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On Siegel paramodular forms of degree 2 with small levels

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In this paper, we show that the graded ring of Siegel paramodular forms of degree 2 with level $N = 2, 3, 4$ has a very simple unified structure, taking with character. All are generated by six modular forms. The first five are obtained by a kind of Maass lift. The last one is obtained by a kind of Rankin–Cohen–Ibukiyama differential operator from the first five. This result is similar to the case of the graded ring of Siegel modular forms of degree 2 with respect to $\Gamma_0(N)$.

Keywords: Siegel modular forms; Jacobi forms; dimension formula; paramodular groups.

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1. Introduction

Generally, for a given discrete subgroup $\Gamma \subset \mathrm{Sp}(2,\mathbb{R})$, to determine the structure of the graded ring of Siegel modular forms with respect to Γ is not easy. However, for some kind of Γ, we have a very simple way to determine the structure, by using Jacobi forms or Witt operators. This way consists of two parts. The first part is to calculate the upper bound of the dimension of the space of Siegel modular forms for each weight. And the second part is to construct sufficiently many modular forms and to show the upper bound obtained in the first part coincides with the true dimension. Originally this way was given by Aoki in [1] as the simple proof of Igusa's theorem ([14, 15]), that is the structure theorem of Siegel modular forms of degree 2 with respect to the full modular group $\mathrm{Sp}(2,\mathbb{Z})$. Next, Ibukiyama and Aoki applied this way to Siegel modular forms of degree 2 with small levels in [3] and then Aoki applied it to vector valued Siegel modular forms of degree 2 with small levels in [2].

To Siegel paramodular forms, recently, Ibukiyama, Poor and Yuen applied this way in [17]. They showed that the upper-bound of the dimension of the space of Siegel paramodular forms of degree 2 coincides with the true dimension when its level is equal or less than 4. Nevertheless they did not mention the ring structure of the graded ring of modular forms. In this paper, we show that the graded ring of Siegel paramodular forms of degree 2 with level $N = 2, 3, 4$ has a very simple unified structure, taking with character.

We denote by $\mathbb{M}_k(\Gamma)$ the space of all Siegel modular forms of weight k with respect to Γ. We denote the graded ring of these modular forms by $\mathbb{M}_*(\Gamma) := \bigoplus_{k\in\mathbb{Z}} \mathbb{M}_k(\Gamma)$. Let $K(N)$ be the Siegel paramodular group of degree 2 with level N. For $N = 2, 3, 4$, we have already known the dimension of $\mathbb{M}_k(K(N))$ (cf. [5, 13, 15, 17]), namely,

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(K(2)))x^k = \frac{(1+x^{10})(1+x^{11})(1+x^{12})}{(1-x^4)(1-x^6)(1-x^8)(1-x^{12})}, \tag{1.1}$$

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(K(3)))x^k = \frac{((1+x^8+x^{10})+x^9(1+x^2+x^{10}))(1+x^{12})}{(1-x^4)(1-x^6)^2(1-x^{12})} \tag{1.2}$$

and

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(K(4)))x^k$$
$$= \frac{((1+x^6+x^8+x^{10})+x^7(1+x^2+x^4+x^{10}))(1+x^{12})}{(1-x^4)^2(1-x^6)(1-x^{12})}. \tag{1.3}$$

These generating functions seem to be bit complicated. But taking with character, these generating functions become very simple and unified. Let Γ_N be a subgroup of $K(N)$ of index N, that is defined by Eq. (2.2) in Sec. 2.3. Our main theorem is as follows:

Theorem 1.1. *When $N = 2, 3, 4$, the graded ring $\mathbb{M}_*(\Gamma_N)$ is generated by six modular forms whose weights are $4, 6, \frac{12}{N}-2, \frac{12}{N}, \frac{24}{N}-1$ and 12. The first five are obtained by a kind of Maass lift. The last one is obtained by a kind of Rankin–Cohen–Ibukiyama differential operator from the first five. The dimension of $\mathbb{M}_k(\Gamma_N)$ is given by the following generating functions*:

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_2))x^k = \frac{(1+x^{11})(1+x^{12})}{(1-x^4)(1-x^4)(1-x^6)(1-x^6)},$$

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_3))x^k = \frac{(1+x^7)(1+x^{12})}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}$$

and

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_4))x^k = \frac{(1+x^5)(1+x^{12})}{(1-x)(1-x^3)(1-x^4)(1-x^6)}.$$

2. Preliminaries

2.1. *Elliptic modular forms*

To define Γ_N explicitly, first we review elliptic modular forms briefly. We denote complex upper half plane by

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}.$$

The special linear group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$\mathbb{H} \ni \tau \mapsto g\langle\tau\rangle := \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \in \mathbb{H},$$

where $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. For a function $f(\tau)$ on $\mathbb{H}$ and $k \in \mathbb{Z}$, define the action of $\mathrm{SL}(2,\mathbb{R})$ by

$$f(\tau) \mapsto (f|_k g)(\tau) := (\gamma\tau+\delta)^{-k} f(g\langle\tau\rangle).$$

Here, we only treat elliptic modular forms with respect to $\mathrm{SL}(2,\mathbb{Z})$. The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by two matrices

$$T := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We note that there are relations

$$S^2 = (ST)^3 = -E_2,$$

where we denote the unit matrix of size $n\times n$ by E_n. Let χ be a character of $\mathrm{SL}(2,\mathbb{Z})$, namely, χ is a homomorphism from Γ to $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$. From the above relations, easily we can show that χ satisfies two conditions $\chi(T)^{12} = 1$ and $\chi(S) = \chi(T)^9$. We say a holomorphic function f on $\mathbb{H}$ is an elliptic modular form of weight k if f satisfies the following two conditions:

(1) $\chi(g)f = (f|_k g)$ for any $g \in \mathrm{SL}(2,\mathbb{Z})$,
(2) f is bounded at $\sqrt{-1}\infty$.

From the first condition, such f has the Fourier expansion $f(\tau) = \sum_n c_f(n)q^n$, where we denote by $q^n := \mathbf{e}(n\tau) := \exp(2\pi\sqrt{-1}n\tau)$. The second condition says that $c_f(n) = 0$ if $n < 0$. We denote the space of all elliptic modular forms of weight k by $\mathrm{M}_k(\chi)$. Especially, when χ is the principle character $\mathbf{1}$, namely the constant map to 1, we denote $\mathrm{M}_k := \mathrm{M}_k(\mathbf{1})$. It is well known that $\mathrm{M}_k = \{0\}$ if $k < 0$ and that $\mathrm{M}_0 = \mathbb{C}$.

The structure theorem of the graded ring of elliptic modular forms is well-known. This is generated by two algebraically independent modular forms of weight 4 and 6:

$$\mathrm{M}_{\mathbb{Z}} := \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k = \mathbb{C}[e_4, e_6].$$

Here, we denote Eisenstein series of weight k by

$$e_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \in \mathrm{M}_k \quad (k \in 2\mathbb{Z}, k \geq 4),$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and the Bernoulli number B_k is defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}x^k.$$

An example of an elliptic modular form with a nontrivial character can be constructed from the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 - q^n).$$

The square of the Dedekind eta function is an example of an elliptic modular form with a nontrivial character. Actually, $\eta^2 \in \mathrm{M}_1(\chi_{\eta^2})$, where χ_{η^2} is a character of $\mathrm{SL}(2,\mathbb{Z})$ defined by

$$\chi_{\eta^2}(T) = \mathbf{e}\left(\frac{1}{12}\right) \quad \text{and} \quad \chi_{\eta^2}(S) = \mathbf{e}\left(\frac{3}{4}\right).$$

Hence any character of $\mathrm{SL}(2,\mathbb{Z})$ is equal to $\chi_{\eta^2}^c$ for some $c \in \{0, 1, \ldots, 11\}$. For $c \in \{0, 1, \ldots, 11\}$, the map $\mathrm{M}_k \ni f \mapsto \eta^{2c}f \in \mathrm{M}_{k+c}(\chi_{\eta^2}^c)$ is isomorphic, because η has no zero on $\mathbb{H}$.

Throughout this paper, we put $c' := \frac{12}{(12,c)}$. It is not easy but well-known fact (cf. [19] (p. 50)) that

$$\mathrm{Ker}(\chi_{\eta^2}^c) \supset \Gamma(c') := \{g \in \mathrm{SL}(2,\mathbb{Z}) \,|\, g \equiv E_2 \pmod{c'}\}. \tag{2.1}$$

Especially, the Ramanujan delta function $\Delta := \eta^{24} \in \mathrm{M}_{12}$ satisfies

$$\Delta = \eta^{24} = \frac{1}{1728}(e_4^3 - e_6^2) = \frac{\sqrt{-1}}{6912\pi}\det\begin{pmatrix} 4e_4 & 6e_6 \\ \frac{\partial}{\partial\tau}e_4 & \frac{\partial}{\partial\tau}e_6 \end{pmatrix}.$$

2.2. *Siegel modular forms*

Here we review Siegel modular forms of degree 2. We denote Siegel upper half space of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}(2,\mathbb{C}) \,|\, \mathrm{Im}\, Z > 0 \right\}.$$

The symplectic group

$$G := \mathrm{Sp}(2,\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}(4,\mathbb{R}) \,|\, {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

For a function $F(Z)$ on $\mathbb{H}_2$ and $k \in \mathbb{Z}$, define the action of G by

$$F(Z) \mapsto (F|_k M)(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle).$$

Sometimes we denote $F(\tau, z, \omega) = F(Z)$. Let Γ be a subgroup of G commensurable with $\mathrm{Sp}(2,\mathbb{Z}) := G \cap \mathrm{M}(4,\mathbb{Z})$. Let ψ be a character of Γ of finite order, namely, ψ is a homomorphism from Γ to $\mathbb{C}\backslash\{0\}$ and $\{M \in \Gamma \,|\, \psi(M) = 1\}$ is a finite index subgroup of Γ. We say F is a Siegel modular form of weight k if F is holomorphic on $\mathbb{H}_2$ and satisfies the condition $\psi(M)F = (F|_k M)$ for any $M \in \Gamma$. We denote the space of all Siegel modular forms of weight k by $\mathbb{M}_k(\Gamma;\psi)$. Especially when ψ is the principle character $\mathbf{1}$, namely the constant map to 1, we denote $\mathbb{M}_k(\Gamma) := \mathbb{M}_k(\Gamma;\mathbf{1})$. It is well-known that $\mathbb{M}_k(\Gamma) = \{0\}$ if $k < 0$ and that $\mathbb{M}_0(\Gamma) = \mathbb{C}$. Let $F \in \mathbb{M}_k(\Gamma;\psi)$. Since ψ is a character of Γ of finite order, F has the Fourier expansion

$$F(Z) = \sum_{n,l,m} c_F(n,l,m) q^n \zeta^l p^m,$$

where $\zeta^l := \mathbf{e}(lz)$, $p^m := \mathbf{e}(m\omega)$ and n, l, m run over $\frac{1}{n_0}\mathbb{Z}$, $\frac{1}{l_0}\mathbb{Z}$, $\frac{1}{m_0}\mathbb{Z}$ for some n_0, l_0, m_0. It is well-known as Koecher principle that $c_F(n,l,m) = 0$ if $4nm - l^2 < 0$ or $m < 0$. For $r \in \mathbb{Q}$, we define

$$\mathbb{M}_k(\Gamma;\psi)[r] := \{F \in \mathbb{M}_k(\Gamma;\psi) \,|\, c_F(n,l,m) = 0 \text{ if } m < r\}.$$

2.3. *Siegel paramodular group*

In this paper, our interest is the case that the discrete subgroup Γ is a paramodular group. Let $N \geq 2$ be an integer. The paramodular group of level N is defined by

$$K(N) := \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap G.$$

Let

$$V_N := \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -N & 0 \end{pmatrix} \in G$$

and we denote by $K^*(N)$ the subgroup of G generated by $K(N)$ and V_N. We can easily show that $[K^*(N) : K(N)] = 2$. We label some elements of $K(N)$:

$$C(g) := \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right)$$

and

$$U(\lambda,\mu) := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\lambda,\mu \in \mathbb{Z}).$$

Easily we can show that the group $K^*(N)$ is generated by above three kinds of elements $C(g), U(\lambda,\mu)$ and V_N.

Let ψ_{V_N} be a character $K^*(N)$ defined by $\psi_{V_N}(C(g)) = 1, \psi_{V_N}(U(\lambda,\mu)) = 1$ and $\psi_{V_N}(V_N) = -1$. By this character, the space of all modular forms with respect to $K(N)$ is decomposed to the direct sum of the space of all modular forms with respect to $K^*(N)$:

$$\mathbb{M}_k(K(N)) = \mathbb{M}_k(K^*(N)) \oplus \mathbb{M}_k(K^*(N); \psi_{V_N}).$$

In [17], Ibukiyama, Poor and Yuen obtained the dimension formula (1.1)–(1.3) by estimating the dimension of each space of the right hand side according to the way in [1]. In this paper, we improve this idea.

Throughout this paper, we put $N' := (N, 12)$. Let ψ_N be a character of $K^*(N)$ defined by $\psi_N(C(g)) = \chi_{\eta^2}(g)^{\frac{12}{N'}}, \psi_N(U(\lambda,\mu)) = 1, \psi_N(V_N) = 1$. We remark that this character is well-defined, because we construct an actual Siegel paramodular modular form with this characters in Sec. 3.2. Let

$$\Gamma_N := \{M \in K(N) \mid \psi_N(M) = 1\}. \tag{2.2}$$

Then we have

$$\mathbb{M}_k(\Gamma_N) = \bigoplus_{a=0}^{1} \bigoplus_{b=0}^{N'-1} \mathbb{M}_k(K^*(N); \psi_{V_N}^a \psi_N^b). \tag{2.3}$$

2.4. *Jacobi forms*

Jacobi forms were first studied by Eichler and Zagier in their book [6]. In their book, mainly they treated Jacobi forms with the trivial character. Here we review Jacobi forms with characters, based on their book.

For a while we fix $k \in \mathbb{Z}$ and $m \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. Let Γ^J be a subgroup of $\mathrm{Sp}(2,\mathbb{R})$ generated by two kinds of elements $C(g)$ $(g \in \mathrm{SL}(2,\mathbb{Z}))$ and $U(\lambda,\mu)(\lambda,\mu \in \mathbb{Z})$. For a function $\varphi(\tau,z)$ on $\mathbb{H} \times \mathbb{C}$ and $M \in \Gamma^J$, define

$$(\varphi|_{k,m}M)(\tau,z) := ((\varphi(\tau,z)p^m)|_k M)\, p^{-m}.$$

It is easy to show that the right-hand side of the above equation is independent on ω. Hence this definition makes sense and Γ^J acts on the set of all functions on $\mathbb{H} \times \mathbb{C}$. More precisely,

$$(\varphi|_{k,m} C(g))(\tau, z) = (\gamma\tau + \delta)^{-k} \mathbf{e}\left(\frac{m\gamma z^2}{\gamma\tau + \delta}\right) \varphi\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \frac{z}{\gamma\tau + \delta}\right),$$

$$(\varphi|_{k,m} U(\lambda, \mu))(\tau, z) = \mathbf{e}(m(\lambda^2\tau + 2\lambda z + \lambda\mu))\varphi(\tau, z + \lambda\tau + \mu).$$

Let c be an integer and suppose φ be a Γ^J-invariant holomorphic function with respect to the character $\chi_{\eta^2}^c$. Namely, we suppose a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ satisfying the following two conditions:

$$\chi_{\eta^2}^c(g)\varphi(\tau, z) = (\varphi|_{k,m} C(g))(\tau, z) \quad \text{for any } g \in \mathrm{SL}(2, \mathbb{Z})$$

$$\text{and} \quad \varphi(\tau, z) = (\varphi|_{k,m} U(\lambda, \mu))(\tau, z) \quad \text{for any } \lambda, \mu \in \mathbb{Z}.$$

Since χ_{η^2} is a character of $\mathrm{SL}(2, \mathbb{Z})$ of finite order, φ has the Fourier expansion

$$(\varphi|_{k,m} M)(\tau, z) = \sum_{n,l} c_\varphi(n, l) q^n \zeta^l,$$

where n runs over $\mathbb{Z} + \frac{c}{12}$ and l runs over $\mathbb{Z}$. We say above φ is a weak Jacobi form of weight k and index m if $c_\varphi(n, l) = 0$ when $n < 0$. We say a weak Jacobi form φ is a Jacobi form if $c_\varphi(n, l) = 0$ when $4nm - l^2 < 0$. We denote the space of all Jacobi forms or all weak Jacobi form of weight k and index m by $\mathbb{J}_{k,m}(\chi_{\eta^2}^c)$ or $\mathbb{J}_{k,m}^{\mathrm{w}}(\chi_{\eta^2}^c)$. Clearly we have $\mathbb{J}_{k,m}(\chi_{\eta^2}^c) \subset \mathbb{J}_{k,m}^{\mathrm{w}}(\chi_{\eta^2}^c)$. According to the book of Eichler and Zagier [6] (Theorem 1.2), We can show that every (weak) Jacobi form of index 0 is an elliptic modular form. Namely, $\mathbb{J}_{k,0}(\chi_{\eta^2}^c) = \mathbb{J}_{k,0}^{\mathrm{w}}(\chi_{\eta^2}^c) = \mathrm{M}_k(\chi_{\eta^2}^c)$. For $r \in \mathbb{Q}$, we define

$$\mathbb{J}_{k,m}(\chi_{\eta^2}^c)[r] := \{\varphi \in \mathbb{J}_{k,m}(\chi_{\eta^2}^c) \,|\, c_\varphi(n, l) = 0 \text{ if } n < r\},$$

$$\mathbb{J}_{k,m}^{\mathrm{w}}(\chi_{\eta^2}^c)[r] := \{\varphi \in \mathbb{J}_{k,m}^{\mathrm{w}}(\chi_{\eta^2}^c) \,|\, c_\varphi(n, l) = 0 \text{ if } n < r\}$$

and $\mathrm{M}_k(\chi_{\eta^2}^c)[r] := \mathbb{J}_{k,0}^{\mathrm{w}}(\chi_{\eta^2}^c)[r]$. When the character $\chi_{\eta^2}^c$ is principle, namely $c \in 12\mathbb{Z}$, we omit the character in the above notations, just as the case of elliptic modular forms: $\mathbb{J}_{k,m} := \mathbb{J}_{k,m}(\mathbf{1})$, $\mathbb{J}_{k,m}[r] := \mathbb{J}_{k,m}(\mathbf{1})[r]$.

2.5. *Structure theorem of Jacobi forms*

In the book of Eichler and Zagier [6], they said the structure of the bi-graded ring of Jacobi forms is complicated. However they also showed that the structure of the bi-graded ring of weak Jacobi forms is quite simple. They constructed some weak Jacobi forms:

$$\varphi_{-2,1}(\tau, z) = (\zeta - 2 + \zeta^{-1}) + (-2\zeta^2 + 8\zeta - 12 + 8\zeta^{-1} - 2\zeta^{-2})q + \cdots \in \mathbb{J}_{-2,1}^{\mathrm{w}},$$

$$\varphi_{0,1}(\tau, z) = (\zeta + 10 + \zeta^{-1}) + (10\zeta^2 - 64\zeta + 108 - 64\zeta^{-1} + 10\zeta^{-2})q + \cdots \in \mathbb{J}_{0,1}^{\mathrm{w}}$$

and

$$\varphi_{-1,2}(\tau,z) = (\zeta - \zeta^{-1}) + (\zeta^3 + 3\zeta - 3\zeta^{-1} - 3\zeta^{-3})q + \cdots \in \mathbb{J}^{\mathrm{w}}_{-1,2}.$$

These three Jacobi forms are unique weak Jacobi forms of each weight and index up to constant. They showed

$$\mathbb{J}^{\mathrm{w}}_{2\mathbb{Z},\mathbb{Z}} := \bigoplus_{k\in 2\mathbb{Z}, m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m} = \mathrm{M}_{\mathbb{Z}}[\varphi_{0,1}, \varphi_{-2,1}]$$

and

$$\mathbb{J}^{\mathrm{w}}_{\mathbb{Z},\mathbb{Z}} := \bigoplus_{k\in \mathbb{Z}, m\in\mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m} = \mathbb{J}^{\mathrm{w}}_{2\mathbb{Z},\mathbb{Z}} \oplus \varphi_{-1,2}\mathbb{J}^{\mathrm{w}}_{2\mathbb{Z},\mathbb{Z}}.$$

We remark that $\varphi_{-2,1}$ and $\varphi_{0,1}$ are algebraically independent on $\mathrm{Hol}(\mathbb{H})$, the ring of all holomorphic functions on $\mathbb{H}$. On the other hand, $\varphi_{-1,2}$ satisfies the equation

$$\varphi_{-1,2}(\tau,z)^2 = \frac{1}{432}\left(2e_6\varphi_{-2,1}^4 - 3e_4\varphi_{-2,1}^3\varphi_{0,1} + \varphi_{-2,1}\varphi_{0,1}^3\right) \tag{2.4}$$

and

$$\varphi_{-1,2} = \frac{\sqrt{-1}}{24\pi}\det\begin{pmatrix} \varphi_{-2,1} & \varphi_{0,1} \\ \dfrac{\partial}{\partial z}\varphi_{-2,1} & \dfrac{\partial}{\partial z}\varphi_{0,1}\end{pmatrix}.$$

On the dimension of the space of Jacobi forms, the generating functions are given by

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,0})x^k = \frac{1}{(1-x^4)(1-x^6)},$$

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,1})x^k = \frac{x^{-2}+1}{(1-x^4)(1-x^6)}$$

and

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,m})x^k = \frac{(x^{-2m} + x^{-2m+2} + \cdots + x^{-2} + 1)}{(1-x^4)(1-x^6)} + \frac{(x^{-2m+3} + x^{-2m+5} + \cdots + x^{-3} + x^{-1})}{(1-x^4)(1-x^6)} \quad (m \geq 2).$$

In the case of Jacobi forms with characters, the linear map

$$\mathbb{J}^{\mathrm{w}}_{k,m} \ni \varphi \overset{\sim}{\mapsto} \eta^{2c}\varphi \in \mathbb{J}^{\mathrm{w}}_{k+c,m}(\chi^c_{\eta^2})\left[\frac{c}{12}\right]$$

is isomorphic for $c \in \mathbb{N}_0$. Hence we have

$$\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k+c,m}(\chi^c_{\eta^2})\left[\frac{c}{12}\right] = \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,m}.$$

2.6. *Fourier–Jacobi expansion*

In this subsection, we see the relation between Siegel paramodular forms and Jacobi forms via the Fourier–Jacobi expansion. Let $F \in \mathbb{M}_k(K^*(N); \psi_{V_N}^a \psi_N^b)$. Since

$$\psi_{V_N}^a \psi_N^b \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dfrac{1}{N} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \psi_{V_N}^a \psi_N^b \left(V_N C(T) V_N^3 \right)$$

$$= \chi_{\eta^2}(T)^{\frac{12b}{N'}} = \mathbf{e}\left(\frac{b}{N'} \right),$$

we have $\mathbf{e}\left(\frac{b}{N'}\right) F(\tau, z, \omega) = F(\tau, z, \omega + \frac{1}{N})$. Therefore, F has the expansion (Fourier–Jacobi expansion)

$$F(Z) = \sum_{m=0}^{\infty} \varphi_{N(m+\frac{b}{N'})}(\tau, z) p^{N(m+\frac{b}{N'})}.$$

Because of the Koecher principle, we have $\varphi_{N(m+\frac{b}{N'})} \in \mathbb{J}_{k, N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}})$ for each m. We denote the projection $F \mapsto \varphi_{N(m+\frac{b}{N'})}$ by

$$(\mathrm{FJ})_{N(m+\frac{b}{N'})} : \mathbb{M}_k(K^*(N); \psi_{V_N}^a \psi_N^b) \to \mathbb{J}_{k, N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}}). \tag{2.5}$$

Therefore, we can regard the Fourier–Jacobi expansion as the injective map $(\mathrm{FJ}) := \prod_{m=0}^{\infty} (\mathrm{FJ})_{N(m+\frac{b}{N'})}$. The Fourier coefficients of each $\varphi_{N(m+\frac{b}{N'})}$ are directly come from the Fourier coefficients of F. Since $\mathbf{e}(\frac{b}{N'}) F(\tau, z, \omega) = F(\tau + 1, z, \omega)$ and $F(\tau, z, \omega) = F(\tau, z + 1, \omega)$, F has the Fourier expansion

$$F(Z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_F \left(n + \frac{b}{N'}, l, N\left(m + \frac{b}{N'} \right) \right) q^{n+\frac{b}{N'}} \zeta^l p^{N(m+\frac{b}{N'})}$$

and then we have

$$\varphi_{N(m+\frac{b}{N'})}(\tau, z) = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_F \left(n + \frac{b}{N'}, l, N\left(m + \frac{b}{N'} \right) \right) q^{n+\frac{b}{N'}} \zeta^l. \tag{2.6}$$

On the above Fourier expansion, since $V_N C(-E_2) \in K^*(N)$ induces the equality

$$(-1)^{k+a+\frac{12b}{N'}} F(\tau, z, \omega) = F\left(N\omega, z, \frac{\tau}{N} \right),$$

we have

$$c_F \left(n + \frac{b}{N'}, l, N\left(m + \frac{b}{N'} \right) \right) = (-1)^{k+a+\frac{12b}{N'}} c_F \left(m + \frac{b}{N'}, l, N\left(n + \frac{b}{N'} \right) \right). \tag{2.7}$$

Here we define the space of formal series of Jacobi forms with the involution condition:

$$\mathbb{FM}_k(K^*(N);\psi_{V_N}^a\psi_N^b)$$
$$:= \left\{ (\varphi_{N(m+\frac{b}{N'})})_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}}) \;\middle|\; \begin{array}{c}\text{On the Fourier expansions of}\\ \varphi_{N(m+\frac{b}{N'})} \text{ like as (2.6), their}\\ \text{Fourier coefficients satisfies}\\ \text{the condition (2.7).}\end{array} \right\}.$$

Then we can regard the Fourier–Jacobi expansion as the injective map

$$(\mathrm{FJ}) : \mathbb{M}_k(K^*(N);\psi_{V_N}^a\psi_N^b) \to \mathbb{FM}_k(K^*(N);\psi_{V_N}^a\psi_N^b).$$

Generally, on the theory of modular forms on the symmetric domain of type IV, we can induce an injection of this kind. The big interest of the author is whether this map is surjective or not. And the author consider that this map is always surjective under some good conditions. In the case of $\mathrm{SO}(2,3)$ (Siegel modular forms of degree 2), we have already known some examples whose Fourier–Jacobi expansion is surjective (cf. [1–3, 17]), and we have no example whose Fourier–Jacobi expansion is not surjective. In this paper we will show that the Fourier–Jacobi expansion is surjective for $\Gamma = \Gamma_N (N = 2, 3, 4)$.

2.7. *Maass lift*

Maass lift or Saito–Kurokawa lift is a method to construct Siegel modular forms from Jacobi forms. Most well-known one is a lift from Jacobi forms of index 1 to Siegel modular forms with respect to $\mathrm{Sp}(2,\mathbb{Z})$. To Siegel paramodular forms of level N, Gritsenko and Hulek [10] constructed Maass lift (so called Gritsenko lift) from Jacobi forms of index N. The case with characters, Ibukiyama [12] constructed Maass lift from Jacobi forms of index 1 to Siegel modular forms with levels. Recently, by combining these results, Cléry and Gritsenko [4] constructed a lift from Jacobi forms to Siegel paramodular forms with characters. The following two propositions are a slight modification of their construction.

Proposition 2.1 (cf. [4, Lemma 2.1]). *Let $\varphi \in \mathbb{J}_{k,m}(\chi_{\eta^2}^c)$ and let t be a positive integer satisfying $t \equiv 1 \pmod{c'}$. Then we have*

$$(\varphi|_k T_-(t))(\tau,z) := t^{-1} \sum_{\alpha\delta=t} \sum_{\beta=0}^{\delta-1} \alpha^k \chi_c(\alpha)\varphi\left(\frac{\alpha\tau+\beta c'}{\delta}, \alpha z\right) \in \mathbb{J}_{k,mt}(\chi_{\eta^2}^c),$$

where χ_c is the character of $(\mathbb{Z}/c'\mathbb{Z})^\times$ defined as follows:

$$\chi_c(\alpha) = 1 \quad ((\alpha,c')=1) \quad (c'=1,2,3,6),$$

$$\chi_c(\alpha) = \begin{cases} 1 & (\alpha \equiv 1 \pmod 4) \\ -1 & (\alpha \equiv 3 \pmod 4) \end{cases} \quad (c'=4),$$

$$\chi_c(\alpha) = \begin{cases} 1 & (\alpha \equiv 1,5 \pmod{12}) \\ -1 & (\alpha \equiv 7,11 \pmod{12}) \end{cases} \quad (c'=12).$$

We need to say where the character χ_c comes from. Roughly, this operator $T_-(t)$ is a Hecke operator of

$$\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{M}(2,t) = \left\{ \mathrm{SL}(2,\mathbb{Z}) \begin{pmatrix} \alpha & \beta c' \\ 0 & \delta \end{pmatrix} \middle| \begin{array}{l} \alpha\delta = t,\ \alpha > 0, \\ \beta \in \{0, 1, \ldots, \delta - 1\} \end{array} \right\},$$

where we put

$$\mathrm{M}(2,t) := \{g \in \mathrm{GL}(2,\mathbb{Z}) \mid \det g = t\}.$$

We can easily show that for any $g \in \mathrm{GL}(2,\mathbb{Z})$ satisfying $\det g \equiv 1 \pmod{c'}$ there exists $g' \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \equiv g' \pmod{c'}$. Hence, by (2.1), the character $\chi_{\eta^2}^c$ of $\mathrm{SL}(2,\mathbb{Z})$ can be extended to the multiplier system of $\bigcup_{t\equiv 1 \pmod{c'}} \mathrm{M}(2,t)$. $\chi_c(a)$ is the value of this multiplier system at $\left(\begin{smallmatrix} \alpha & \beta c' \\ 0 & \delta \end{smallmatrix}\right)$.

Proposition 2.2 (cf. [4, Theorem 2.2]). *Let* $c \in \{0,1,2,3,4,6\}$ *and*

$$\varphi(\tau,z) = \sum_{n=0}^{\infty} \sum_{l\in\mathbb{Z}} c_\varphi\left(n + \frac{c}{12}, l\right) q^{n+\frac{c}{12}} \zeta^l \in \mathbb{J}_{k,m}(\chi_{\eta^2}^c).$$

If $c = 0$, *we put*

$$(\varphi|_k T_-(0))(\tau,z) := -\frac{B_k}{2k} c_\varphi(0,0) e_k(\tau).$$

Then we have

$$\mathrm{ML}(\varphi)(Z) := \sum_{\substack{t\equiv 1 \pmod{c'} \\ t\geq 0}} (\varphi|_k T_-(t))(\tau,z) p^{mt} \in \mathbb{M}_k\left(K(mc'); \psi_{V_{mc'}}^{k+c} \psi_{mc'}^{(m,c)}\right).$$

The Fourier coefficients of

$$\mathrm{ML}(\varphi)(Z) = \sum_{\substack{n,t\equiv 1 \pmod{c'} \\ n,t\geq 0}} \sum_{l\in\mathbb{Z}} c_{\mathrm{ML}(\varphi)}\left(\frac{n}{c'}, l, mt\right) q^{\frac{n}{c'}} \zeta^l p^{mt}$$

are given by

$$c_{\mathrm{ML}(\varphi)}\left(\frac{n}{c'}, l, mt\right) = \begin{cases} -\dfrac{B_k}{2k} c_\varphi(0,0) & (c = 0, (n,l,t) = (0,0,0)), \\ 0 & (c \neq 0, (n,l,t) = (0,0,0)), \\ \displaystyle\sum_{\alpha|(n,l,t)} \alpha^k \chi_c(\alpha) c_\varphi\left(\frac{nt}{\alpha^2 c'}, \frac{l}{\alpha}\right) & ((n,l,t) \neq (0,0,0)). \end{cases}$$

2.8. *Differential operator*

On the theory of elliptic modular forms, the Rankin–Cohen differential operator is a method of constructing new modular forms from given two modular forms. Recently, Ibukiyama [11] constructed similar method on the theory of modular forms of multi-variables. This method is very simple but very useful to construct modular forms which is not obtained by Maass lift. Actually, in [3], we constructed some generators of the ring of Siegel modular forms with levels.

Proposition 2.3 (cf. [3, Proposition 2.1]). *Let* $F_j \in \mathbb{M}_{k_j}(K^*(N); \psi_{V_N}^{a_j} \psi_N^{b_j})$ *for* $j = 1, 2, 3, 4$. *Then we have*

$$\{F_1, F_2, F_3, F_4\}_3 := \det \begin{pmatrix} k_1 F_1 & k_2 F_2 & k_3 F_3 & k_4 F_4 \\ \dfrac{\partial}{\partial \tau} F_1 & \dfrac{\partial}{\partial \tau} F_2 & \dfrac{\partial}{\partial \tau} F_3 & \dfrac{\partial}{\partial \tau} F_4 \\ \dfrac{\partial}{\partial z} F_1 & \dfrac{\partial}{\partial z} F_2 & \dfrac{\partial}{\partial z} F_3 & \dfrac{\partial}{\partial z} F_4 \\ \dfrac{\partial}{\partial \omega} F_1 & \dfrac{\partial}{\partial \omega} F_2 & \dfrac{\partial}{\partial \omega} F_3 & \dfrac{\partial}{\partial \omega} F_4 \end{pmatrix} \in \mathbb{M}_{\mathbf{k}}(K^*(N); \psi_{V_N}^{\mathbf{a}} \psi_N^{\mathbf{b}}),$$

where we put

$$\begin{aligned} \mathbf{k} &:= k_1 + k_2 + k_3 + k_4 + 3, \\ \mathbf{a} &:= a_1 + a_2 + a_3 + a_4, \\ \textit{and} \quad \mathbf{b} &:= b_1 + b_2 + b_3 + b_4. \end{aligned}$$

We can easily prove this proposition by elementary calculation.

3. Proof of Our Main Theorem

3.1. *Estimation*

Fourier–Jacobi expansion gives us an information about the upper bound of the dimension of the space of Siegel paramodular forms. Sometimes this information is very useful for determining the structure of modular forms. This idea was first appeared in [1], where Aoki applied this idea to the case $\Gamma = \mathrm{Sp}(2, \mathbb{Z})$ and gave a new proof of Igusa's theorem. Recently, Ibukiyama, Poor and Yuen applied this idea to the case $\Gamma = K(N)$ and they showed that the above upper bound coincides with the true dimension of $\mathbb{M}_k(K(N))$ when $N = 2, 3, 4$ ([17]). Here we apply this idea to the case Γ_N.

Let $F \in \mathbb{M}_k(K^*(N); \psi_{V_N}^a \psi_N^b)$. From the involution condition (2.7) in Sec. 2.6, the Fourier–Jacobi expansion of F (i.e. (2.5) in Sec. 2.6) induces the exact sequence

$$\begin{aligned} \{0\} &\to \mathbb{M}_k\left(K^*(N); \psi_{V_N}^a \psi_N^b\right)\left[N\left(m + \frac{b}{N'} + 1\right)\right] \\ &\to \mathbb{M}_k\left(K^*(N); \psi_{V_N}^a \psi_N^b\right)\left[N\left(m + \frac{b}{N'}\right)\right] \\ &\xrightarrow{(\mathrm{FJ})_{N(m+\frac{b}{N'})}} \begin{cases} \mathbb{J}_{k, mN+b}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[N\left(m + \dfrac{b}{N'}\right)\right] & \left(\text{if } k + a + \dfrac{12b}{N'} \text{ is even}\right), \\ \mathbb{J}_{k, mN+b}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[N\left(m + \dfrac{b}{N'} + 1\right)\right] & \left(\text{if } k + a + \dfrac{12b}{N'} \text{ is odd}\right). \end{cases} \end{aligned}$$

This exact sequence induces an estimation

$$\begin{aligned}
&\dim_{\mathbb{C}} \mathbb{M}_k(K^*(N);\psi_{V_N}^a\psi_N^b)\\
&\quad\le \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}_{k,N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[m+\frac{b}{N'}\right]\\
&\quad\le \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[m+\frac{b}{N'}\right]\\
&\quad= \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-12(m+\frac{b}{N'}),N(m+\frac{b}{N'})} \quad \left(\text{if } k+a+\frac{12b}{N'} \text{ is even}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\dim_{\mathbb{C}} \mathbb{M}_k(K^*(N);\psi_{V_N}^a\psi_N^b)\\
&\quad\le \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}_{k,N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[m+\frac{b}{N'}+1\right]\\
&\quad\le \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k,N(m+\frac{b}{N'})}(\chi_{\eta^2}^{\frac{12b}{N'}})\left[m+\frac{b}{N'}+1\right]\\
&\quad= \sum_{m=0}^{\infty} \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-12(1+m+\frac{b}{N'}),N(m+\frac{b}{N'})} \quad \left(\text{if } k+a+\frac{12b}{N'} \text{ is odd}\right).
\end{aligned}$$

Therefore, by using the Eq. (2.3), we have an estimation

$$\begin{aligned}
\dim_{\mathbb{C}} \mathbb{M}_k(\Gamma_N) &= \sum_{a=0}^{1}\sum_{b=0}^{N'-1} \dim_{\mathbb{C}} \mathbb{M}_k(K^*(N);\psi_{V_N}^a\psi_N^b)\\
&\le \sum_{b=0}^{N'-1}\sum_{m=0}^{\infty} (\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-12(m+\frac{b}{N'}),N(m+\frac{b}{N'})}\\
&\qquad + \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-12(1+m+\frac{b}{N'}),N(m+\frac{b}{N'})})\\
&= \sum_{m=0}^{\infty} (\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-\frac{12}{N'}m,\frac{N}{N'}m} + \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}_{k-12-\frac{12}{N'}m,\frac{N}{N'}m}).
\end{aligned}$$

This estimation bound seems to be very rough. Actually, when $N \ge 6$, the infinite sum of this upper bound does not converge. However, if $N \le 4$, this upper bound coincides with the actual dimension of $\mathbb{M}_k(\Gamma_N)$.

From now on, we assume $N \in \{2,3,4\}$. Then we have $N = N'$. Hence the dimension of $\mathbb{M}_k(\Gamma_N)$ is not greater than the coefficient of x^k on the formal power series (Laurent) development of the function

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{\infty}\left(\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k-\frac{12}{N}m,m}+\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k-12-\frac{12}{N}m,m}\right)x^k$$

$$=(1+x^{12})\sum_{k\in\mathbb{Z}}\sum_{m=0}^{\infty}(\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k,m})x^{k+\frac{12}{N}m}$$

$$=\frac{(1+x^{12})(1+x^{\frac{24}{N}-1})}{(1-x^4)(1-x^6)(1-x^{\frac{12}{N}-2})(1-x^{\frac{12}{N}})}.$$

Hence, to prove our main theorem, it is sufficient to construct six modular forms

$$E_{(N),4}\in\mathbb{M}_4(K^*(N)),\quad E_{(N),6}\in\mathbb{M}_6(K^*(N)),$$

$$X_{(N)}\in\mathbb{M}_{\frac{12}{N}-2}(K^*(N);\psi_N),\quad Y_{(N)}\in\mathbb{M}_{\frac{12}{N}}(K^*(N);\psi_N),$$

$$Z_{(N)}\in\mathbb{M}_{\frac{24}{N}-1}(K^*(N);\psi_{V_N}\psi_N^2)\quad\text{and}\quad W_{(N)}\in\mathbb{M}_{12}(K^*(N);\psi_{V_N})$$

and to show that they generates $\mathbb{M}_k(\Gamma_N)$ by seeing their algebraic relations.

To show our main theorem, we prepare a subspace of $\mathbb{M}_k(\Gamma_N)$. Let

$$B_{(N),k}:=\left\{0\le b\le N'-1\,\middle|\,k\equiv\frac{12b}{N'}\pmod 2\right\}$$

and

$$R_k(\Gamma_N):=\bigoplus_{b\in B_{(N),k}}\mathbb{M}_k(K^*(N);\psi_N^b).$$

If k is odd and $\frac{12}{N'}$ is even, we define $R_k(\Gamma_N)=\{0\}$. In the same way, we have an estimation

$$\dim_{\mathbb{C}}R_k(\Gamma_N)=\sum_{b\in B_{(N),k}}\dim_{\mathbb{C}}\mathbb{M}_k(K^*(N);\psi_N^b)$$

$$\le\sum_{b\in B_{(N),k}}\sum_{m=0}^{\infty}\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k-12(m+\frac{b}{N'}),N(m+\frac{b}{N'})}.$$

Hence, if $N\in\{2,3,4\}$, the dimension of $R_k(\Gamma_N)$ is not greater than the coefficient of x^k on the formal power series (Laurent) development of the function

$$\sum_{k\in\mathbb{Z}}\sum_{b\in B_{(N),k}}\sum_{m=0}^{\infty}(\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k-12(m+\frac{b}{N}),N(m+\frac{b}{N})})x^k$$

$$=\sum_{k\in2\mathbb{Z}}\sum_{m=0}^{\infty}\left(\dim_{\mathbb{C}}\mathbb{J}^{\mathrm{w}}_{k,m}\right)x^{k+\frac{12}{N}m}$$

$$=\frac{1}{(1-x^4)(1-x^6)(1-x^{\frac{12}{N}-2})(1-x^{\frac{12}{N}})}.$$

3.2. *Construction of the generators*

We know several methods to construct modular forms. Here we construct the generators $E_{(N),4}, E_{(N),6}, X_{(N)}, Y_{(N)}, Z_{(N)}, W_{(N)}$ by using Maass lift and Rankin–Cohen differential operators. The advantage of this construction is the easiness of calculating the Fourier coefficients of modular forms.

3.2.1. *Case* $N = 2$

Let $\varphi_{4,2} = 1 + O(q) \in \mathbb{J}_{4,2}$ and $\varphi_{6,2} = 1 + O(q) \in \mathbb{J}_{6,2}$ be Jacobi forms of index 2. As $\dim_{\mathbb{C}} \mathbb{J}_{4,2} = \dim_{\mathbb{C}} \mathbb{J}_{6,2} = 1$, $\varphi_{4,2}$ and $\varphi_{6,2}$ are unique Jacobi forms of each weight and index up to constant. When $N = 2$, the first five generators can be constructed directly by Maass lift.

$$\begin{aligned}
E_{(2),4} &:= \mathrm{ML}(\varphi_{4,2}) = e_4(\tau) + O(p^2) \in \mathbb{M}_4(K^*(2)),\\
E_{(2),6} &:= \mathrm{ML}(\varphi_{6,2}) = e_6(\tau) + O(p^2) \in \mathbb{M}_6(K^*(2)),\\
X_{(2)} &:= \mathrm{ML}(\eta^{12}\varphi_{-2,1}) = \eta(\tau)^{12}\varphi_{-2,1}(\tau,z)p + O(p^3) \in \mathbb{M}_4(K^*(2);\psi_2),\\
Y_{(2)} &:= \mathrm{ML}(\eta^{12}\varphi_{0,1}) = \eta(\tau)^{12}\varphi_{0,1}(\tau,z)p + O(p^3) \in \mathbb{M}_6(K^*(2);\psi_2),\\
Z_{(2)} &:= \mathrm{ML}(\eta^{24}\varphi_{-1,2}) = \eta(\tau)^{24}\varphi_{-1,2}(\tau,z)p^2 + O(p^4) \in \mathbb{M}_{11}(K^*(2);\psi_{V_2}).
\end{aligned}$$

Because $e_4, e_6, \varphi_{-2,1}$ and $\varphi_{0,1}$ are algebraically independent on $\mathbb{C}$, the first four generators $E_{(2),4}, E_{(2),6}, X_{(2)}$ and $Y_{(2)}$ are algebraically independent on $\mathbb{C}$. Therefore, we have

$$\bigoplus_{k\in\mathbb{Z}} R_k(\Gamma_2) = \mathbb{C}[E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}].$$

On the other hand, from (2.4), we have

$$Z_{(2)}^2 = \frac{1}{432}(2E_{(2),6}X_{(2)}^4 - 3E_{(2),4}X_{(2)}^3Y_{(2)} + X_{(2)}Y_{(2)}^3).$$

The last generator $W_{(2)}$ is given by

$$W_{(2)} := \frac{\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3}{331776\pi^3\sqrt{-1}Z_{(2)}} = \Delta(\tau) + O(p^2) \in \mathbb{M}_{12}(K^*(2);\psi_{V_2}).$$

We need to show that $\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3$ can be divided by $Z_{(2)}$. As

$$\begin{aligned}
2Z_{(2)}\{E_{(2),4}, Z_{(2)}, X_{(2)}, Y_{(2)}\}_3 &= \{E_{(2),4}, Z_{(2)}^2, X_{(2)}, Y_{(2)}\}_3\\
&= \frac{1}{216}X_{(2)}^4\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3,
\end{aligned}$$

we have

$$\begin{aligned}
&(2E_{(2),6}X_{(2)}^4 - 3E_{(2),4}X_{(2)}^3Y_{(2)} + X_{(2)}Y_{(2)}^3)\{E_{(2),4}, Z_{(2)}, X_{(2)}, Y_{(2)}\}_3^2\\
&\quad = \frac{1}{432}X_{(2)}^8\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3^2.
\end{aligned}$$

Each side on the above equation is in $\mathbb{M}_{78}(K^*(2)) \subset \mathbb{C}[E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}]$. Therefore

$$\frac{432}{X_{(2)}^8}\{E_{(2),4}, Z_{(2)}, X_{(2)}, Y_{(2)}\}_3^2 = \frac{\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3^2}{2E_{(2),6}X_{(2)}^4 - 3E_{(2),4}X_{(2)}^3 Y_{(2)} + X_{(2)}Y_{(2)}^3}$$

is holomorphic. Hence

$$\frac{432}{X_{(2)}^4}\{E_{(2),4}, Z_{(2)}, X_{(2)}, Y_{(2)}\}_3 = \frac{\{E_{(2),4}, E_{(2),6}, X_{(2)}, Y_{(2)}\}_3}{Z_{(2)}}$$

is holomorphic.

From the above, we have finished to show that the estimation in Sec. 3.1 gives the actual dimension of the space of paramodular forms when $N = 2$.

3.2.2. *Case* $N = 3$

Let $\varphi_{4,3} = 1 + O(q) \in \mathbb{J}_{4,3}$ and $\varphi_{6,3} = 1 + O(q) \in \mathbb{J}_{6,3}$ be Jacobi forms of index 3. As $\dim_{\mathbb{C}} \mathbb{J}_{4,3} = 1$, $\varphi_{4,3}$ is a unique Jacobi form of weight 4 and index 3 up to constant. However, as $\dim_{\mathbb{C}} \mathbb{J}_{6,3} = 2$, $\varphi_{6,3}$ is not determined uniquely. We remark that the space $\mathbb{J}_{6,3}$ is spanned by $\varphi_{6,3}$ and $\eta^{24}\varphi_{-2,1}^3$. When $N = 3$, the first four generators can be constructed directly by Maass lift.

$$\begin{aligned}
E_{(3),4} &:= \mathrm{ML}(\varphi_{4,3}) = e_4(\tau) + O(p^3) \in \mathbb{M}_4(K^*(3)), \\
E_{(3),6} &:= \mathrm{ML}(\varphi_{6,3}) = e_6(\tau) + O(p^3) \in \mathbb{M}_6(K^*(3)), \\
X_{(3)} &:= \mathrm{ML}(\eta^8\varphi_{-2,1}) = \eta(\tau)^8\varphi_{-2,1}(\tau, z)p + O(p^4) \in \mathbb{M}_2(K^*(3); \psi_3), \\
Y_{(3)} &:= \mathrm{ML}(\eta^8\varphi_{0,1}) = \eta(\tau)^8\varphi_{0,1}(\tau, z)p + O(p^4) \in \mathbb{M}_4(K^*(3); \psi_3).
\end{aligned}$$

Because $e_4, e_6, \varphi_{-2,1}$ and $\varphi_{0,1}$ are algebraically independent on $\mathbb{C}$, these four generators $E_{(3),4}, E_{(3),6}, X_{(3)}$ and $Y_{(3)}$ are algebraically independent on $\mathbb{C}$. Therefore, we have

$$\bigoplus_{k \in \mathbb{Z}} R_k(\Gamma_3) = \mathbb{C}[E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}].$$

The fifth generator $Z_{(3)}$ is given by

$$Z_{(3)} := \frac{\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1})}{X_{(3)}} = \eta(\tau)^{16}\varphi_{-1,2}(\tau, z)p^2 + O(p^5) \in \mathbb{M}_7(K^*(3); \psi_{V_3}\psi_3^2).$$

We need to show that $\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1})$ can be divided by $X_{(3)}$. As

$$(\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1}))^2 = \eta(\tau)^{48}\varphi_{-1,2}(\tau, z)^2\varphi_{-2,1}(\tau, z)^2 p^6 + O(p^9),$$

we have

$$\begin{aligned}
&(\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1}))^2 \\
&\quad - \frac{X_{(3)}^2}{432}(2E_{(3),6}X_{(3)}^4 - 3E_{(3),4}X_{(3)}^3 Y_{(3)} + X_{(3)}Y_{(3)}^3) \in \mathbb{M}_{18}(K^*(3))[10].
\end{aligned}$$

Therefore, $(\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1}))^2$ can be divided by $X_{(3)}^2$, because the space $\mathbb{M}_{18}(K^*(3))[10]$ is one dimensional spanned by $X_{(3)}^9$. Hence $\mathrm{ML}(\eta^{24}\varphi_{-1,2}\varphi_{-2,1})$ can be divided by $X_{(3)}$.

The last generator $W_{(3)}$ is given by

$$W_{(3)} := \frac{\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3}{331776\pi^3\sqrt{-1}Z_{(3)}} = \Delta(\tau) + O(p^3).$$

We need to show that $\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3$ can be divided by $Z_{(3)}$. As

$$\begin{aligned} 2Z_{(3)}\{E_{(3),4}, Z_{(3)}, X_{(3)}, Y_{(3)}\}_3 &= \{E_{(3),4}, Z_{(3)}^2, X_{(3)}, Y_{(3)}\}_3 \\ &= \frac{1}{216}X_{(3)}^4\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3, \end{aligned}$$

we have

$$Z_{(3)}^2\{E_{(3),4}, Z_{(3)}, X_{(3)}, Y_{(3)}\}_3^2 = \frac{1}{432}X_{(3)}^8\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3^2.$$

Each side on the above equation is in $\mathbb{M}_{54}(K^*(3)) \subset \mathbb{C}[E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}]$. Therefore

$$\frac{432}{X_{(3)}^8}\{E_{(3),4}, Z_{(3)}, X_{(3)}, Y_{(3)}\}_3^2 = \frac{\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3^2}{Z_{(3)}^2}$$

is holomorphic. Hence $\{E_{(3),4}, E_{(3),6}, X_{(3)}, Y_{(3)}\}_3$ can be divided by $Z_{(3)}$.

From the above, we have finished to show that the estimation in Sec. 3.1 gives the actual dimension of the space of paramodular forms when $N = 3$.

3.2.3. *Case $N = 4$*

Let $\varphi_{4,4} = 1+O(q) \in \mathbb{J}_{4,4}$ and $\varphi_{6,4} = 1+O(q) \in \mathbb{J}_{6,4}$ be Jacobi forms of index 4. As $\dim_{\mathbb{C}} \mathbb{J}_{4,4} = \dim_{\mathbb{C}} \mathbb{J}_{6,4} = 2$, $\varphi_{4,4}$ and $\varphi_{6,4}$ are not determined uniquely. We remark that the space $\mathbb{J}_{4,4}$ is spanned by $\varphi_{4,4}$ and $\eta^{24}\varphi_{-2,1}^4$ and that the space $\mathbb{J}_{6,4}$ is spanned by $\varphi_{6,4}, \eta^{24}$ and $\varphi_{-2,1}^3\varphi_{0,1}$. When $N = 4$, the first five generators can be constructed directly by Maass lift.

$$\begin{aligned} E_{(4),4} &:= \mathrm{ML}(\varphi_{4,4}) = e_4(\tau) + O(p^4) \in \mathbb{M}_4(K^*(4)), \\ E_{(4),6} &:= \mathrm{ML}(\varphi_{6,4}) = e_6(\tau) + O(p^4) \in \mathbb{M}_6(K^*(4)), \\ X_{(4)} &:= \mathrm{ML}(\eta^6\varphi_{-2,1}) = \eta(\tau)^6\varphi_{-2,1}(\tau, z)p + O(p^5) \in \mathbb{M}_1(K^*(4); \psi_4), \\ Y_{(4)} &:= \mathrm{ML}(\eta^6\varphi_{0,1}) = \eta(\tau)^6\varphi_{0,1}(\tau, z)p + O(p^5) \in \mathbb{M}_3(K^*(4); \psi_4), \\ Z_{(4)} &:= \mathrm{ML}(\eta^{12}\varphi_{-1,2}) = \eta(\tau)^{12}\varphi_{-1,2}(\tau, z)p^2 + O(p^6) \in \mathbb{M}_5(K^*(4); \psi_{V_4}\psi_4^2). \end{aligned}$$

Because $e_4, e_6, \varphi_{-2,1}$ and $\varphi_{0,1}$ are algebraically independent on $\mathbb{C}$, the first four generators $E_{(4),4}, E_{(4),6}, X_{(4)}$ and $Y_{(4)}$ are algebraically independent on $\mathbb{C}$. Therefore, we have

$$\bigoplus_{k\in\mathbb{Z}} R_k(\Gamma_4) = \mathbb{C}[E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}].$$

On the other hand, $Z_{(4)} \notin R_5(\Gamma_4)$ and $Z_{(4)}^2 \in R_{10}(\Gamma_4)$.

The last generator $W_{(4)}$ is given by

$$W_{(4)} := \frac{\{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3}{331776\pi^3\sqrt{-1}Z_{(4)}} = \Delta(\tau) + O(p^4) \in \mathbb{M}_{12}(K^*(4); \psi_{V_4}).$$

We need to show that $\{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3$ can be divided by $Z_{(4)}$. As

$$\begin{aligned} 2Z_{(4)}\{E_{(4),4}, Z_{(4)}, X_{(4)}, Y_{(4)}\}_3 &= \{E_{(4),4}, Z_{(4)}^2, X_{(4)}, Y_{(4)}\}_3 \\ &= \frac{1}{216} X_{(4)}^4 \{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3, \end{aligned}$$

we have

$$Z_{(4)}^2\{E_{(4),4}, Z_{(4)}, X_{(4)}, Y_{(4)}\}_3^2 = \frac{1}{432} X_{(4)}^8 \{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3^2.$$

Each side on the above equation is in $\mathbb{M}_{42}(K^*(4)) \subset \mathbb{C}[E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}]$. Therefore

$$\frac{432}{X_{(4)}^8}\{E_{(4),4}, Z_{(4)}, X_{(4)}, Y_{(4)}\}_3^2 = \frac{\{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3^2}{Z_{(4)}^2}$$

is holomorphic. Hence $\{E_{(4),4}, E_{(4),6}, X_{(4)}, Y_{(4)}\}_3$ can be divided by $Z_{(4)}$.

From the above, we have finished to show that the estimation in Sec. 3.1 gives the actual dimension of the space of paramodular forms when $N = 4$.

4. Remarks

4.1. *Eisenstein series*

In our paper, we construct all generators by using Maass lifts and Rankin–Cohen–Ibukiyama differential operators. Here we remark that some generators can be constructed by Eisenstein series.

Usual Siegel Eisenstein series with the principal character are non-cuspital modular forms of even weights. Hence the first two generators of weight 4 and 6 can be given by Eisenstein series instead of $E_{(N),4}$ and $E_{(N),6}$. Especially, by seeing the dimension of the space of modular forms, the generators $E_{(2),4}, E_{(2),6}$ and $E_{(3),4}$ coincide with the (normalized) Eisenstein series.

Moreover, Iwahori constructed some paramodular forms by Eisenstein series of Klingen type in [16]. He showed that Klingen Eisenstein series obtained from the Ramanujan delta function is in the space $\mathbb{M}_{12}(K(N)) = \mathbb{M}_{12}(K^*(N)) \oplus \mathbb{M}_{12}(K^*(N); \psi_{V_N})$ but not in the space $\mathbb{M}_{12}(K^*(N))$, when $N = 2, 3$. Hence the last generators $W_{(2)}$ and $W_{(3)}$ coincide with the $\mathbb{M}_{12}(K^*(N); \psi_{V_N})$-parts of the Klingen Eisenstein series obtained from the Ramanujan delta function up to a constant.

4.2. *dd-modular forms*

In the paper by Cléry and Gritsenko [4], they defined dd-modular forms, which have the simplest divisor, and determined all dd-modular forms explicitly. On Siegel paramodular forms of degree 2 with level $N = 2, 3, 4$, there is exactly one dd-modular form (with a nontrivial character) for each level. These dd-modular forms had been constructed by Gritsenko and Nikulin in [8, 9]. They denote these forms by Δ_2, Δ_1 and $\Delta_{1/2}$, whose weights are $2, 1$ and $\frac{1}{2}$. It is easy to see that

$$X_{(2)} = \Delta_2^2, \quad X_{(3)} = \Delta_1^2 \quad \text{and} \quad X_{(4)} = \Delta_{1/2}^2.$$

Hence each $X_{(N)}$ has the simplest divisor with multiplicity 2.

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