 4 \\ 1 & 1 & 4 & 22 \end{smallmatrix}\right); C = \left(\begin{smallmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 42 & 0 \\ 0 & -1 & 0 & 42 \end{smallmatrix}\right); D = \left(\begin{smallmatrix} 2 & 1 & -1 & 0 \\ 1 & 12 & 3 & -2 \\ -1 & 3 & 16 & 3 \\ 0 & -2 & 3 & 22 \end{smallmatrix}\right).$$

For convenience, consider the index matrix $T_0 = \frac{1}{2}\left(\begin{smallmatrix} 5478 & 148 \\ 148 & 4 \end{smallmatrix}\right)$ in ${}^{83}\mathcal{X}_2$ of smallest determinant. For a cusp form in $S_2^4\left(K(83)\right)$, we shall call its coefficient at T_0 its leading coefficient. By Theorem 3.13, the form $\frac{1}{4}\left(\mathrm{Tr}\left(\vartheta_Q\vartheta_B\right) - \mathrm{Tr}\left(\vartheta_Q\vartheta_B\right)|\mu\right)$ is integral; it has leading coefficient 51. Also by Theorem 3.13, the form $\frac{1}{4}\left(\mathrm{Tr}\left(\vartheta_C\vartheta_D\right) - \mathrm{Tr}\left(\vartheta_C\vartheta_D\right)|\mu\right)$ is integral; it has leading coefficient $16 \cdot 23$. Some integer linear combination of the above two integral forms has leading coefficient 1 because 51 and $16 \cdot 23$ are relatively prime. Since the leading coefficient of f is 1 and the space $S_2^4\left(K(83)\right)^-$ is one-dimensional, f is itself integral.

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