International Journal of Mathematics
Vol. 26, No. 1 (2015) 1550004 (16 pages)

DOI: 10.1142/S0129167X15500044

Vector-valued Siegel modular forms of weight $\det^k \otimes \mathrm{Sym}(8)$

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Received 27 February 2013
Accepted 10 December 2014
Published 28 January 2015

We determine the structures of modules of vector-valued Siegel modular forms of weight $\det^k \otimes \mathrm{Sym}(8)$ with respect to the full Siegel modular group of degree two.

Keywords: Siegel modular forms; differential operators.

Mathematics Subject Classification 2010: 11F46, 11F60

1. Introduction

The structures of spaces of Siegel modular forms have been studied by many mathematicians, for example, Igusa [12], Satoh [15], Freitag and Salvati Manni [6, 7], Gunji [8], Aoki and Ibukiyama [1], Kitayama [13], Ibukiyama [10] and van Dorp [18]. Particularly noteworthy of the above works is that of Igusa, in which he obtains explicit generators of the graded ring of scalar-valued Siegel modular forms with respect to the full Siegel modular group Γ_2 of degree two. In this paper, we carry out a similar study, investigating the structures of modules of certain vector-valued Siegel modular forms with respect to Γ_2.

We denote by $A_{k,j}(\Gamma_2)$ the vector space over $\mathbb{C}$ of vector-valued Siegel modular forms of weight $\det^k \otimes \mathrm{Sym}(j)$ with respect to Γ_2. Here, $\det^k$ is the determinant representation of $\mathrm{GL}_2(\mathbb{C})$ to the kth power, and $\mathrm{Sym}(j)$ is the symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$ of degree j. We define the sets $A^{\mathrm{even}}_{\mathrm{Sym}(j)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(j)}(\Gamma_2)$ as $A^{\mathrm{even}}_{\mathrm{Sym}(j)}(\Gamma_2) := \bigoplus_{k=0}^{\infty} A_{2k,j}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(j)}(\Gamma_2) := \bigoplus_{k=0}^{\infty} A_{2k+1,j}(\Gamma_2)$. It is clear that the sets $A^{\mathrm{even}}_{\mathrm{Sym}(j)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(j)}(\Gamma_2)$ are modules over the ring $A^{\mathrm{even}}(\Gamma_2) := A^{\mathrm{even}}_{\mathrm{Sym}(0)}(\Gamma_2)$. Satoh [15] determined the structure of $A^{\mathrm{even}}_{\mathrm{Sym}(2)}(\Gamma_2)$, and Ibukiyama [10] determined the structures of $A^{\mathrm{odd}}_{\mathrm{Sym}(2)}(\Gamma_2)$, $A^{\mathrm{even}}_{\mathrm{Sym}(4)}(\Gamma_2)$, $A^{\mathrm{odd}}_{\mathrm{Sym}(4)}(\Gamma_2)$ and $A^{\mathrm{even}}_{\mathrm{Sym}(6)}(\Gamma_2)$. Recently, van Dorp [18] determined the structure of $A^{\mathrm{odd}}_{\mathrm{Sym}(6)}(\Gamma_2)$. In this work, we add to these results by determining the structures of $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$.

The motivation for this work is provided by the following. First, we are interested in the direct sum $\bigoplus_{k,j=0}^{\infty} A_{k,j}(\Gamma_2)$ because, although it is the fundamental, mathematical object in Siegel modular forms of degree two, very little is known about it. Also, this set is related to the question raised by Ibukiyama [10] concerning the feasibility of constructing a theory of "weak vector-valued Siegel modular forms" similar to that of weak Jacobi forms presented in [4]. We are also interested in a separate conjecture made by Ibukiyama [10] (see Sec. 4).

This paper is organized as follows. In Sec. 2, we review the theory of Siegel modular forms, summarize existing results, present the dimension formula for $A_{k,8}(\Gamma_2)$ and outline methods for constructing Siegel modular forms (employing theta functions and Rankin–Cohen-type differential operators). We also construct a Rankin–Cohen-type differential operator that maps three scalar-valued Siegel modular forms of even weights to a vector-valued Siegel modular form of weight $\det^k \otimes \mathrm{Sym}(8)$ for odd k. The main result of the paper is given in Sec. 3. We give evidence supporting the conjecture made by Ibukiyama [10] in Sec. 4.

2. Preliminaries

2.1. *Vector-valued Siegel modular forms*

First, we present definitions. Let $\mathrm{Sp}(2,\mathbb{R})$ be the real symplectic group of degree two and H_2 the Siegel upper half space of degree two:

$$\mathrm{Sp}(2,\mathbb{R}) := \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{R}) \,\middle|\, {}^t\gamma \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \gamma = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\},$$

$$H_2 := \left\{ Z = X + iY \in \mathrm{Mat}_2(\mathbb{C}) \,|\, Z = {}^tZ, Y \text{ is positive definite} \right\}.$$

The group $\mathrm{Sp}(2,\mathbb{R})$ acts on H_2 as $\left(\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z\right) \mapsto \gamma Z := (AZ+B)(CZ+D)^{-1}$. Let $\Gamma_2 := \mathrm{Sp}(2,\mathbb{Z})$ be the full Siegel modular group of degree two:

$$\Gamma_2 = \mathrm{Sp}(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Z}) \,\middle|\, {}^t\gamma \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \gamma = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}.$$

Fixing non-negative integers k and j, let $\rho_{k,j} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{j+1}(\mathbb{C})$ be the irreducible rational representation of the signature $(j+k,k)$; i.e.

$$\rho_{k,j} = \det{}^k \otimes \mathrm{Sym}(j),$$

where $\det^k$ is the determinant representation of $\mathrm{GL}_2(\mathbb{C})$ to the kth power and $\mathrm{Sym}(j)$ is the symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$ of degree j. For any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2,\mathbb{R})$ and $\mathbb{C}^{j+1}$-valued function f on H_2, we define

$$(f|_{k,j}[\gamma])(Z) := \rho_{k,j}(CZ+D)^{-1} f(\gamma Z) \quad (Z \in H_2).$$

In the case that $j = 0$, we write $|_{k,j}$ as $|_k$. A vector-valued holomorphic function $f : H_2 \to \mathbb{C}^{j+1}$ is a vector-valued Siegel modular form of weight $\rho_{k,j}$ with respect to Γ_2 if $f\big|_{k,j}[\gamma] = f$ for all $\gamma \in \Gamma_2$. Note that f is a scalar-valued Siegel modular form of weight k with respect to Γ_2 when $j = 0$. Then, because $-1_4 \in \Gamma_2$, we see that f is identically zero if j is odd. To treat cusp forms, we define the Siegel operator $\Phi : A_{k,j}(\Gamma_2) \to S_{k+j}(\mathrm{SL}_2(\mathbb{Z}))$. For $f \in A_{k,j}(\Gamma_2)$, we define $\Phi(f)(\tau) := \lim_{\lambda\to+\infty} f\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}$. As shown by Arakawa [2], we have

$$\Phi(f)(\tau) = \begin{pmatrix} \tilde{f}(\tau) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{f} \in S_{k+j}(\mathrm{SL}_2(\mathbb{Z})),$$

where $S_{k+j}(\mathrm{SL}_2(\mathbb{Z}))$ is the $\mathbb{C}$-vector space of elliptic cusp forms of weight $k+j$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. We say that a Siegel modular from $f \in A_{k,j}(\Gamma_2)$ is a cusp form if it satisfies $\Phi(f) = 0$. We denote by $A_{k,j}(\Gamma_2)$ (respectively, $S_{k,j}(\Gamma_2)$) the vector space over $\mathbb{C}$ of Siegel modular forms (respectively, cusp forms) of weight $\rho_{k,j}$ with respect to Γ_2. When $j = 0$, we simply write these as $A_k(\Gamma_2)$ and $S_k(\Gamma_2)$. Recall the definitions of $A^{\mathrm{even}}_{\mathrm{Sym}(j)}(\Gamma_2)$, $A^{\mathrm{odd}}_{\mathrm{Sym}(j)}(\Gamma_2)$ and $A^{\mathrm{even}}(\Gamma_2)$ given above.

Next, we summarize existing results concerning spaces of Siegel modular forms with respect to Γ_2. In [11, 12], Igusa proved the following theorem.

Theorem 2.1. (i) *The graded ring of scalar-valued Siegel modular forms of even weight with respect to* Γ_2 *is given by*

$$A^{\mathrm{even}}(\Gamma_2) = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}],$$

where ϕ_k *is the Eisenstein series of weight* k, *and* χ_{10} *and* χ_{12} *are the cusp forms of weights* 10 *and* 12, *respectively.*

(ii) *The graded ring of scalar-valued Siegel modular forms with respect to* Γ_2 *is given by*

$$\bigoplus_{k=0}^{\infty} A_k(\Gamma_2) = A^{\mathrm{even}}(\Gamma_2) \oplus \chi_{35} A^{\mathrm{even}}(\Gamma_2),$$

where χ_{35} *is the cusp form of weight* 35.

We normalize the Eisenstein series ϕ_k of weight k so that the constant term is equal to 1 and the cusp form χ_{10} or χ_{12} so that the Fourier coefficient for $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ is equal to 1. The cusp form χ_{35} is defined as

$$\chi_{35} := 2^{-9} \cdot 3^{-4} \begin{vmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\phi_4 & \partial_1\phi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ \partial_2\phi_4 & \partial_2\phi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ \partial_3\phi_4 & \partial_3\phi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{vmatrix} \in S_{35}(\Gamma_2),$$

where for $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, we define

$$\partial_1 := \frac{1}{2\pi i}\frac{\partial}{\partial \tau}, \quad \partial_2 := \frac{1}{2\pi i}\frac{\partial}{\partial z}, \quad \partial_3 := \frac{1}{2\pi i}\frac{\partial}{\partial \omega}.$$

Then, the Fourier coefficient of χ_{35} for $\begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$ is -1.

The structures of the following $A^{\mathrm{even}}(\Gamma_2)$-modules of vector-valued Siegel modular forms are known:

- $A^{\mathrm{even}}_{\mathrm{Sym}(2)}(\Gamma_2) \cdots$ [15].
- $A^{\mathrm{odd}}_{\mathrm{Sym}(2)}(\Gamma_2) \cdots$ [10].
- $A^{\mathrm{even}}_{\mathrm{Sym}(4)}(\Gamma_2) \cdots$ [10].
- $A^{\mathrm{odd}}_{\mathrm{Sym}(4)}(\Gamma_2) \cdots$ [10].
- $A^{\mathrm{even}}_{\mathrm{Sym}(6)}(\Gamma_2) \cdots$ [10].
- $A^{\mathrm{odd}}_{\mathrm{Sym}(6)}(\Gamma_2) \cdots$ [18].

Here, we study the $A^{\mathrm{even}}(\Gamma_2)$-modules $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$.

2.2. *Dimension formula*

The generating function of $\dim_{\mathbb{C}} A_{k,8}(\Gamma_2)$ is known (cf. [10, 17, 19]), and is given by the following.

Theorem 2.2. *We have*

$$\sum_{k=0}^{\infty} \dim_{\mathbb{C}} A_{k,8}(\Gamma_2)t^k = \frac{t^4 + t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16} + t^{18}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})} + \frac{t^9 + t^{11} + t^{13} + 2t^{15} + 2t^{17} + t^{19} + t^{23}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

From Theorem 2.2, we can calculate $\dim_{\mathbb{C}} A_{k,8}(\Gamma_2)$. We obtain the following:

k	4	8	9	10	11	12	13	14	15	16	17	18	19	23
$\dim A_{k,8}$	1	2	1	3	1	4	2	6	4	9	5	10	7	13

With the relation

$$\sum_{k:\mathrm{even}} \dim_{\mathbb{C}} A_k(\Gamma_2)t^k = \frac{1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

(obtained from Theorem 2.1(i)), Theorem 2.2 implies that $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$ are free modules over $A^{\mathrm{even}}(\Gamma_2)$ and that $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ is spanned by nine elements of weights $\rho_{4,8}, \rho_{8,8}, \rho_{10,8}, \rho_{10,8}, \rho_{12,8}, \rho_{12,8}, \rho_{14,8}, \rho_{16,8}$ and $\rho_{18,8}$, while $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$ is spanned by nine elements of weights $\rho_{9,8}$, $\rho_{11,8}$, $\rho_{13,8}$, $\rho_{15,8}$, $\rho_{15,8}$, $\rho_{17,8}$, $\rho_{17,8}$, $\rho_{19,8}$ and $\rho_{23,8}$.

2.3. *Construction of Siegel modular forms*

There exist three methods to construct vector-valued Siegel modular forms with respect to Γ_2, which employ the following:

(i) Theta functions with pluri-harmonic polynomials.
(ii) Rankin–Cohen-type differential operators.
(iii) Klingen-type Eisenstein series (cf. [2]).

We can construct all generators of $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ and $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$ by using methods (i) and (ii). Below we explain this procedure.

2.3.1. *Theta functions with pluri-harmonic polynomials*

Let k and j be non-negative integers. For a natural number d and vectors $x = (x_i), y = (y_i) \in \mathbb{C}^d$, we define the inner product $\langle x, y\rangle$ as $\langle x, y\rangle := \sum_{i=1}^{d} x_i y_i$. Next, let L be an even unimodular lattice in $\mathbb{R}^d$. We identify an element of L^2 with a $2 \times d$ matrix according to $X = {}^t(x, y)$ for $x, y \in L$.

Proposition 2.3 ([5, 10]). *We choose* $a, b \in \mathbb{C}^d$ *such that* $\langle a, a\rangle = \langle a, b\rangle = \langle b, b\rangle = 0$. *For* $X = {}^t(x, y) \in \mathrm{Mat}_{2,d}(\mathbb{C})$, *we define*

$$P_v(X) := \binom{j}{v} \langle x, a\rangle^{j-v} \langle y, a\rangle^{v} \begin{vmatrix} \langle x, a\rangle & \langle x, b\rangle \\ \langle y, a\rangle & \langle y, b\rangle \end{vmatrix}^{k} \quad (0 \le v \le j)$$

and $P(X) := {}^t(P_0(X), \ldots, P_j(X))$. *Then, we have*

$$\Theta_{L,a,b,(k,j)}(Z) := \sum_{X \in L^2} P(X) \exp(\pi i \,\mathrm{Tr}({}^t\! XZX)) \in A_{\frac{d}{2}+k,j}(\Gamma_2).$$

For $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in H_2$, we introduce $q_1 := e^{2\pi i\tau}, \zeta := e^{2\pi i z}$ and $q_2 := e^{2\pi i\omega}$. Then, from the definition of theta functions, we find

$$\begin{aligned} \Theta_{L,a,b,(k,j)}(Z) &= \sum_{x,y \in L} P(X) q_1^{\frac{\langle x,x\rangle}{2}} \zeta^{\langle x,y\rangle} q_2^{\frac{\langle y,y\rangle}{2}} \\ &= \sum_{n,m=0}^{\infty} \left(\sum_{\substack{x,y \in L \\ \langle x,x\rangle = 2n, \langle y,y\rangle = 2m}} P(X) q_1^{\frac{\langle x,x\rangle}{2}} \zeta^{\langle x,y\rangle} q_2^{\frac{\langle y,y\rangle}{2}} \right) \\ &= \sum_{n,m=0}^{\infty} \left(\sum_{\substack{x,y \in L \\ \langle x,x\rangle = 2n, \langle y,y\rangle = 2m}} P(X) \zeta^{\langle x,y\rangle} \right) q_1^n q_2^m. \end{aligned}$$

Therefore, the $(v+1)$th component of $\Theta_{L,a,b,(k,j)}(Z)$ $(0 \le v \le j)$ is

$$\sum_{n,m=0}^{\infty} \left(\sum_{\substack{x,y \in L \\ \langle x,x\rangle = 2n, \langle y,y\rangle = 2m}} P_v(X) \zeta^{\langle x,y\rangle} \right) q_1^n q_2^m.$$

In order to prove the main result, Theorem 3.1, we consider the case in which $L = E_8$, where E_8 is the well-known eight-dimensional root lattice:

$$E_8 := \left\{ x = (x_1, \dots, x_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 \,\middle|\, \sum_{i=1}^{8} x_i \in 2\mathbb{Z} \right\}.$$

2.3.2. *Rankin–Cohen-type differential operators*

Ibukiyama constructed the general theory of Rankin–Cohen-type differential operators in [9]. Here, we summarize this theory. (For further details, see [9].)

For a natural number r, we write $H_2^r := H_2 \times \cdots \times H_2$ (r copies of H_2). Then, we fix non-negative integers k and j and natural numbers k_i $(1 \le i \le r)$. Let $\mathbb{D}$ be a $\mathbb{C}^{j+1}$-valued linear homogeneous holomorphic differential operator with constant coefficients acting on functions on H_2^r. We impose the following condition on $\mathbb{D}$.

Condition 2.4.

$$\operatorname*{Res}_{Z_i = Z} \mathbb{D}\big((f_1|_{k_1}[\gamma])(Z_1) \cdots (f_r|_{k_r}[\gamma])(Z_r)\big)$$
$$= \operatorname*{Res}_{Z_i = Z} \mathbb{D}\big(f_1(Z_1) \cdots f_r(Z_r)\big)|_{k_1+\cdots+k_r+k,j}[\gamma]$$

for all holomorphic functions f_i $(1 \le i \le r)$ on H_2 and all $\gamma \in \mathrm{Sp}(2,\mathbb{R})$, where $\mathrm{Res}_{Z_i=Z}$ represents the restriction of all $Z_i \in H_2$ to the same $Z \in H_2$.

Condition 2.4 implies that if $f_i \in A_{k_i}(\Gamma_2)$ $(1 \le i \le r)$, then we have

$$\operatorname*{Res}_{Z_i = Z} \mathbb{D}\left(f_1(Z_1) \cdots f_r(Z_r)\right) \in A_{k_1+\cdots+k_r+k,j}(\Gamma_2).$$

Here, we call a differential operator $\mathbb{D}$ satisfying Condition 2.4 a "Rankin–Cohen-type differential operator".

To characterize Rankin–Cohen-type differential operators, we need a further condition, namely, Condition 2.5. For a variable $X = (x_{i,j}) \in \mathrm{Mat}_{2,2(k_1+\cdots+k_r)}(\mathbb{C})$, we define the operators

$$\Delta_{i,j} := \sum_{\nu=1}^{2(k_1+\cdots+k_r)} \frac{\partial^2}{\partial x_{i,\nu} \partial x_{j,\nu}} \quad (1 \le i, j \le 2).$$

A polynomial $P(X)$ in $X = (x_{i,j}) \in \mathrm{Mat}_{2,2(k_1+\cdots+k_r)}(\mathbb{C})$ is called pluri-harmonic with respect to X if $\Delta_{i,j}(P) = 0$ for all $1 \le i, j \le 2$. A polynomial vector $Q(X) = {}^t(Q_v(X))$ is pluri-harmonic with respect to X if each component $Q_v(X)$

is pluri-harmonic with respect to X. Let $Q(R_1, \ldots, R_r)$ be a polynomial vector in symmetric matrices $R_1, \ldots, R_r \in \mathrm{Mat}_2(\mathbb{C})$. We impose the following condition on $Q(R_1, \ldots, R_r)$.

Condition 2.5.

(i) For all $A \in \mathrm{GL}_2(\mathbb{C})$,

$$Q(AR_1{}^tA, \ldots, AR_r{}^tA) = \rho_{k,j}(A)Q(R_1, \ldots, R_r).$$

(ii) $P(X) := Q(X_1{}^tX_1, \ldots, X_r{}^tX_r)$ is pluri-harmonic with respect to X, where we write $X = (x_{i,j}) := (X_1, \ldots, X_r) \in \mathrm{Mat}_{2,2(k_1+\cdots+k_r)}(\mathbb{C})$ for $X_i \in \mathrm{Mat}_{2,2k_i}(\mathbb{C})\,(1 \le i \le r)$.

Let $\mathcal{H}_{2,\rho_{k,j}}(2k_1, \ldots, 2k_r)$ denote the set of polynomial vectors satisfying Condition 2.5. We now present a characterization of Rankin–Cohen-type differential operators. For $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, we define

$$\partial_Z := \begin{pmatrix} \partial_1 & \dfrac{\partial_2}{2} \\ \dfrac{\partial_2}{2} & \partial_3 \end{pmatrix} \quad \left(\text{with } \partial_1 = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}, \partial_2 = \frac{1}{2\pi i}\frac{\partial}{\partial z}, \partial_3 = \frac{1}{2\pi i}\frac{\partial}{\partial \omega}\right).$$

Then there exists a $\mathbb{C}^{j+1}$-valued polynomial vector $Q(R_1, \ldots, R_r)$ depending on $(r \times 2(2+1))/2 = 3r$ variables such that

$$\mathbb{D} = Q(\partial_{Z_1}, \ldots, \partial_{Z_r}).$$

Theorem 2.6 ([9]). *The differential operator* $\mathbb{D}$ *satisfies Condition* 2.4 *if and only if the polynomial vector* Q *satisfies Condition* 2.5.

Next, we give explicit examples of Rankin–Cohen-type differential operators. (By Theorem 2.6, this is equivalent to giving explicit polynomial vectors satisfying Condition 2.5.) More specifically, we consider the following three cases:

(a) $\mathcal{H}_{2,\rho_{0,j}}(2k_1, 2k_2)$ for an even number j.
(b) $\mathcal{H}_{2,\rho_{2,j}}(2k_1, 2k_2)$ for an even number j.
(c) $\mathcal{H}_{2,\rho_{1,8}}(2k_1, 2k_2, 2k_3)$.

We consider each of these cases below.

First, we consider the cases (a) and (b). We begin by presenting two results obtained by Eholzer and Ibukiyama [3]. They constructed explicit polynomial vectors $Q \in \mathcal{H}_{2,\rho_{k,j}}(2k_1, 2k_2)$ (with $k = 0, 2$) such that $Q \neq 0$.

Considering the case (a), let j be an even number, and let r, s, u_1 and u_2 be independent variables. For symmetric matrices $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}$ and $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$, we define

$$Q_{2k_1,2k_2,1,j}(r,s) := \sum_{i=0}^{\frac{j}{2}} (-1)^i \binom{\frac{j}{2}+k_2-1}{i} \binom{\frac{j}{2}+k_1-1}{\frac{j}{2}-i} r^i s^{\frac{j}{2}-i},$$

$$U := r_{11}u_1^2 + 2r_{12}u_1u_2 + r_{22}u_2^2,$$

$$V := s_{11}u_1^2 + 2s_{12}u_1u_2 + s_{22}u_2^2.$$

Next, we define polynomials $Q_v(R,S)\,(0 \le v \le j)$ through the relation

$$Q_{2k_1,2k_2,1,j}(U,V) = \sum_{v=0}^{j} Q_v(R,S) u_1^{j-v} u_2^v.$$

Further, we define $Q_{2k_1,2k_2,\mathrm{Sym}(j)}(R,S) := {}^t(Q_0(R,S),\dots,Q_j(R,S))$. Then we have $Q_{2k_1,2k_2,\mathrm{Sym}(j)}(R,S) \in \mathcal{H}_{2,\rho_{0,j}}(2k_1,2k_2)$ (see [3, Proposition 6.1]). For $f_i \in A_{k_i}(\Gamma_2)\,(1 \le i \le 2)$, we define

$$\{f_1,f_2\}_{\mathrm{Sym}(j)}(Z) := \operatorname*{Res}_{Z_i=Z} \mathbb{D}_{k_1,k_2,\mathrm{Sym}(j)}\big(f_1(Z_1)f_2(Z_2)\big),$$

where $\mathbb{D}_{k_1,k_2,\mathrm{Sym}(j)} := Q_{2k_1,2k_2,\mathrm{Sym}(j)}(\partial_{Z_1},\partial_{Z_2})$. Then we have $\{f_1,f_2\}_{\mathrm{Sym}(j)} \in A_{k_1+k_2,j}(\Gamma_2)$.

Next, we consider the case (b). First, we define

$$\begin{aligned} Q_{2k_1,2k_2,2,j}(r,s) := {} & \frac{1}{4}\widetilde{Q}_{2k_1,2k_2}(R,S) Q_{2(k_1+1),2(k_2+1),1,j}(r,s) \\ & + \frac{1}{2}\{(2k_2-1)\det(R)s - (2k_1-1)\det(S)r\} \\ & \times \left(\frac{\partial Q_{2(k_1+1),2(k_2+1),1,j}}{\partial r} - \frac{\partial Q_{2(k_1+1),2(k_2+1),1,j}}{\partial s} \right)(r,s), \end{aligned}$$

where

$$\begin{aligned} \widetilde{Q}_{2k_1,2k_2}(R,S) := {} & (2k_1-1)(2k_2-1)\det(R+S) \\ & - (2k_2-1)(2k_1+2k_2-1)\det(R) \\ & - (2k_1-1)(2k_1+2k_2-1)\det(S). \end{aligned}$$

Then, we define polynomials $Q_v(R,S)\,(0 \le v \le j)$ through the relation

$$Q_{2k_1,2k_2,2,j}(U,V) = \sum_{v=0}^{j} Q_v(R,S) u_1^{j-v} u_2^v.$$

Further, we define $Q_{2k_1,2k_2,\det^2\mathrm{Sym}(j)}(R,S) := {}^t(Q_0(R,S),\dots,Q_j(R,S))$. Then we have $Q_{2k_1,2k_2,\det^2\mathrm{Sym}(j)}(R,S) \in \mathcal{H}_{2,\rho_{2,j}}(2k_1,2k_2)$ (see [3, p. 461; 14, Remark 2.5]).

Next, we define $\mathbb{D}_{k_1,k_2,\det^2\mathrm{Sym}(j)} := Q_{2k_1,2k_2,\det^2\mathrm{Sym}(j)}(\partial_{Z_1}, \partial_{Z_2})$, and for $f_i \in A_{k_i}(\Gamma_2)\ (1 \le i \le 2)$, we define

$$\{f_1, f_2\}_{\det^2\mathrm{Sym}(j)}(Z) := \operatorname*{Res}_{Z_i=Z} \mathbb{D}_{k_1,k_2,\det^2\mathrm{Sym}(j)}\big(f_1(Z_1)f_2(Z_2)\big).$$

Then we have $\{f_1, f_2\}_{\det^2\mathrm{Sym}(j)} \in A_{k_1+k_2+2,j}(\Gamma_2)$.

Finally, we consider the case (c). Here, we give a new example of a Rankin–Cohen-type differential operator. Let $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}$, $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$ and $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}$ be 2×2 symmetric matrices, and let u be an independent variable. We define

$$\begin{aligned} Q_0(R,S,T) :=\ & (2k_2+2)(2k_2+4)(2k_2+6)r_{11}^3 \begin{vmatrix} r_{11} & s_{11} & t_{11} \\ r_{12} & s_{12} & t_{12} \\ 2k_1+6 & 2k_2 & 2k_3 \end{vmatrix} \\ & - 3(2k_1+6)(2k_2+4)(2k_2+6)r_{11}^2 s_{11} \begin{vmatrix} r_{11} & s_{11} & t_{11} \\ r_{12} & s_{12} & t_{12} \\ 2k_1+4 & 2k_2+2 & 2k_3 \end{vmatrix} \\ & + 3(2k_1+4)(2k_1+6)(2k_2+6)r_{11} s_{11}^2 \begin{vmatrix} r_{11} & s_{11} & t_{11} \\ r_{12} & s_{12} & t_{12} \\ 2k_1+2 & 2k_2+4 & 2k_3 \end{vmatrix} \\ & - (2k_1+2)(2k_1+4)(2k_1+6)s_{11}^3 \begin{vmatrix} r_{11} & s_{11} & t_{11} \\ r_{12} & s_{12} & t_{12} \\ 2k_1 & 2k_2+6 & 2k_3 \end{vmatrix}. \end{aligned}$$

Then, we define polynomials $Q_v(R,S,T)\ (0 \le v \le 8)$ through the relation

$$Q_0(AR^tA, AS^tA, AT^tA) = \sum_{v=0}^{8} Q_v(R,S,T)u^v,$$

where $A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Further, we define

$$Q_{2k_1,2k_2,2k_3,\det\mathrm{Sym}(8)}(R,S,T) := {}^t(Q_0(R,S,T), \ldots, Q_8(R,S,T)).$$

Lemma 2.7. *We have* $Q_{2k_1,2k_2,2k_3,\det\mathrm{Sym}(8)}(R,S,T) \in \mathcal{H}_{2,\rho_{1,8}}(2k_1, 2k_2, 2k_3)$.

Proof. Note that the group $\mathrm{GL}_2(\mathbb{C})$ is generated by the matrices

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad (a, d \in \mathbb{C}\backslash\{0\}), \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad (b \in \mathbb{C}), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Because $\rho_{1,8}$ is a representation of $\mathrm{GL}_2(\mathbb{C})$, it is sufficient to prove that $Q_{2k_1,2k_2,2k_3,\det\mathrm{Sym}(8)}(R,S,T)$ satisfies Condition 2.5(i) with the above matrices. It can also be shown that $Q_{2k_1,2k_2,2k_3,\det\mathrm{Sym}(8)}(R,S,T)$ satisfies Condition 2.5(ii) by a direct calculation. ∎

We define $\mathbb{D}_{k_1,k_2,k_3,\det \mathrm{Sym}(8)} := Q_{2k_1,2k_2,2k_3,\det \mathrm{Sym}(8)}(\partial_{Z_1},\partial_{Z_2},\partial_{Z_3})$. Next, for $f_i \in A_{k_i}(\Gamma_2)$ $(1 \le i \le 3)$, we define

$$\{f_1,f_2,f_3\}_{\det \mathrm{Sym}(8)}(Z) := \operatorname*{Res}_{Z_i=Z} \mathbb{D}_{k_1,k_2,k_3,\det \mathrm{Sym}(8)}\big(f_1(Z_1)f_2(Z_2)f_3(Z_3)\big).$$

Then we have $\{f_1,f_2,f_3\}_{\det \mathrm{Sym}(8)} \in A_{k_1+k_2+k_3+1,8}(\Gamma_2)$.

3. Main Result

In this section, we present the theorem that represents the main result of this paper, along with its proof. First, as preparation, let us define the vectors

$$\begin{aligned} a_1 &:= (1,i,0,0,0,0,0,0), & b_1 &:= (0,0,1,i,0,0,0,0),\\ a_2 &:= (2,i,i,i,i,0,0,0), & b_2 &:= (0,i,i,-i,-i,2,0,0),\\ a_3 &:= (2,1,i,i,i,i,i,0), & b_3 &:= (1,-1,i,i,1,-1,-i,i) \end{aligned}$$

and the Siegel modular forms $X_{k,8}, Y_{k,8} \in A_{k,8}(\Gamma_2)$ as follows:

$$X_{4,8} := \frac{\Theta_{E_8,a_1,b_1,(0,8)}}{120} \in A_{4,8}(\Gamma_2), \quad X_{8,8} := \frac{\Theta_{E_8,a_1,b_1,(4,8)}}{22400} \in A_{8,8}(\Gamma_2),$$

$$X_{10,8} := \frac{\Theta_{E_8,a_2,b_2,(6,8)}}{275251200} \in A_{10,8}(\Gamma_2), \quad Y_{10,8} := \frac{\{\phi_4,\phi_4\}_{\det^2 \mathrm{Sym}(8)}}{19756800} \in A_{10,8}(\Gamma_2),$$

$$X_{12,8} := \frac{\Theta_{E_8,a_1,b_1,(8,8)}}{400} \in A_{12,8}(\Gamma_2), \quad Y_{12,8} := \frac{\{\phi_4,\phi_6\}_{\det^2 \mathrm{Sym}(8)}}{9313920} \in A_{12,8}(\Gamma_2),$$

$$X_{14,8} := \frac{\{\phi_4,\chi_{10}\}_{\mathrm{Sym}(8)}}{5} \in A_{14,8}(\Gamma_2), \quad X_{16,8} := \frac{\{\phi_6,\chi_{10}\}_{\mathrm{Sym}(8)}}{126} \in A_{16,8}(\Gamma_2),$$

$$X_{18,8} := \frac{\{\phi_6,\chi_{12}\}_{\mathrm{Sym}(8)}}{126} \in A_{18,8}(\Gamma_2),$$

$$X_{9,8} := \frac{\Theta_{E_8,a_3,b_3,(5,8)}}{350563200} \in A_{9,8}(\Gamma_2), \quad X_{11,8} := \frac{\Theta_{E_8,a_3,b_3,(7,8)}}{12096000000} \in A_{11,8}(\Gamma_2),$$

$$X_{13,8} := \frac{\{\phi_4,\phi_4,\phi_4\}_{\det \mathrm{Sym}(8)}}{6502809600} \in A_{13,8}(\Gamma_2),$$

$$X_{15,8} := \frac{\{\phi_4,\phi_4,\phi_6\}_{\det \mathrm{Sym}(8)}}{1393459200} \in A_{15,8}(\Gamma_2),$$

$$Y_{15,8} := \frac{\{\phi_6,\phi_4,\phi_4\}_{\det \mathrm{Sym}(8)}}{418037760} \in A_{15,8}(\Gamma_2),$$

$$X_{17,8} := \frac{\{\phi_4,\phi_6,\phi_6\}_{\det \mathrm{Sym}(8)}}{1463132160} \in A_{17,8}(\Gamma_2),$$

$$Y_{17,8} := \frac{\{\phi_6,\phi_6,\phi_4\}_{\det \mathrm{Sym}(8)}}{7023034368} \in A_{17,8}(\Gamma_2),$$

$$X_{19,8} := \frac{\{\chi_{10},\phi_4,\phi_4\}_{\det \mathrm{Sym}(8)}}{599040} \in A_{19,8}(\Gamma_2),$$

$$X_{23,8} := \frac{\{\phi_4,\phi_6,\chi_{12}\}_{\det \mathrm{Sym}(8)}}{483840} \in A_{23,8}(\Gamma_2).$$

Theorem 3.1. *Both* $A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2)$ *and* $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$ *are free* $A^{\mathrm{even}}(\Gamma_2)$*-modules of rank* 9, *and their generators over* $A^{\mathrm{even}}(\Gamma_2)$ *are given by the above modular forms*:

$$\begin{aligned} A^{\mathrm{even}}_{\mathrm{Sym}(8)}(\Gamma_2) &= A^{\mathrm{even}}(\Gamma_2)X_{4,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{8,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{10,8} \\ &\quad \oplus A^{\mathrm{even}}(\Gamma_2)Y_{10,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{12,8} \oplus A^{\mathrm{even}}(\Gamma_2)Y_{12,8} \\ &\quad \oplus A^{\mathrm{even}}(\Gamma_2)X_{14,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{16,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{18,8}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2) &= A^{\mathrm{even}}(\Gamma_2)X_{9,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{11,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{13,8} \\ &\quad \oplus A^{\mathrm{even}}(\Gamma_2)X_{15,8} \oplus A^{\mathrm{even}}(\Gamma_2)Y_{15,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{17,8} \\ &\quad \oplus A^{\mathrm{even}}(\Gamma_2)Y_{17,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{19,8} \oplus A^{\mathrm{even}}(\Gamma_2)X_{23,8}. \end{aligned} \tag{3.2}$$

Proof. (i) The first part of this theorem follows from Theorem 2.2, as pointed out in Sec. 2.2. We first prove the linear independence of the generators over $A^{\mathrm{even}}(\Gamma_2)$. Assume that

$$\begin{aligned} &f_1X_{4,8} + f_2X_{8,8} + f_3X_{10,8} + f_4Y_{10,8} + f_5X_{12,8} \\ &\quad + f_6Y_{12,8} + f_7X_{14,8} + f_8X_{16,8} + f_9X_{18,8} = 0, \end{aligned} \tag{3.3}$$

with $f_i \in A^{\mathrm{even}}(\Gamma_2)\,(1 \le i \le 9)$. Next, we introduce the 9×9 matrix

$$\begin{aligned} D^{\mathrm{even}}(Z) := (&X_{4,8}(Z), X_{8,8}(Z), X_{10,8}(Z), Y_{10,8}(Z), X_{12,8}(Z), \\ &Y_{12,8}(Z), X_{14,8}(Z), X_{16,8}(Z), X_{18,8}(Z)). \end{aligned}$$

Then, from (3.3), we have

$$D^{\mathrm{even}}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Through computer-aided symbolic manipulation, we obtain

$$\begin{aligned} -\frac{\det D^{\mathrm{even}}(Z)}{3009045915000373248000} &= (\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^{12}q_2^8 \\ &\quad + (\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^8q_2^{12} \\ &\quad + 4(-\zeta^{-4} + 4\zeta^{-2} - 6 + 4\zeta^2 - \zeta^4)q_1^{11}q_2^9 \\ &\quad + 4(-\zeta^{-4} + 4\zeta^{-2} - 6 + 4\zeta^2 - \zeta^4)q_1^9q_2^{11} \\ &\quad + 6(\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^{10}q_2^{10} \\ &\quad + \cdots, \end{aligned} \tag{3.4}$$

where for $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in H_2$, we write $q_1 = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$ and $q_2 = e^{2\pi i\omega}$. Hence, we find that $\det D^{\mathrm{even}}(Z)$ is not identically zero as a holomorphic function, and therefore there exists a domain $\Omega \subset H_2$ such that $\det D^{\text{even}}(Z) \neq 0$ for all $Z \in \Omega$. Next, by the holomorphy of f_i, we see that each function f_i is identically zero on Ω, and hence on H_2. Then, from the linear independence of the generators over $A^{\text{even}}(\Gamma_2)$ and Theorem 2.2, we obtain (3.1).

(ii) Through computer-aided symbolic manipulation, we obtain

$$\begin{aligned}\frac{\det D^{\text{odd}}(Z)}{2^{28}\cdot 3^{15}\cdot 5\cdot 7^{6}} &= (\zeta^{-5} - 5\zeta^{-3} + 10\zeta^{-1} - 10\zeta + 5\zeta^{3} - \zeta^{5})q_1^{15}q_2^{10}\\ &\quad + 5(-\zeta^{-5} + 5\zeta^{-3} - 10\zeta^{-1} + 10\zeta - 5\zeta^{3} + \zeta^{5})q_1^{14}q_2^{11}\\ &\quad + \cdots,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}D^{\text{odd}}(Z) := (&X_{9,8}(Z), X_{11,8}(Z), X_{13,8}(Z), X_{15,8}(Z), Y_{15,8}(Z),\\ &X_{17,8}(Z), Y_{17,8}(Z), X_{19,8}(Z), X_{23,8}(Z)).\end{aligned}$$

Hence, $\det D^{\text{odd}}(Z)$ is not identically zero as a holomorphic function. From this fact, and using the same argument as in the proof of (i), we arrive at (3.2). This completes the proof of Theorem 3.1. □

4. Ibukiyama's Conjecture

Let $k_1, \ldots, k_{j+1}$ be non-negative integers such that $k_1 \equiv \cdots \equiv k_{j+1} \pmod 2$. Let us define the integer k as

$$k := k_1 + \cdots + k_{j+1} + \frac{j(j+1)}{2}.$$

If $f_i \in A_{k_i,j}(\Gamma_2)\,(1 \le i \le j+1)$, then the determinant $\det(f_1, \ldots, f_{j+1})$ is a scalar-valued Siegel modular form of weight k with respect to Γ_2. We write $k = 35q + r$ $(q \in \mathbb{Z}_{\ge 0}, 0 \le r < 35)$. Next, we assume that $f_i \in A_{k_i,j}(\Gamma_2)$ $(1 \le i \le j+1)$ are linearly independent over $A^{\text{even}}(\Gamma_2)$. Ibukiyama made the following conjecture in [10].

Conjecture 4.1. *The determinant* $\det(f_1, \ldots, f_{j+1})$ *is divisible by* χ_{35}^{q}, *where* χ_{35} *is the cusp form of weight* 35 *with respect to* Γ_2.[a]

Assuming that this conjecture holds, we can conclude that the determinants $\det D^{\text{even}}(Z)$ and $\det D^{\text{odd}}(Z)$ are equal to χ_{35}^4 and χ_{35}^5 (up to a constant), respectively, because we have the following:

$$4 + 8 + 10 + 10 + 12 + 12 + 14 + 16 + 18 + \frac{8\times 9}{2} = 140 = 35\times 4,$$

$$9 + 11 + 13 + 15 + 15 + 17 + 17 + 19 + 23 + \frac{8\times 9}{2} = 175 = 35\times 5.$$

[a]Takemori [16] proved Conjecture 4.1 for $A^{\text{even}}(\Gamma_2)$-bases of $A^{\text{even}}_{\text{Sym}(j)}(\Gamma_2)$ and $A^{\text{odd}}_{\text{Sym}(j)}(\Gamma_2)$ $(j = 4, 6)$.

Although we have not been able to demonstrate rigorously these relations between $\det D^{\mathrm{even}}(Z)$ and χ_{35}^4 and $\det D^{\mathrm{odd}}(Z)$ and χ_{35}^5, we have obtained results that suggest their validity, as we now discuss. First, note that the first several Fourier coefficients of $\det D^{\mathrm{even}}(Z)$ and $\det D^{\mathrm{odd}}(Z)$ are given in (3.4) and (3.5). Then, note that χ_{35} has the following Fourier expansion:

$$\begin{aligned}\chi_{35}(Z) = {}&(\zeta^{-1} - \zeta)q_1^3 q_2^2 + (-\zeta^{-1} + \zeta)q_1^2 q_2^3 + 0 \times q_1^3 q_2^3\\ &+ 0 \times q_1^4 q_2 + 0 \times q_1 q_2^4 + (-\zeta^{-3} - 69\zeta^{-1} + 69\zeta + \zeta^3)q_1^4 q_2^2\\ &+ (\zeta^{-3} + 69\zeta^{-1} - 69\zeta - \zeta^3)q_1^2 q_2^4 + \cdots,\end{aligned}$$

where for $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, we write $q_1 = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$ and $q_2 = e^{2\pi i\omega}$. Hence, we have

$$\begin{aligned}\chi_{35}^4(Z) = {}&(\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^{12} q_2^8\\ &+ (\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^8 q_2^{12}\\ &+ 4(-\zeta^{-4} + 4\zeta^{-2} - 6 + 4\zeta^2 - \zeta^4)q_1^{11} q_2^9\\ &+ 4(-\zeta^{-4} + 4\zeta^{-2} - 6 + 4\zeta^2 - \zeta^4)q_1^9 q_2^{11}\\ &+ 6(\zeta^{-4} - 4\zeta^{-2} + 6 - 4\zeta^2 + \zeta^4)q_1^{10} q_2^{10} + \cdots,\\ \chi_{35}^5(Z) = {}&(\zeta^{-5} - 5\zeta^{-3} + 10\zeta^{-1} - 10\zeta + 5\zeta^3 - \zeta^5)q_1^{15} q_2^{10}\\ &+ 5(-\zeta^{-5} + 5\zeta^{-3} - 10\zeta^{-1} + 10\zeta - 5\zeta^3 + \zeta^5)q_1^{14} q_2^{11} + \cdots.\end{aligned}$$

It is thus seen that the first several Fourier coefficients of χ_{35}^4 and χ_{35}^5 are identical to those of $\det D^{\mathrm{even}}(Z)$ and $\det D^{\mathrm{odd}}(Z)$, respectively.

Acknowledgments

Some of the results given in this paper are based on the author's master's thesis, written under the supervision of Professor Tomoyoshi Ibukiyama at Osaka University in 2012. The author would like to thank Professor Ibukiyama for suggesting this problem and for valuable comments. The author also would like to thank Professor Masanobu Kaneko for carefully reading preliminary manuscripts of this paper.

Appendix A

Here, we present Tables A.1 and A.2 of Fourier coefficients of the generators appearing in Theorem 3.1. In the tables, $(a, c, b; i)$ (used as short-hand for $(f; a, c, b; i)$) represents the Fourier coefficient of the $(i+1)$th component of $f \in A_{k,8}(\Gamma_2)$ $(0 \le i \le 8)$ for the half-integral matrix $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

Table A.1. Fourier coefficients of the generators of $A^{\text{even}}_{\text{Sym}(8)}(\Gamma_2)$.

$(a, c, b; i)$	$X_{4,8}$	$X_{8,8}$	$X_{10,8}$	$Y_{10,8}$	$X_{12,8}$	$Y_{12,8}$	$X_{14,8}$	$X_{16,8}$	$X_{18,8}$
$(1, 1, 0; 0)$	126	0	6	2	330	−110	−14	−2	10
$(1, 1, 0; 1)$	0	0	0	0	0	0	0	0	0
$(1, 1, 0; 2)$	−504	−2	448	−44	−112	220	−56	−8	40
$(1, 1, 0; 3)$	0	0	0	0	0	0	0	0	0
$(1, 1, 0; 4)$	420	10	−350	38	2310	−870	−84	−12	60
$(1, 1, 0; 5)$	0	0	0	0	0	0	0	0	0
$(1, 1, 0; 6)$	−504	−2	448	−44	−112	220	−56	−8	40
$(1, 1, 0; 7)$	0	0	0	0	0	0	0	0	0
$(1, 1, 0; 8)$	126	0	6	2	330	−110	−14	−2	10
$(1, 1, 1; 0)$	56	0	−3	−1	33	−11	7	1	1
$(1, 1, 1; 1)$	224	0	−12	−4	132	−44	28	4	4
$(1, 1, 1; 2)$	224	1	28	−10	56	−110	70	10	10
$(1, 1, 1; 3)$	−112	3	126	−16	−294	−176	112	16	16
$(1, 1, 1; 4)$	−280	4	175	−19	−469	−209	133	19	19
$(1, 1, 1; 5)$	−112	3	126	−16	−294	−176	112	16	16
$(1, 1, 1; 6)$	224	1	28	−10	56	−110	70	10	10
$(1, 1, 1; 7)$	224	0	−12	−4	132	−44	28	4	4
$(1, 1, 1; 8)$	56	0	−3	−1	33	−11	7	1	1
$(1, 1, 2; 0)$	1	0	0	0	0	0	0	0	0
$(1, 1, 2; 1)$	8	0	0	0	0	0	0	0	0
$(1, 1, 2; 2)$	28	0	0	0	0	0	0	0	0
$(1, 1, 2; 3)$	56	0	0	0	0	0	0	0	0
$(1, 1, 2; 4)$	70	0	0	0	0	0	0	0	0
$(1, 1, 2; 5)$	56	0	0	0	0	0	0	0	0
$(1, 1, 2; 6)$	28	0	0	0	0	0	0	0	0
$(1, 1, 2; 7)$	8	0	0	0	0	0	0	0	0
$(1, 1, 2; 8)$	1	0	0	0	0	0	0	0	0

Table A.2. Fourier coefficients of the generators of $A^{\mathrm{odd}}_{\mathrm{Sym}(8)}(\Gamma_2)$.

$(a,c,b;i)$	$X_{9,8}$	$X_{11,8}$	$X_{13,8}$	$X_{15,8}$	$Y_{15,8}$	$X_{17,8}$	$Y_{17,8}$	$X_{19,8}$	$X_{23,8}$
$(1,1,0;0)$	0	0	0	0	0	0	0	0	0
$(1,1,0;1)$	−4	−20	0	4	−16	−4	4	0	0
$(1,1,0;2)$	0	0	0	0	0	0	0	0	0
$(1,1,0;3)$	−14	14	−1	−7	49	21	−9	0	0
$(1,1,0;4)$	0	0	0	0	0	0	0	0	0
$(1,1,0;5)$	14	−14	1	7	−49	−21	9	0	0
$(1,1,0;6)$	0	0	0	0	0	0	0	0	0
$(1,1,0;7)$	4	20	0	−4	16	4	−4	0	0
$(1,1,0;8)$	0	0	0	0	0	0	0	0	0
$(1,1,1;0)$	0	0	0	0	0	0	0	0	0
$(1,1,1;1)$	2	−2	0	0	0	0	0	0	0
$(1,1,1;2)$	7	−7	0	0	0	0	0	0	0
$(1,1,1;3)$	7	−7	0	0	0	0	0	0	0
$(1,1,1;4)$	0	0	0	0	0	0	0	0	0
$(1,1,1;5)$	−7	7	0	0	0	0	0	0	0
$(1,1,1;6)$	−7	7	0	0	0	0	0	0	0
$(1,1,1;7)$	−2	2	0	0	0	0	0	0	0
$(1,1,1;8)$	0	0	0	0	0	0	0	0	0
$(2,1,0;0)$	0	0	0	0	0	0	0	0	0
$(2,1,0;1)$	−936	6312	−60	−180	1980	−5220	2052	−4	−100
$(2,1,0;2)$	0	0	0	0	0	0	0	0	0
$(2,1,0;3)$	−1428	−3948	174	−1134	5418	882	774	0	0
$(2,1,0;4)$	0	0	0	0	0	0	0	0	0
$(2,1,0;5)$	−84	1428	102	−966	11298	−4662	558	−84	0
$(2,1,0;6)$	0	0	0	0	0	0	0	0	0
$(2,1,0;7)$	−72	−264	0	−408	1632	−1416	1416	0	0
$(2,1,0;8)$	0	0	0	0	0	0	0	0	0
$(2,1,1;0)$	0	1008	0	−168	168	−336	120	1	−5
$(2,1,1;1)$	416	3088	28	−476	1316	−1356	372	2	−10
$(2,1,1;2)$	1148	3052	112	−560	4424	−2520	528	21	0
$(2,1,1;3)$	448	1400	168	−952	8344	−3416	968	84	0
$(2,1,1;4)$	−420	−1260	140	−1260	9660	−3220	1540	105	0
$(2,1,1;5)$	−56	−1120	56	−1288	7000	−2520	1848	42	0
$(2,1,1;6)$	112	−616	0	−784	3136	−1232	1232	0	0
$(2,1,1;7)$	32	−176	0	−224	896	−352	352	0	0
$(2,1,1;8)$	0	0	0	0	0	0	0	0	0
$(2,1,2;0)$	0	0	0	0	0	0	0	0	0
$(2,1,2;1)$	52	92	2	−10	−2	−18	−6	0	0
$(2,1,2;2)$	182	322	7	−35	−7	−63	−21	0	0
$(2,1,2;3)$	266	574	9	−77	119	−49	−59	0	0
$(2,1,2;4)$	210	630	5	−105	315	35	−95	0	0
$(2,1,2;5)$	98	406	1	−77	287	63	−75	0	0
$(2,1,2;6)$	28	140	0	−28	112	28	−28	0	0
$(2,1,2;7)$	4	20	0	−4	16	4	−4	0	0
$(2,1,2;8)$	0	0	0	0	0	0	0	0	0

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