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RESEARCH

Conjectures on correspondence of symplectic modular forms of middle parahoric type and Ihara lifts

Tomoyoshi Ibukiyama*

*Correspondence: ibukiyam@math.sci.osaka-u.ac.jp
Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka 560-0043, Japan

Abstract

By Ihara (J Math Soc Jpn 16:214–225, 1964) and Langlands (Lectures in modern analysis and applications III, lecture notes in math, vol 170. Springer, Berlin, pp 18–61, 1970), it is expected that automorphic forms of the symplectic group $Sp(2,\mathbb{R}) \subset GL_4(\mathbb{R})$ of rank 2 and those of its compact twist have a good correspondence preserving L functions. Aiming to give a neat classical isomorphism between automorphic forms of this type for concrete discrete subgroups like Eichler (J Reine Angew Math 195:156–171, 1955) and Shimizu (Ann Math 81(2):166–193, 1965) (and not aiming the general representation theory), in our previous papers Hashimoto and Ibukiyama (Adv Stud Pure Math 7:31–102, 1985) and Ibukiyama (J Fac Sci Univ Tokyo Sect IA Math 30:587–614, 1984; Adv Stud Pure Math 7:7–29 1985; in: Furusawa (ed) Proceedings of the 9-th autumn workshop on number theory, 2007), we have given two different conjectures on precise isomorphisms between Siegel cusp forms of degree two and automorphic forms of the symplectic compact twist $USp(2)$, one is the case when subgroups of both groups are maximal locally, and the other is the case when subgroups of both groups are minimal parahoric. We could not give a good conjecture at that time when the discrete subgroups for Siegel cusp forms are middle parahoric locally (like $\Gamma_0^{(2)}(p)$ of degree two). Now a subject of this paper is a conjecture for such remaining cases. We propose this new conjecture with strong evidence of relations of dimensions and also with numerical examples. For the compact twist, it is known by Ihara that there exist liftings of Saito–Kurokawa type and of Yoshida type. It was not known about the description of the image of these liftings, but we can give here also a very precise conjecture on the image of the Ihara liftings.

Keywords: Siegel modular form, Middle parahoric, Compact twist, Ihara lift

Mathematics Subject Classification: Primary 11F46, Secondly 11F66

1 Introduction

In this paper, we first show certain new dimensional relations between Siegel cusp forms of degree two and automorphic forms belonging to the compact twist of $Sp(2,\mathbb{R}) \subset SL(4,\mathbb{R})$. More precisely, the discrete subgroups we treat here for Siegel modular forms are mainly those of prime level p whose p-adic completions are parahoric subgroups of $Sp(2,\mathbb{Q}_p)$ which are not maximal, but not minimal. On the other hand, the adelic open subgroups of the compact twist are those which correspond with the subgroup stabilizing a fixed lattice in the principal genus of binary quaternion hermitian lattices over the definite quaternion algebra over $\mathbb{Q}$ with prime discriminant p. We give a new dimensional relation between Siegel cusp forms and automorphic forms of the compact twist belonging to such groups (§3 Main Theorem). Such relation leads naturally to conjectures on correspondence between automorphic forms (See Conjectures 3.2, 3.3.) In particular, we give a precise conjecture on the image of Ihara lifts in Conjecture 3.2. The Ihara lift is a compact analogue of Saito–Kurokara lift and Yoshida lift, though actually it appeared much earlier in the paper [27] (See also [22]). In §4, we give numerical examples of the lifts and the correspondence for the most cases explained in §3. In particular, Ihara gave in [27] an example consisting of two automorphic forms of the compact twist which seem to be very different but to have the same Hecke eigenforms. We will explain in §4.3 that both forms should correspond to essentially the same Siegel modular form belonging to middle parahoric subgroups, and this fact supports the last claim of Conjecture 3.3, though we do not know if this is a counter example to the multiplicity one property or not. In §5, we give a proof of the Main Theorem, first for $p \neq 2, 3$ and then for $p = 2, 3$ and $j = 0$.

2 Definitions and notation

2.1 Siegel modular forms

We denote by H_n the Siegel upper half space of degree n and $Sp(n, \mathbb{R}) \subset M_{2n}(\mathbb{R})$ the real symplectic group of degree n. For any irreducible representation (ρ, V) of $GL_n(\mathbb{C})$, any V-valued holomorphic function $f : H_n \to V$, and any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, we write

$$(f|_\rho[g])(\tau) = \rho(C\tau + D)^{-1} f(g\tau) \qquad (\tau \in H_n).$$

For any discrete subgroup $\Gamma \subset Sp(n, \mathbb{Q})$ with $\mathrm{vol}(\Gamma \backslash H_n) < \infty$, a V-valued holomorphic function f is said to be a Siegel modular form of weight ρ of degree n with respect to Γ if $f|_\rho[\gamma] = f$ for any $\gamma \in \Gamma$, with extra condition that it is holomorphic at cusps when $n = 1$, which is automatic when $n > 1$ by the Koecher principle. We denote the space of such modular forms by $A_\rho(\Gamma)$. For $f \in A_\rho(\Gamma)$, we define the Siegel Φ operator by

$$(\Phi f)(\tau_1) = \lim_{t \to +\infty} f \begin{pmatrix} \tau_1 & 0 \\ 0 & it \end{pmatrix} \qquad (\tau_1 \in H_{n-1}, t \in \mathbb{R}).$$

If we have $\Phi(f|_\rho[g]) = 0$ for any $g \in Sp(n, \mathbb{Q})$, we say that f is a cusp form. We denote by $S_\rho(\Gamma)$ the space of cusp forms. In the paper, we mainly consider the case $n = 2$. Any rational irreducible representation of $GL_2(\mathbb{C})$ is given by $\rho = \det^k Sym(j)$ for some $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 0}$, where $Sym(j)$ is the symmetric tensor representation of degree j and $\det^k$ is the k-th power of the determinant. For the sake of simplicity, we write $A_\rho(\Gamma) = A_{k,j}(\Gamma)$ and $S_\rho(\Gamma) = S_{k,j}(\Gamma)$ for such representations. When $n = 2$, if $-1_4 \in \Gamma$, then we have $A_{k,j}(\Gamma) = 0$ if j is odd. For $n = 1$, we denote as usual by $A_k(\Gamma)$ and $S_k(\Gamma)$ the space of modular forms and cusp forms of weight k belonging to Γ, respectively.

2.2 Discrete subgroups

We explain discrete subgroups of $Sp(2, \mathbb{R}) \subset GL_4(\mathbb{R})$ that we want to treat in the paper. We fix a prime p and define a subgroup $B(p)$ of $Sp(2, \mathbb{Q})$ by

$$B(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2, \mathbb{Q}).$$

The p-adic completion of $B(p)$ is the minimal parahoric subgroup (i.e., the Iwahori subgroup) of $Sp(2, \mathbb{Q}_p)$. There are seven (proper) standard parahoric subgroups of $Sp(2, \mathbb{Q}_p)$ containing $B(p)$. Corresponding to those local subgroups, we can define global subgroups as follows. We define

$$S_0 = \begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

We put

$$\Gamma_0(p) = B(p) \cup B(p)S_1B(p),$$
$$\Gamma_0'(p) = B(p) \cup B(p)S_2B(p), \qquad \Gamma_0''(p) = B(p) \cup B(p)S_0B(p).$$

We call these subgroups the subgroups of middle parahoric type, since these are not maximal but not minimal p-adically. More concretely, these groups are given as follows.

$$\Gamma_0(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & Z & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2, \mathbb{Q}),$$

$$\Gamma_0'(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & Z & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2, \mathbb{Q}),$$

$$\Gamma_0''(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2, \mathbb{Q}).$$

We have $\Gamma_0''(p) = \omega\Gamma_0'(p)\omega^{-1}$. There are three other subgroups which are maximal p-adically. Two are $Sp(2, \mathbb{Z}) = M_4(\mathbb{Z}) \cap Sp(2, \mathbb{Q})$ and $\omega Sp(2, \mathbb{Z})\omega^{-1}$ and the remaining one is the paramodular group $K(p)$ generated by $\Gamma_0'(p)$ and $\Gamma_0''(p)$. We can write $K(p) = B(p) \cup B(p)S_0B(p) \cup B(p)S_2B(p) \cup B(p)S_0S_2B(p)$ and more concretely we have

$$K(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & Z & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2, \mathbb{Q}).$$

2.3 Automorphic forms on the compact twist

We fix a prime p as before. We denote by $\mathbb{H}$ the unique division quaternion algebra over $\mathbb{R}$. Let D be the definite quaternion algebra over $\mathbb{Q}$ with discriminant p. For any prime q, we put $D_q = D \otimes_{\mathbb{Q}} \mathbb{Q}_q$, where $\mathbb{Q}_q$ is the q-adic number field and also put $D_\infty = D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$. We define the group of similitudes of the positive definite binary quaternion hermitian form over D by

$$G = \{g \in M_2(D); g\, {}^t\overline{g} = n(g)1_2 \text{ for some } n(g) \in \mathbb{Q}_+^\times\}$$

and call it shortly a quaternion hermitian group. Here, we denote by $\overline{*}$ the main involution of D and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(D)$, we write ${}^t\overline{g} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$. We denote by G_A the adelization of G and by G_v the v-component of G_A for places $v \leq \infty$. For example, we have

$$G_\infty = \{g \in M_2(\mathbb{H}); g\, {}^t\overline{g} = n(g)1_2, n(g) > 0\}$$

and if we put $G_\infty^1 = \{g \in G_\infty; n(g) = 1\}$, then this is the compact twist $USp(2)$ of $Sp(2, \mathbb{R})$. For a prime q, we also have

$$G_q = \{g \in M_2(D_q); g\, {}^t\overline{g} = n(g)1_2 \text{ for some } n(g) \in \mathbb{Q}_q^\times\}.$$

For any prime $q \neq p$, we have

$$G_q \cong GSp(2, \mathbb{Q}_q) = \{g \in M_4(\mathbb{Q}_q);\ {}^tgJg = n(g)J \text{ for some } n(g) \in \mathbb{Q}_q^\times\},$$

where $J = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$.

Now we describe a subgroup of G_A for which we define automorphic forms of G_A in this paper. Let $\mathcal{O}$ be a maximal order of D. For a prime q, we write $\mathcal{O}_q = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_q$ where $\mathbb{Z}_q$ is the ring of q-adic integers. We write

$$GL_2(\mathcal{O}_q) = \{g \in GL_2(D_q); g \in M_2(\mathcal{O}_q) \text{ and } g^{-1} \in M_2(\mathcal{O}_q)\},$$

the group of invertible elements of $M_2(\mathcal{O}_q)$. We put $U_q = G_q \cap GL_2(\mathcal{O}_q)$. This is the subgroup of G_q which stabilizes the maximal lattice $\mathcal{O}_q^2$, and a maximal compact subgroup of G_q. When $q \neq p$, then we have

$$U_q \cong GSp(2, \mathbb{Z}_q) = \{g \in GSp(2, \mathbb{Q}_q) \cap M_4(\mathbb{Z}_q); n(g) \in \mathbb{Z}_q^\times\}.$$

We define an open subgroup $\mathcal{U}_{pr}(p)$ of G_A by

$$\mathcal{U}_{pr}(p) = G_\infty \prod_{q;\text{prime}} U_q.$$

We explain that this group is related to the genus of maximal quaternion hermitian lattices containing $\mathcal{O}^2$. A lattice $L \subset D^2$ is called an $\mathcal{O}$ lattice if $\mathcal{O}L \subset L$. The set Λ of quaternion hermitian $\mathcal{O}$ lattices L in D^2 such that $L_v := L \otimes_{\mathbb{Z}} \mathbb{Z}_v = O_v^2 g_v$ for some $g_v \in G_v$ for all $v < \infty$ is called the principal genus. Two $\mathcal{O}$ lattices L_1 and L_2 in Λ are said to be isomorphic if $L_1 = L_2 g$ for some $g \in G$. The number of isomorphism classes of $\mathcal{O}$ lattices in Λ is finite and called the class number $h = h(\Lambda)$ of Λ. The group G_A acts on Λ by

$L \to Lg := \bigcap_{v<\infty}(L_v g_v \cap D^2)$ for $g = (g_v) \in G_A$. Then the group $\mathcal{U}_{pr}(p)$ is the stabilizer of the lattice $\mathcal{O}^2$. We use the subscript "pr" for $\mathcal{U}_{pr}(p)$ to indicate the principal genus. We have $h(\Lambda) = \#(\mathcal{U}_{pr}(p)\backslash G_A/G)$ (cardinality). Here as usual G is identified with the image of the diagonal embedding of G into G_A ([39]).

We explain weights of our automorphic forms. Let (τ_∞, V) be a (finite dimensional) irreducible representation of the compact group G_∞^1. Any such representation of G_∞^1 corresponds with the Young diagram parameter (ν_1, ν_2) with $\nu_1 \ge \nu_2 \ge 0$, $\nu_i \in \mathbb{Z}$. We denote this representation by $\tau_\infty = \tau_{\nu_1,\nu_2}$. We assume that $\tau_{\nu_1,\nu_2}(-1_2) = id_V$, which is equivalent to the condition that $\nu_1 \equiv \nu_2 \bmod 2$. Associated with such τ_{ν_1,ν_2}, we define a representation of G_A by

$$G_A \to G_\infty \to G_\infty/\text{center} = G_\infty^1/\{\pm 1_2\} \xrightarrow{\tau_{\nu_1,\nu_2}} \mathrm{GL}(V).$$

This is denoted by the same letter τ_{ν_1,ν_2}. Any V-valued function f of G_A is said to be an automorphic form of G_A of weight $\tau = \tau_{\nu_1,\nu_2}$ with respect to $\mathcal{U}_{pr}(p)$ if it satisfies the following condition.

$$f(uga) = \tau_{\nu_1,\nu_2}(u)f(g) \qquad \text{for any } a \in G, g \in G_A, u \in \mathcal{U}_{pr}(p).$$

We denote the space of such automorphic forms by $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$. More down to earth description of this space is given as follows (See [27] and [10]). We take a double coset decomposition $G_A = \cup_{\kappa=1}^h \mathcal{U}_{pr}(p) g_\kappa G$ and for each κ, put $\Gamma_\kappa = G \cap g_\kappa^{-1}\mathcal{U}_{pr}(p) g_\kappa$. This is a finite group. We put

$$\begin{aligned} V^{\Gamma_\kappa} &= \{v \in V; \tau_{\nu_1,\nu_2}(\gamma_\kappa)v = v \text{ for all } \gamma_\kappa \in \Gamma_\kappa\}, \\ M_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p)) &= \oplus_{\kappa=1}^h V^{\Gamma_\kappa}. \end{aligned}$$

Then we have a following isomorphism.

$$\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p)) \cong M_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p)). \tag{1}$$

An explicit isomorphism of (1) is given by mapping $f \in \mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ to $\sum_{\kappa=1}^h \tau(g_\kappa)^{-1} f(g_\kappa)$. We often identify these two spaces in Sect. 4. The action of Hecke operators is defined as follows. For any $z \in G_A$, we take a double coset

$$\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p) = \bigcup_i z_i \mathcal{U}_{pr}(p).$$

Then the action of this double coset on $f \in \mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ is defined by

$$([\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)] f)(g) = \sum_i \tau_{\nu_1,\nu_2}(z_i) f(z_i^{-1} g).$$

On the element $(f_\kappa) \in M_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ corresponding to f in the isomorphism (1), the action of $\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)$ is given by

$$([\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)] f)_\kappa = \sum_{\mu=1}^h \sum_{t \in T_{\kappa\mu}/\Gamma_\mu} \tau_{\nu_1,\nu_2}(t) f_\mu \qquad (1 \le \kappa \le h),$$

where $([\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)] f)_\kappa$ denotes the κ component of $[\mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)] f$ in $M_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ and the set $T_{\kappa\mu}$ is defined by

$$T_{\kappa\mu} = G \cap g_\kappa^{-1} \mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p) g_\mu.$$

For any positive integer n, we put $T(n) = \sum_z \mathcal{U}_{pr}(p) z \mathcal{U}_{pr}(p)$, where z runs over elements $z = (z_v) \in G_A \cap G_\infty \prod_q M_2(O_q)$ with $n(z_q) = n\mathbb{Z}_q^\times$ for all primes q. The action of $T(n)$ on $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p)) \cong M_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ is defined by the linear prolongation of the action of double cosets. The Hecke algebra $\mathbb{C}[T(n); p \nmid n]$ is isomorphic to that of the split group $GSp(2,\mathbb{Q})$, since $G_q \cong GSp(2,\mathbb{Q}_q)$ and $U_q \cong GSp(2,\mathbb{Z}_q)$ for $q \neq p$. If f is an eigenform of all the Hecke operators $T(n)$, we denote by $\lambda(n)$ the eigenvalues of $T(n)$ on f and define an Euler q-factor for $q \neq p$ by

$$1 - \lambda(q)q^{-s} + (\lambda(q)^2 - \lambda(q^2) - q^{\nu_1+\nu_2+2})q^{-2s} - \lambda(q)q^{\nu_2+\nu_2+3-3s} + q^{2\nu_1+2\nu_2+6-4s}.$$

(The Euler q factor for $q = p$ is given in Ihara [27] and we omit it here.) We define a spinor L function of f as the product over primes of the inverse of all these Euler factors. (See [10,22,27].)

2.4 Ihara lifts

We review shortly a theory of Ihara lifts ([22,27]) to $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$. Although they are compact versions of the Saito–Kurokawa lifts and the Yoshida lifts (published in 1978 and 1980, respectively), Ihara's lifts appeared much earlier (around in 1963 in Ihara's master thesis in Univ. Tokyo and partly published in [27]). We take a basis of $\mathbb{H}$ over $\mathbb{R}$ as $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. We identify $\mathbb{H}^2 = \mathbb{R}^8$ for $(x, y) \in \mathbb{H}^2$ by taking $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ for $x = x_1 + x_2 i + x_3 j + x_4 k$, $y = y_1 + y_2 i + y_3 j + y_4 k$, and consider a polynomial $f(x, y)$ in eight variables. We denote by $\Delta_{x,y}$ the usual Laplacian of these eight variables:

$$\Delta_{x,y} = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{4} \frac{\partial^2}{\partial y_i^2}.$$

We say that a polynomial $f(x, y)$ is harmonic if $\Delta_{x,y} f = 0$. We denote by $\mathcal{H}m_l(\mathbb{H}^2)$ the space of harmonic polynomials in $(x, y) \in \mathbb{R}^8$ of homogeneous degree l. For $\lambda \in \mathbb{H}$, we denote by Δ_λ the Laplacian of $\mathbb{H} \cong \mathbb{R}^4$ in the same sense. We assume that l is even. For each (a, b) with $a \geq b \geq 0$ and $a + b = l$, we define a subspace $V_{a,b}$ of $\mathcal{H}m_l(\mathbb{H}^2)$ by

$$V_{a,b} = \{f(x, y) \in \mathcal{H}m_l(\mathbb{H}^2);\ f(\lambda x, \lambda y) = n(\lambda)^b \varphi(x, y, \lambda) \text{ for any } \lambda, x, y \in \mathbb{H},$$
$$\text{where } \varphi \text{ is a polynomial with } \Delta_\lambda \varphi = 0\},$$

where $n(\lambda) = \lambda\bar{\lambda}$. This space is invariant by the action $f \to f(\bar{h}(x, y)g)$ for $h \in \mathbb{H}^\times$ and $g \in G_\infty$. This is the representation $Sym_{a-b} \otimes \tau_{a,b}$ of $USp(1) \times USp(2)$, where Sym_{a-b} is the symmetric tensor representation of $USp(1) \cong SU(2)$ (the compact twist of $SL_2(\mathbb{R})$) of degree $a - b$. So as a representation of $USp(2)$, $V_{a,b}$ is not multiplicity free unless $a = b$. We denote by h_0 the class number of D. We take representatives $\{b_i\}$ of the double coset decomposition $D_A^\times = \cup_{i=1}^{h_0} (\mathbb{H}^\times \prod_{q:\text{prime}} \mathcal{O}_q^\times) b_i D^\times$ and put $\mathcal{O}_i = \cap_{q:\text{prime}} (b_i^{-1} O_q b_i \cap D)$ for i with $1 \leq i \leq h_0$. We denote by $V_{a,b}^{(i,\kappa)} = V_{a,b}^{O_i^\times \times \Gamma_\kappa}$ the space of polynomials in $V_{a,b}$ such that $f(\bar{u}(x, y)\gamma) = f(x, y)$ for all $(u, \gamma) \in \mathcal{O}_i^\times \times \Gamma_\kappa$. Then the space

$$\mathcal{V} = \oplus_{i=1}^{h_0} \oplus_{\kappa=1}^{h} V_{a,b}^{(i,\kappa)}$$

is naturally identified with the tensor product of the space of automorphic forms on $D_A^\times$ with respect to $\mathcal{O}_A^\times = \mathbb{H}^\times \prod_q \mathcal{O}_q^\times$ of weight Sym_{a-b} and the space $\mathfrak{M}_{a,b}(\mathcal{U}_{pr}(p))$. For an element $F = (F_{i\kappa})_{1 \leq i \leq h_0, 1 \leq \kappa \leq h}$ of this space $\mathcal{V}$, we define a theta function of $\tau \in H_1$ by

$$\vartheta_F(\tau) = \sum_{i=1}^{h_0} \sum_{\kappa=1}^{h} |\mathcal{O}_i^\times|^{-1} |\Gamma_\kappa|^{-1} \vartheta_F^{(i\kappa)}(\tau),$$

where

$$\vartheta_F^{(i\kappa)}(\tau) = \sum_{m=0}^{\infty} \sum_{\substack{x \in L_{i\kappa}, \\ n_{i\kappa}(x)=m}} F_{i\kappa}(x) e^{2\pi i m\tau}$$

and $L_{i\kappa} = \overline{b_i} O^2 g_\kappa$, $n_{ik}(x) = n(x)/n(L_{i\kappa})$ and $n(L_{i\kappa})$ is the fractional $\mathbb{Z}$ ideal spanned by all $n(x)$ with $x \in L_{i\kappa}$. For any integer $N \geq 1$, we define a subgroup $\Gamma_0^{(1)}(N)$ of $SL_2(\mathbb{Z})$ by

$$\Gamma_0^{(1)}(N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod N \right\}.$$

Then we see that $\vartheta_F(\tau) \in A_{a+b+4}(\Gamma_0^{(1)}(p))$ and this is also a cusp form unless $a = b = 0$. Now assume that F is an Hecke eigenform as an automorphic form $F_1 \times F_2$ on $D_A^\times \times G$. The theory of Ihara lifts claims that if ϑ_F does not vanish, then ϑ_F is also an eigenform and we have

$$L(s, F_2) = L(s - b - 1, F_1) L(s, \vartheta_F) = L(s - k + 2, F_1) L(s, \vartheta_F),$$

where $L(s, F_2)$ is the spinor-type L function of F_2, $L(s, F_1)$ is usual Hecke's L function of the elliptic modular form of $\Gamma_0^{(1)}(p)$ associated with F_1 by the Eichler–Jacquet–Langlands correspondence. Here we put $k = b + 3$. For more details, see [22,27].

3 Main theorem and conjectures

3.1 Main theorem

Note that we write superscript $^{(1)}$ always for subgroups $\Gamma_0^{(1)}(p)$ of $SL_2(\mathbb{Z})$. By $\Gamma_0(p)$ without superscript, we always mean a subgroup of $Sp(2, \mathbb{Z}) \subset GL_4(\mathbb{Z})$ defined before. We denote by $S_k(\Gamma_0^{(1)}(N))$ the space of elliptic cusp forms of $\Gamma_0^{(1)}(N)$ of weight k and by $S_k^{\text{new}}(\Gamma_0^{(1)}(N))$ the subspace of new cusp forms.

Theorem 3.1 *We fix a prime p. Notation being the same as before, for integer $k \geq 3$ and nonnegative even integer j, we have the following dimensional identities*

$$\begin{aligned}
&\dim S_{k,j}(\Gamma_0'(p)) + \dim S_{k,j}(\Gamma_0''(p)) - \dim S_{k,j}(\Gamma_0(p)) - 2 \dim S_{k,j}(K(p)) \\
&\quad = \dim \mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p)) - \delta_{j0}\delta_{k3} \\
&\qquad - (\dim S_{j+2}^{\text{new}}(\Gamma_0^{(1)}(p)) + \delta_{j0}) \times \left(\dim S_{2k+j-2}^{\text{new}}(\Gamma_0^{(1)}(p)) + \dim S_{2k+j-2}(SL_2(\mathbb{Z})) \right)
\end{aligned}$$

under the following additional assumptions: $k \geq 5$ if $j \geq 2$, and $p \neq 2, 3$ if $j \geq 2$. Here δ_{j0} and δ_{k3} are the Kronecker deltas.

The condition $k \geq 3$ is essential, but the conditions $k \geq 5$ for $j \geq 2$ and $p \neq 2, 3$ for $j \geq 2$ are technical. We believe that the same equality holds for any $k \geq 3$ with even $j \geq 0$ for any primes p without extra conditions. We note that since $\Gamma_0''(p) = \omega \Gamma_0'(p) \omega^{-1}$, we have $\dim S_{k,j}(\Gamma_0'(p)) = \dim S_{k,j}(\Gamma_0''(p))$, and the sum of the first two terms of LHS in Theorem 3.1 can be written as $2 \dim S_{k,j}(\Gamma_0'(p))$.

Of course we expect that the same sort of relation as Theorem 3.1 hold also for Hecke operators, interpreting the endoscopic part coming from elliptic modular forms into appropriate combinations of Hecke operators of one variable, but at the moment we do not have results for that since the trace formula is much more complicated in that case.

The proof of Theorem 3.1 will be given in the last two sections.

3.2 Conjectures on lifts and non-lifts

After explaining the possible appearance of lifts, we give a concrete conjecture on Ihara lifts (See Conjecture 3.2). We also give some conjectural correspondence for non-lifts (See Conjecture 3.3). Numerical evidence of automorphic forms will be given in Sect. 4. About local representations which have the Iwahori subgroup fixed vector, there is a work on classification by Roberts and Schmidt [33]. The author learned more details on this from Professor Ralf Schmidt. He confirmed author's partly conjectural tables of the number of local fixed vectors in the first manuscript by secure results of the classification of local representations in [33,36,37] and made it much clearer. The author would like to thank him for this discussion.

For $f \in S_k(\Gamma_0^{(1)}(p))$, we define the Atkin–Lehner involution $W_{p,k}$ by

$$(f|_k W_{p,k}) = p^{-k/2}\tau^{-1}f(-(p\tau)^{-1}).$$

For each $\epsilon = +$ or $-$, define a subspace $S_k^{\epsilon}(\Gamma_0^{(1)}(p))$ of $S_k(\Gamma_0^{(1)}(p))$ by

$$S_k^{\epsilon}(\Gamma_0^{(1)}(p)) = \{f \in S_k(\Gamma_0^{(1)}(p)); f|_k W_{p,k} = \epsilon f\}.$$

We also put $S_k^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p)) = S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p)) \cap S_k^{\epsilon}(\Gamma_0^{(1)}(p))$. If f is a Hecke eigenform and $f|_k W_{p.k} = \epsilon f$, then the functional equation of $L(s,f)$ is given by

$$L(k-s,f) = (-1)^{k/2}\epsilon L(s,f).$$

It is known that there exist Saito–Kurokawa type lifts to $S_k(\Gamma_0(p))$ from $S_{2k-2}(\Gamma_0^{(1)}(p))$ for even k, and Yoshida type lifts from $S_{j+2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p))$ to $S_{k,j}(\Gamma_0(p))$ for the same $\epsilon = \pm$ (see [3,21].) There also exist Gritsenko lifts (an analogue of Saito–Kurokawa lifts) to $S_k(K(p))$. This last lift is a little more complicated. Although this lift does not play an essential role in our lifting conjecture compared with the former two liftings, we review this shortly for readers' convenience since we use this later in our numerical calculation. We denote by $J_{k,p}^{\mathrm{cusp}}(SL_2(\mathbb{Z}))$ the space of Jacobi cusp forms of index p with respect to the group $SL_2(\mathbb{Z})$ (See Eichler–Zagier in [6].) By Skoruppa–Zagier [40], we have an isomorphism

$$J_{k,p}^{\mathrm{cusp}}(SL_2(\mathbb{Z})) \cong S_{2k-2}^{\mathrm{new},(-1)^k}(\Gamma_0^{(1)}(p)) \oplus \delta_{k,\mathrm{even}} S_{2k-2}(SL_2(\mathbb{Z}))$$

where $\delta_{k,\mathrm{even}} = 1$ if k is even and 0 otherwise. There exists a lifting from $J_{k,p}(SL_2(\mathbb{Z}))$ to $S_k(K(p)) \subset S_k(\Gamma_0'(p)), S_k(\Gamma_0''(p))$ by Gritsenko (See [9]).

Now we see how both sides of the dimensional relation in the Main Theorem 3.1 are explained for various types of liftings.

(1) Saito–Kurokawa lifts of level 1 and Ihara lifts.

We consider the case $j = 0$ now. When k is even, for a Hecke eigenform f of $S_{2k-2}(SL_2(\mathbb{Z}))$, there exists the Saito–Kurokawa lift from f to $F \in S_k(Sp(2,\mathbb{Z}))$. By virtue of [36], the number of local (p-adic) fixed vectors belonging to this automorphic representation containing F is as follows. (See also [33] Table A15 II b.)

$Sp(2,\mathbb{Z})$	$\Gamma_0(p)$	$\Gamma_0'(p) \cong \Gamma_0''(p)$	$K(p)$
1	3	2	1

So the contribution of this part to the LHS of Theorem 3.1 is

$$2 + 2 - 3 - 2 \cdot 1 = -1$$

On the other hand, in RHS of Theorem 3.1, we have $-\dim S_{2k-2}(SL_2(\mathbb{Z}))$, which should cancel with the above. This means that there should not exist an Ihara lift to $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$ from $S_{2k-2}(SL_2(\mathbb{Z}))$ when k is even. On the other hand, when k is odd, there is no Saito–Kurokawa lift to LHS. This means that the term $-\dim S_{2k-2}(SL_2(\mathbb{Z}))$ of the RHS should be canceled with the Ihara lift in $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$, so there should exist the Ihara lift of Saito–Kurokawa type in this case.

(2) Saito–Kurokawa lifts of level p and Ihara lifts.

Again we assume that $j=0$. We consider the Saito–Kurokawa lifting from $S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ (See [21,36]). If we assume that k is even, then by [33] A15, the number of local fixed vectors is given by

$\Gamma_0(p)$	$\Gamma_0^{'}(p)\cong\Gamma_0^{''}(p)$	$K(p)$
1	a	a

Here $a=1$ if $f\in S_{2k-2}^{\mathrm{new},(-1)^k}(\Gamma_0^{(1)}(p))$ (The case Vb of A15, loc.cit.,) and $a=0$ otherwise (the case VIb of A15). Anyway the contribution to the LHS is $2a-2a-1=-1$. So in RHS, there should not exist an Ihara lift from $S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$. If k is odd, then there exists no Saito–Kurokara lift to LHS. So judging from the dimensional relation, there should exist an injective Ihara lift from $S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$ if k is odd.

(3) Yoshida lifts of level p and Ihara lifts.

Yoshida constructed in [44] a lift from pairs of elliptic new cusp forms of level p to Siegel cusp forms of degree two belonging to $\Gamma_0(p)$, which vanishes when the signs of the Atkin-Lehner involution are not equal, and Boecherer and Schulze-Pillot proved injectivity of the lift in [3] when the signs are the same. More precisely, if we set $\epsilon=\pm$, there exists an injective lift from $(f_1,f_2)\in S_{j+2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p))\times S_{2k+j-2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p))$ to $F\in S_{k,j}(\Gamma_0(p))$ such that, if (f_1,f_2) are Hecke eigenforms, then F is also a Hecke eigenform and we have $L(s,F)=L(s-k+2,f_1)L(s,f_2)$, where $L(s,F)$ is the spinor L function. For a fixed pair of Hecke eigenforms $(f_1,f_2)\in S_{j+2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\times S_{2k+j-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ with the Atkin-Lehner sign ϵ_i for $i=1,2$, denote the dimensions of forms in $S_{k,j}(\Gamma_0^{'}(p))$, in $S_{k,j}(K(p))$, and in $S_{k,j}(\Gamma_0(p))$ coming from lifts of the fixed pair by a, b, and c, respectively. Here we have $a\geq b\geq 0$ since $\Gamma_0^{'}(p)\subset K(p)$. So the contribution of lifts to the dimension of LHS is given by $2(a-b)-c$, where we have $c=0$ and 1 when $\epsilon_1=-\epsilon_2$ and $\epsilon_1=\epsilon_2$, respectively. Actually, by virtue of [37] Table (30), for Yoshida lifts, we have $a=b=1$ and $c=0$ or 1, or $a=b=0$ and $c=1$. (See also [33] A 15 VIc, Vb, VIb.) The counter part of this lift for the compact twist is the Ihara lift ([22,27]). For $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$, there is the theory of lift from the same pair, but there is no theory how big the image of these lifts is. It is very plausible that the image is at most one dimensional from a fixed pair. If so, the contributions from the pair to the dimension of RHS is 0 if there exists a lift to $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ and -1 if there does not exist. This fits the result $2(a-b)-c=0$ or -1 for LHS. Hence there should exists an Ihara lift to the compact twist if and only if the parity of the signs of the Atkin-Lehner involution of new cusp forms of level p are not the same.

(4) Ihara lifts from pairs of level 1 and level p.

There is one remaining term in the dimensional relation in the Main Theorem suggesting lifts: the term $S_{j+2}^{\mathrm{new}}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$. Numerical examples in Sect. 4 strongly suggest that there is an injective Ihara lift from whole of this space to $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$.

Summing up all these, it is natural to propose the following conjecture on the image of the Ihara lifting.

Conjecture 3.2 *(1) When k is even, there should be no Ihara lift from $S_{2k-2}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$. When k is odd, then the Ihara lifting from $S_{2k-2}(SL_2(\mathbb{Z})) + S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$ is injective.*

(2) There exists an injective Ihara lifting from $S_{j+2}^{\mathrm{new},+}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}^{\mathrm{new},-}(\Gamma_0^{(1)}(p))$ and $S_{j+2}^{\mathrm{new},-}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}^{\mathrm{new},+}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ and no lift from $S_{j+2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}^{\mathrm{new},\epsilon}(\Gamma_0^{(1)}(p))$ for the same sign $\epsilon = \pm$.

(3) There exists an injective Ihara lifting from $S_{j+2}^{\mathrm{new}}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$.

Now we give some description of non-lifts. The local admissible representations at p of $Sp(2,\mathbb{Q}_p)$ which have Iwahori subgroup fixed vectors are classified by Roberts and Schmidt [33]. If we write the dimension of vectors of a local admissible representation fixed by $\Gamma_0'(p)$, $\Gamma_0(p)$, $K(p)$ by a, c, b and put $c_0 = 2a - c - 2b$, then by table A.15 of [33], we have $c_0 = -1, 0, 1, 2$. As we have seen already, the case $c_0 = -1$ appears for lifts in several cases. The case $c_0 = 0$ appears in an example in the next section for non-lifts and there is no corresponding form in the compact twist in this case except for lifts. Interesting cases for the correspondence with RHS is the case $c_0 = 1$ or 2. If $c_0 = 2$, which happens for Va in their table, then it means that there should exist two automorphic forms $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ corresponding to a form in $S_{k,j}(\Gamma_0'(p))$ with the same L function, though we do not know if two forms in $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ generate the same automorphic representation or not. Anyway, the correspondence for $c_0 \neq -1$ should be as in the following conjecture.

Conjecture 3.3 *Notation being as above, for a global representation coming from a holomorphic Siegel cusp form such that $c_0 = 0$ locally at p, there is no corresponding automorphic form in $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ except for the case obtained by lifting. If $c_0 = 1$, then global automorphic representations coming from holomorphic Siegel cusp forms having $\Gamma_0'(p)$ fixed vector should correspond one to one to eigenforms in $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$. If $c_0 = 2$, then a Siegel cusp eigenform in $S_{k,j}(\Gamma_0'(p))$ should correspond with two eigenforms in $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$ having the same L function.*

Here we excluded lifts for $c_0 = 0$ because of the following reason. If k is odd, then there is no Saito–Kurokawa lift to $S_k(\Gamma_0(p))$ but there would exist a lift to a paramodular form in $S_k(K(p))$ as we explained before. This lift also belongs to $S_k(\Gamma_0'(p))$ and $S_k(\Gamma_0''(p))$ and the contribution to LHS of Theorem 3.1 is zero. On the other hand, there should exist an Ihara lift to $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(p))$ by Conjecture 3.2 (1) and the contribution to RHS is also zero as a total. So even if $c_0 = 0$, there should exist a correspondence between a lift to $S_k(\Gamma_0'(p))$ and that to $M_{k-3,k-3}(\mathcal{U}_{pr}(p))$. Later we give concrete examples of $c_0 = 2$ for the above conjectural correspondence, including an explanation of Ihara's example in [27]. There he gave two concrete automorphic forms in $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$ which looks very different but have the same Euler 2 factors, and seem to have the same Euler factors for all primes.

We will give a corresponding Siegel cusp form concretely. It is an interesting problem to see if these two forms in $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$ span the same representation or not. This is a problem if the multiplicity one theorem holds for $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ or not.

4 Numerical examples

Before giving numerical examples, we quote the trace formula of the Atkin-Lehner involution $W_{p,k}$ for prime p on $S_k(\Gamma_0^{(1)}(p))$ from [43]. For $p \neq 2, 3$, we have

$$Tr(W_{p,k}) = -\frac{(-1)^{k/2-1}}{2}\left(\frac{h(-4p)}{w(-4p)} + \frac{h(-p)}{w(-p)}\right) + \delta_{k2},$$

where δ_{k2} is the Kronecker delta and $w(d)$ is the half of the number of roots of unity in the quadratic order of the discriminant d. We also have

$$Tr(W_{2,k}) = \frac{(-1)^{k/2} - (-1)^{(k-4)(k-2)/8}}{2} + \delta_{2k},$$

$$Tr(W_{3,k}) = \delta_{2k} + \begin{cases} -1 & \text{if } k \equiv 2, 6 \bmod 12, \\ 0 & \text{if } k \equiv 4, 10 \bmod 12, \\ -1 & \text{if } k \equiv 0, 8 \bmod 12. \end{cases}$$

For $f(\tau) \in S_k(SL_2(\mathbb{Z}))$, the signs of the forms $f(\tau) \pm p^{k/2} f(p\tau)$ w.r.t. $W_{p,k}$ are $\pm$, respectively, so this part cancels in $Tr(W_{p,k})$. For new forms, we have

$$\dim S_k^{\mathrm{new},\pm}(\Gamma_0^{(1)}(p)) = \frac{1}{2}\left(\dim S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p)) \pm Tr(W_{p,k})\right).$$

4.1 The scalar-valued case for $p = 2$

In this subsection, we consider the case $p = 2$ and $j = 0$. We have examples of Euler 3-factors in [14], though the emphasis there is on the minimal parahoric subgroups. Here we review the results there and add some more examples. First we give a table of dimensions of $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(2))$ and $\dim \mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(2)) - (\dim S_{2k-2}(SL_2(\mathbb{Z})) + \dim S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(2))) - \delta_{k3}$. The latter is indicated by *RHS* in the following table.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
dim	1	0	1	0	1	0	2	0	3	0	3	0	6	1	7
RHS	0	0	0	−1	0	−2	0	−2	0	−3	0	−4	2	−3	2

We give a table of dimensions of Siegel cusp forms and LHS of Theorem 3.1.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\Gamma_0(2)$	0	0	0	1	0	2	0	4	0	7	0	10	0	15	0
$\Gamma_0'(2)$	0	0	0	0	0	1	0	2	1	4	0	5	2	10	2
$K(2)$	0	0	0	0	0	1	0	1	1	2	0	2	1	4	1
LHS	0	0	0	−1	0	−2	0	−2	0	−3	0	−4	2	−3	2

Of course we have LHS = RHS as we claimed in Theorem 3.1. The point is the correspondence of automorphic forms. We see each k more precisely. For $k \leq 11$, all the forms are lifts. We can check this fact by giving elements of $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(2))$ concretely, but we omit it here since there is nothing new. When $k = 12$, an interesting thing happens and we will explain that. The following table is a resumé extracted from [14], where the column $T(3)$ indicates the eigenvalues of the Hecke operator $T(3)$ and the column under the group indicates the dimension having that eigenvalue.

k	$T(3)$	$\Gamma_0^{(2)}(2)$	$\Gamma_0'(2)$	$\Gamma_0''(2)$	$K(2)$	Lift from
6	168	1	0	0	0	$S_{10}(\Gamma_0^{(1)}(2))$
8	4152	1	0	0	0	$S_{14}^{+}(\Gamma_0^{(1)}(2))$
8	1080	1	1	1	1	$S_{14}^{-}(\Gamma_0^{(1)}(2))$
10	21960	3	2	2	1	$S_{18}(SL_2(\mathbb{Z}))$
10	32328	1	0	0	0	$S_{18}^{\mathrm{new},+}(\Gamma_0^{(1)}(2))$
12	107352	3	2	2	1	$S_{22}(SL_2(\mathbb{Z}))$
12	307800	1	1	1	1	$S_{22}^{\mathrm{new},-}(\Gamma_0^{(1)}(2))$
12	-88488	2	1	1	0	Non-lift

Here all the examples fit the conjecture in Sect. 3. The most interesting examples in the above table are the non-lift case of weight 12. In $\dim S_{12}(\Gamma_0'(2)) + \dim S_{12}(\Gamma_0''(2)) - \dim S_{12}(\Gamma_0^{(2)}(2))$, the contribution of non-lift is $1 + 1 - 2 = 0$. So these Siegel cusp forms exist but should not correspond with automorphic forms of the compact twist. Indeed we have $\dim \mathfrak{M}_{9,9}(\mathcal{U}_{pr}(2)) = 0$ and there is no corresponding automorphic form.

We add here one more interesting example of weight $k = 15$. First we give a table of dimensions of Siegel cusp forms for each eigenvalues of $T(3)$.

$T(3)$	$S_{15}(\Gamma_0'(2))$	$S_{15}(\Gamma_0''(2))$	$S_{15}(\Gamma_0(2))$	$S_{15}(K(2))$	LHS	Lift from
5360904	1	1	0	1	0	$S_{28}^{-}(\Gamma_0^{(1)}(2))$
-4260600	1	1	0	0	2	Non-lift

Next we give a table for the compact twist $\mathfrak{M}_{12,12}(\mathcal{U}_{pr}(2))$.

$T(3)$	$\mathcal{U}_{pr}(2)$	RHS	Lift from
-4260600	2	2	Non-Lift
10362120	1	0	$S^{\mathrm{new}}(\Gamma_0^{(1)}(2))$
5360904	1	0	$S_{28}^{\mathrm{new}}(\Gamma_0^{(1)}(2))$
$648(8849 + 32\sqrt{18209})$	1	0	$S_{28}(SL_2(\mathbb{Z}))$
$648(8849 - 32\sqrt{18209})$	1	0	$S_{28}(SL_2(\mathbb{Z}))$

Here the case LHS = RHS = 2 is an interesting new example. This is the case $c_0 = 2$ in the notation of Sect. 3, where both eigenvalues of $T(3)$ are -4260600.

Concrete automorphic forms in each of the above two tables are newly obtained this time, so we give them here. We know that $\dim S_{11}(K(2)) = \dim S_{11}(B(2)) = 1$ and a nonzero form $\chi_{11} \in S_{11}(B(2))$ is explicitly given in [17] and [24] by theta constants (See also [1].) On the other hand, we put

$$F_4 = X^2 + 3Y + 3072Z + 960T$$

where X, Y, Z, T are Siegel modular forms of weight 2, 4, 4, 4 of $B(2)$ given explicitly in [17]. Then we have $F_4 \in S_4(K(2))$ ([24]). We denote by φ_4 the Siegel Eisenstein series of weight 4 of $Sp(2,\mathbb{Z})$ with constant term 1. Since $\dim S_{15}(\Gamma_0'(2)) = 2$ ([14]), we see that $\varphi_4\chi_{11}$ and $F_4\chi_{11}$ spans $S_{15}(\Gamma_0'(2))$ and $F_4\chi_{11} \in S_{15}(K(2))$. We have $Tr(W_{2,28}) = 0$ and $\dim S_{28}^{\mathrm{new}}(\Gamma_0^{(1)}(2)) = 2$, so we have $\dim S_{28}^{\mathrm{new},\pm}(\Gamma_0^{(1)}(2)) = 1$ for each $\pm$. So $F_4\chi_{11}$ should be the Gritsenko lift from $S_{28}^{\mathrm{new},-}(\Gamma_0^{(1)}(2))$. By using Fourier coefficients, we have

$$\begin{aligned}
&T(3)\phi_4\chi_{11} = 57729024F_4\chi_{11} - 4260600\phi_4\chi_{11},\\
&T(3)F_4\chi_{11} = 5360904F_4\chi_{11}.
\end{aligned}$$

So eigenforms are $F_4\chi_{11}$ and $\phi_4\chi_{11} - 6F_4\chi_{11}$ and we have the above table.

For the compact twist, we have $\dim \mathfrak{M}_{12,12}(\mathcal{U}_{pr}(2)) = 6$ and a basis is given as follows. For any $w \in \mathbb{H}$, we denote by w_1, w_2, w_3, w_4 the coefficients of $1, i, j, k$ of w, respectively, and $n(w) = w_1^2 + w_2^2 + w_3^2 + w_4^2$. For $x, y \in H$, we put $z = \bar{y}x$. For any positive integer i, we put $t_i = z_1^i + z_2^i + z_3^i + z_4^i$. Then a basis of $\mathfrak{M}_{12,12}(\mathcal{U}_{pr}(2))$ is given by the following 6 automorphic forms.

$$\begin{aligned}
P_{12a} &= (z_1^2 - z_2^2)(z_1^2 - z_3^2)(z_1^2 - z_4^2)(z_2^2 - z_3^2)(z_2^2 - z_4^2)(z_3^2 - z_4^2),\\
P_{12b} &= 226512n(x)^6n(y)^6 - 169884n(x)^5n(y)^5(n(x)^2 + n(y)^2)\\
&\quad + 70785n(x)^4n(y)^4(n(x)^4 + n(y)^4) - 15730n(x)^3n(y)^3(n(x)^6 + n(y)^6)\\
&\quad + 1716n(x)^2n(y)^2(n(x)^8 + n(y)^8) - 78n(x)n(y)(n(x)^{10} + n(y)^{10}) + n(x)^{12} + n(y)^{12},\\
P_{12c} &= 279(n(x)^{12} + n(y)^{12})/169380640 - 837n(x)n(y)(n(x)^{10} + n(y)^{10})/6514640\\
&\quad + 837n(x)^2n(y)^2(n(x)^{10} + n(y)^{10})/296120 - 1915n(x)^3n(y)^3(n(x)^6 + n(y)^6)/139984\\
&\quad + 6963n(x)^4n(y)^4(n(x)^4 + n(y)^4)/279968 - 137n(x)^5n(y)^5(n(x)^2 + n(y)^2)/5384\\
&\quad - 428n(x)n(y)(n(x)^6 + n(y)^6)t_4/8749 + 2568n(x)^2n(y)^2(n(x)^4 + n(y)^4)t_4/8749\\
&\quad - 428/673n(x)^3n(y)^3(n(x)^2 + n(y)^2)t_4 + n(x)^4n(y)^4t_4 + 1070(n(x)^4 + n(y)^4)t_4^2/8749\\
&\quad - 428/673n(x)n(y)(n(x)^2 + n(y)^2)t_4^2 + 330/673nx^2ny^2t_4^2 + 476t_4^3/2019\\
&\quad + 1712(n(x)^6 + n(y)^6)t_6/43745 - 5992n(x)n(y)(n(x)^4 + n(y)^4)t_6/43745\\
&\quad + 448/673n(x)n(y)t4t_6 - 12840(n(x)^4 + n(y)^4)t_8/61243\\
&\quad + 5136n(x)n(y)(n(x)^2 + n(y)^2)t_8)/4711 - 7656n(x)^2n(y)^2t_8/4711 - 336t_4t_8/673,\\
P_{12d} &= 181(n(x)^{12} + n(y)^{12})/215575360 - 543n(x)n(y)(n(x)^{10} + n(y)^{10})/8291360\\
&\quad + 543n(x)^2n(y)^2(n(x)^8 + n(y)^8)/376880 - 13661n(x)^3n(y)^3(n(x)^6 + n(y)^6)/1959776\\
&\quad - 35181n(x)^4n(y)^4(n(x)^4 + n(y)^4)/3919552 + 3753n(x)^5n(y)^5(n(x)^2 + n(y)^2)/37688\\
&\quad - 873n(x)n(y)(n(x)^6 + n(y)^6)t_4/34996 + 20571n(x)^2n(y)^2(n(x)^4 + n(y)^4)t_4/69992\\
&\quad - 723/673n(x)^3n(y)^3(n(x)^2 + n(y)^2)t_4 - 2365(n(x)^4 + n(y)^4)t_4^2/69992\\
&\quad + 473n(x)n(y)(n(x)^2 + n(y)^2)t_4^2/2692 + 4851n(x)^2n(y)^2t_4^2/1346\\
&\quad + 473n(x)n(y)(n(x)^2 + n(y)^2)t_4^2/2692 - 1122t_4^3/673 + 873(n(x)^6 + n(y)^6)t_6/43745\\
&\quad - (11468n(x)n(y)(n(x)^4 + n(y)^4)t_6/43745 + n(x)^2n(y)^2(n(x)^2 + n(y)^2)t_6\\
&\quad - 3168/673n(x)n(y)t_4t_6 + 7095(n(x)^4 + n(y)^4)t_8/122486\\
&\quad - 1419n(x)n(y)(n(x)^2 + n(y)^2)t_8/4711 - 2970n(x)^2n(y)^2t_8/4711 + 2376t_4t_8/673,\\
P_{12e} &= 1943(n(x)^{12} + n(y)^{12})/1552142592 - 1943n(x)n(y)(n(x)^{10} + n(y)^{10})/19899264\\
&\quad + 1943n(x)^2n(y)^2)(n(x)^8 + n(y)^8)/904512\\
&\quad - 473225n(x)^3n(y)^3(n(x)^6 + n(y)^6))/70551936\\
&\quad - 137455n(x)^4n(y)^4(n(x)^4 + n(y)^4)/15678208\\
&\quad + 17305n(x)^5n(y)^5(n(x)^2 + n(y)^2)/301504\\
&\quad - 3635n(x)n(y)(n(x)^6 + n(y)^6)t_4/69992 + 10905n(x)^2n(y)^2(n(x)^4 + n(y)^4)t_4/34996
\end{aligned}$$

$$
\begin{aligned}
&- 3635n(x)^3n(y)^3(n(x)^2 + n(y)^2)t_4/5384 + 18175(n(x)^4 + n(y)^4)t_4^2/139984\\
&- 3635n(x)n(y)(n(x)^2 + n(y)^2)t_4^2/5384 + 8265n(x)^2n(y)^2t_4^2/2692\\
&- 1700t_4^3/2019 + 727(n(x)^6 + n(y)^6)t_6/17498 - 5089n(x)n(y)(n(x)^4 + n(y)^4)t_6/34996\\
&+ n(x)^3n(y)^3t_6 - 5089n(x)n(y)(n(x)^4 + n(y)^4)t_6/34996\\
&- 1600/673n(x)n(y)t_4t_6 - 54525(n(x)^4 + n(y)^4)t_8/244972\\
&+ 10905n(x)n(y)(n(x)^2 + n(y)^2)t_8/9422 - 11595n(x)^2n(y)^2t_8/4711 + 1200t_4t_8/673),\\
P_{12f} &= 9557(n(x)^{12} + n(y)^{12})/34147137024 - 9557n(x)n(y)(n(x)^{10} + n(y)^{10})/437783808\\
&+ 9557n(x)^2n(y)^2(n(x)^8 + n(y)^8))/19899264\\
&- 204145n(x)^3n(y)^3(n(x)^6 + n(y)^6))/141103872\\
&- 73895n(x)^4n(y)^4(n(x)^4 + n(y)^4)/31356416\\
&+ 8401n(x)^5n(y)^5(n(x)^2 + n(y)^2)/603008\\
&- 1655n(x)n(y)(n(x)^6 + n(y)^6)t_4/139984 + 4965n(x)^2n(y)^2(n(x)^4 + n(y)^4)t_4/69992\\
&- 1655n(x)^3n(y)^3(n(x)^2 + n(y)^2)t_4/10768 + 8275(n(x)^4 + n(y)^4)t_4^2/279968\\
&- 1655n(x)n(y)(n(x)^2 + n(y)^2)t_4^2/10768 + 1215n(x)^2n(y)^2t_4^2/1346\\
&- 1655n(x)n(y)(n(x)^2 + n(y)^2)t_4^2/10768 + 8275(n(x)^4 + n(y)^4)t_4^2/279968\\
&+ 2465t_4^3/8076 + 331(n(x)^6 + n(y)^6)t_6/34996 - 2317n(x)n(y)(n(x)^4 + n(y)^4)t_6/69992\\
&- 2205n(x)n(y)t_4t_6/1346 + t_6^2 - 24825(n(x)^4 + n(y)^4)t_8/489944\\
&+ 4965n(x)n(y)(n(x)^2 + n(y)^2)t_8/18844 + 2175n(x)^2n(y)^2t_8/18844 - 435t_4t_8/673.
\end{aligned}
$$

Then the calculation of the Hecke operator of $T(3)$ on this basis is done as in [14] and the representation matrix is given by

$$
\begin{pmatrix}
-4260600 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{24471033464}{4719} & 6514848604160 & 5210296565760 & -9797033000960 & 1484944179200\\
0 & \frac{7065888393688}{524545376355} & \frac{4894275883592}{22231209} & -\frac{30934808064}{7403} & \frac{25469246095360}{22231209} & \frac{300405188472832}{111156045}\\
0 & \frac{749043686198}{15895314435} & -\frac{36811760958272}{673673} & -\frac{22903592184}{673} & \frac{41544307693568}{673673} & \frac{2182786574336}{3368365}\\
0 & \frac{1059206285026}{28611565983} & -\frac{234578046469280}{6063057} & -\frac{18952326720}{673} & \frac{283013189393288}{6063057} & \frac{10809831953920}{6063057}\\
0 & \frac{827764143017}{44961032259} & -\frac{186715877065360}{9527661} & -\frac{109169658720}{7403} & \frac{219950269849600}{9527661} & \frac{46482782141288}{9527661}
\end{pmatrix}.
$$

The eigenvalues of $T(3)$ are given by

$$
\begin{array}{ll}
-4260600 \text{ (2 -dimensional)}, & 10362120,\\
5360904, & 648(8849 \pm 32\sqrt{18209}).
\end{array}
$$

The eigenvalues at 3 of Hecke eigenforms of $S_{28}(SL_2(\mathbb{Z}))$ are given by $-643140 \pm 20736\sqrt{18209}$ and the eigenvalues of $T(3)$ of the associated Ihara lifts are given by

$$3^{13} + 3^{14} - 643140 \pm 20736\sqrt{18209} = 648(8849 \pm 32\sqrt{18209}).$$

Also the eivenvalues at 3 of the Hecke eigenforms in the two-dimensional space $S_{28}^{\mathrm{new}}(\Gamma_0^{(1)}(2))$ are given by -1016338 and 3984828. The eigenvalues of $T(3)$ of the associated Ihara lifts are 5360904 and 10362120. This fits the above data. So we explained all the lifts which appears in the table. The remaining two-dimensional eigenvalues -4260600 are non-lifts. The eigenvalue suggests that these should correspond with $S_{15}(\Gamma_0'(2))$ and $S_{15}(\Gamma_0''(2))$.

4.2 The vector-valued case for $p = 2$ and $p = 3$

In this subsection, we consider the case $(k, j) = (3, 6), (4, 6)$ for $p = 2$ and $(k, j) = (3, 8)$ for $p = 3$. These cases correspond to the cases $M_{6,0}(\mathcal{U}_{pr}(2))$, $\mathfrak{M}_{7,1}(\mathcal{U}_{pr}(2))$, and $M_{8,0}(\mathcal{U}_{pr}(3))$. These examples give good evidence for our Conjectures 3.2 on Ihara lifts, so we do not explain Siegel cusp forms here. Before giving concrete examples of automorphic forms, we explain the results. First we give a table of various dimensions.

(k, j)	$\mathfrak{M}_{6,0}(\mathcal{U}_{pr}(2))$ (3,6)	$\mathfrak{M}_{7,1}(\mathcal{U}_{pr}(2))$ (4,6)	$\mathfrak{M}_{8,0}(\mathcal{U}_{pr}(3))$ (3,8)
$\dim \mathfrak{M}$	1	1	3
$\dim S^{\mathrm{new}}_{j+2}(\Gamma_0^{(1)}(p))$	1+	1+	1+,1−
$\dim S^{\mathrm{new}}_{2k+j-2}(\Gamma_0^{(1)}(p))$	1−	0	1+
$\dim S_{2k+j-2}(SL_2(\mathbb{Z}))$	0	1	1

Here we put $k = \nu_2 + 3$, $j = \nu_1 - \nu_2$ for the column $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$. For $\Gamma_0^{(1)}(p)$, we put $p = 2$ for the first two columns and $p = 3$ for the last one, and $a+$, $b-$ means the dimension of the Atkin-Lehner plus (resp. minus) is a (resp. b). By concrete automorphic forms given below, we see the following results. The space $\mathfrak{M}_{6,0}(\mathcal{U}_{pr}(2))$ consists of the Ihara lift from $S_8^{\mathrm{new},+}(\Gamma_0^{(1)}(2)) \times S_{10}^{\mathrm{new},-}(\Gamma_0^{(1)}(2))$. The space $\mathfrak{M}_{7,1}(\mathcal{U}_{pr}(2))$ consists of the Ihara lift from $S_8^{\mathrm{new},+}(\Gamma_0^{(1)}(2)) \times S_{12}(SL_2(\mathbb{Z}))$. The space $\mathfrak{M}_{8,0}(\mathcal{U}_{pr}(3))$ consists of the Ihara lifts from $S_{10}^{\mathrm{new},-}(\Gamma_0^{(1)}(3)) \times S_{12}^{\mathrm{new},+}(\Gamma_0^{(1)}(3))$ and $S_{10}^{\mathrm{new}}(\Gamma_0^{(1)}(3)) \times S_{12}(SL_2(\mathbb{Z}))$. More precisely speaking, the Ihara lifts from $S_{10}^{\mathrm{new}}(\Gamma_0^{(1)}(3)) \times S_{12}(SL_2(\mathbb{Z}))$ consist of eigenforms from pairs of $(f_{10}^{+}, \Delta(\tau) - 3^6\Delta(3\tau))$ and $(f_{10}^{-}, \Delta(\tau) + 3^6\Delta(\tau))$, where $f_{10}^{\pm}$ are elements in $S_{10}(\Gamma_0^{(1)}(3))$ which have $\pm$ as the sign of the Atkin-Lehner involution. Here $\Delta(\tau) \pm 3^6\Delta(3\tau)$ are old forms in $S_{12}(\Gamma_0^{(1)}(3))$ with Atkin-Lehner plus and minus, so the signs of the pairs are opposite in this case too.

Now we give concrete automorphic forms of $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ in the above cases. For $\lambda \in \mathbb{H}$, we consider a polynomial $P(\lambda)$ in four variables identifying $H \cong \mathbb{R}^4$ by $\lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k$ with (λ_i). It is well known that any homogeneous polynomial $P(\lambda)$ is a direct sum of products of a power of $n(\lambda)$ and harmonic polynomials. We can explicitly give the harmonic projection Π_λ of this direct sum decomposition to the harmonic part by the following map.

$$\Pi_\lambda(P(\lambda)) = \sum_{0 \le b \le d/2} (-1)^b \frac{1}{2^{2b}(b!)^2\binom{d}{b}} n(\lambda)^b \Delta_\lambda^b(P),$$

where $\binom{d}{b}$ is the binomial coefficient and we put $d = \deg(P)$. (See [25]).

First we give automorphic forms when $p = 2$. The definite quaternion algebra with discriminant 2 is given by

$$D_{2,\infty} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$$

with $i^2 = j^2 = -1$, $ij = -ji = k$. The class number of $D_{2,\infty}$ is one, and any maximal order of $D_{2,\infty}$ is conjugate to

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2}.$$

We have

$$\mathcal{O}^\times = \left\{\pm 1, \pm i, \pm j, \pm k, \ \frac{\pm 1 \pm i \pm j \pm k}{2}\right\}.$$

The class number of the principal genus in $(D_{2,\infty})^2$ is one (See [12] I) and the automorphism group of $\mathcal{O}^2$ is given by

$$\Gamma = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} 0 & u_1 \\ u_2 & 0 \end{pmatrix}; u_1, u_2 \in \mathcal{O}^\times \right\}.$$

First we explain the case $(k, j) = (3, 6)$. For $x \in \mathbb{H}$, we put

$$\begin{aligned} h(x) &= 6(x_1^6 + x_2^6 + x_3^6 + x_4^6) - 5(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1^4 + x_2^4 + x_3^4 + x_4^4) \\ &\quad + 30(x_1^2x_2^2x_3^2 + x_1^2x_2^2x_4^2 + x_1^2x_3^2x_4^2 + x_2^2x_3^2x_4^2). \end{aligned}$$

This is harmonic with respect to $x \in \mathbb{H}$. For $(x, y) \in \mathbb{H}^2$, we put

$$F(x, y) = h(x) + h(y),$$

then we see that this is harmonic with respect to $(x, y) \in \mathbb{H}^2$ and invariant by Γ, and it is obvious that $f(\lambda x, \lambda y)$ is also harmonic for λ. So we have $f \in \mathfrak{M}_{6,0}(\mathcal{U}_{pr}(2))$. If we put

$$\vartheta_F(\tau) = \sum_{x,y \in \mathcal{O}} F(x, y) e^{2\pi i (n(x)+n(y))\tau},$$

then we see that this is nonzero and $\vartheta_F \in S_{10}^{\mathrm{new}}(\Gamma_0^{(1)}(2)) = S_{10}^{\mathrm{new},-}(\Gamma_0^{(1)}(2))$. As a function of λ, $F(\lambda x, \lambda y)$ corresponds with $S_8^{\mathrm{new},+}(\Gamma_0^{(1)}(2))$ and this gives the Ihara lift from $S_8^{\mathrm{new},+}(\Gamma_0^{(1)}(2)) \times S_{10}^{\mathrm{new},-}(\Gamma_0^{(1)}(2))$.

Next we explain the case $(k, j) = (4, 6)$. We put

$$\begin{aligned} A(x, y) &= x_1^6 - 15x_1^4y_1^2 + 15x_1^2y_1^4 - y_1^6, \\ B(x, y) &= 3x_1^5y_1 - 10x_1^3y_1^3 + 3x_1y_1^5, \\ f(x, y) &= (n(x) - n(y))A(x, y) - 4(x, y)B(x, y), \\ h(x, y, \lambda) &= (n(x) - n(y))A(\lambda x, \lambda y) - 4(x, y)B(\lambda x, \lambda y), \\ \phi(x, y, \lambda) &= \Pi_\lambda(h(x, y, \lambda)). \end{aligned}$$

Here if we put $F(x, y) = \phi(x, y, 1)$, then we see that F is harmonic w.r.t. (x, y), and $F(\lambda x, \lambda y) = n(\lambda)\phi(x, y, \lambda)$. (This can be proved directly or by an easy abstract calculation coming from the shape of Π_λ and h). This $F(x, y)$ is not invariant by Γ, so we must take an average over Γ. We see that this average does not vanish and give a nonzero element of $\mathfrak{M}_{7,1}(\mathcal{U}_{pr}(2))$. We also write this average by F. By calculating ϑ_F as before, up to constant we have

$$\vartheta_F = \Delta(\tau) - 64\Delta(2\tau).$$

By calculating the action of the Hecke operators on λ, we see that $\phi(x, y, \lambda)$ corresponds with an element in $S_8^{\mathrm{new},+}(\Gamma_0^{(1)}(2)) = S_8^{\mathrm{new}}(\Gamma_0^{(1)}(2))$. So this gives a lift from $S_8^{\mathrm{new}}(\Gamma_0^{(1)}(2)) \times S_{12}(SL_2(\mathbb{Z}))$.

Next we give an example for $(k, j) = (3, 8)$ and $p = 3$. The definite quaternion algebra which ramifies exactly at 3 and ∞ is given by

$$D_{3,\infty} = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\beta + \mathbb{Q}\alpha\beta$$

with $\alpha^2 = -3$, $\beta^2 = -1$, $\alpha\beta = -\beta\alpha$. So in $D_{3,\infty} \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$, we may regard $\beta = i$, $\alpha = \sqrt{3}j$, $\alpha\beta = \sqrt{3}k$. The class number of $D_{3,\infty}$ is one, and any maximal order of $D_{3,\infty}$ is conjugate to

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\beta + \mathbb{Z}\frac{1+\alpha}{2} + \mathbb{Z}\frac{(1+\alpha)\beta}{2}.$$

As before we identify $\mathbb{H} \cong \mathbb{R}^4$ and for $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$, we define

$$\begin{aligned} h_1(x) &= 54x_1^2x_2^2x_3^2x_4^2 - 3(x_1^4x_2^2x_3^2 + x_1^2x_2^4x_3^2 + x_1^2x_2^2x_3^4 + x_2^4x_3^2x_4^2 \\ &\quad + x_2^2x_3^4x_4^2 + x_2^2x_3^2x_4^4 + x_1^4x_3^2x_4^2 + x_1^2x_3^4x_4^2 + x_1^2x_3^2x_4^4 + x_1^4x_2^2x_4^2 \\ &\quad + x_1^2x_2^4x_4^2 + x_1^2x_2^2x_4^4) + (x_1^4x_2^4 + x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4 + x_3^4x_4^4), \\ h_2(x) &= x_1^8 - 28x_1^6x_2^2 + 70x_1^4x_2^4 - 28x_1^2x_2^6 + x_2^8, \\ h_3(x,y) &= x_1^8 - 28x_1^6y_1^2 + 70x_1^4y_1^4 - 28x_1^2y_1^6 + y_1^8, \\ \phi(x,y,\lambda) &= \Pi_\lambda(h_3(\lambda x, \lambda y)). \end{aligned}$$

The polynomials h_1, h_2 are harmonic with respect to x, so $h_i(\lambda x)$ for $i = 1, 2$ are also harmonic with respect to λ. The polynomial $h_3(x,y)$ is harmonic with respect to $(x,y) \in \mathbb{H}^2$ and $\phi(x,y,\lambda)$ is harmonic with respect to (x,y) and λ. We also have $\phi(\lambda x, \lambda y, 1) = \phi(x,y,1)$. We put

$$\begin{aligned} G_1(x,y) &= h_1(x) + h_1(y), \\ G_2(x,y) &= h_2(x) + h_2(y), \\ G_3(x,y) &= \phi(x,y,1). \end{aligned}$$

The class number of the principal genus in $(D_{3,\infty})^2$ is one ([12] I), but these functions G_i are not invariant by $\Gamma = GL_2(O) \cap G$, and we take the average

$$F_i(x,y) = \sum_{\gamma \in \Gamma} G_i((x,y)\gamma) \qquad (i = 1, 2, 3).$$

Then we see that these are linearly independent and form a basis of $\mathfrak{M}_{8,0}(\mathcal{U}_{pr}(3))$. If we define

$$\begin{aligned} P_1(x,y) &= 84F_1(x,y) - 4F_2(x,y) + 64F_3(x,y), \\ P_2(x,y) &= 5F_1(x,y) + 3F_2(x,y) - 48F_3(x,y), \\ P_3(x,y) &= -70F_1(x,y) + 6F_2(x,y), \end{aligned}$$

then these are Hecke eigenforms, and the associated theta functions are given up to constant by

$$\begin{aligned} \chi_{12}(\tau) &= q + 78q^2 - 243q^3 + 4036q^4 + \cdots, \\ \Delta(\tau) + 3^6\Delta(3\tau) &= q - 24q^2 + 981q^3 - 1472q^4 + \cdots, \\ \Delta(\tau) - 3^6\Delta(3\tau) &= q - 24q^2 - 477q^3 - 1472q^4, \end{aligned}$$

where $S_{12}^{\mathrm{new}}(\Gamma_0^{(1)}(3)) = S_{12}^{\mathrm{new},+}(\Gamma_0^{(1)}(3)) = \mathbb{C}\chi_{12}$. On the other hand, a basis of each $S_{10}^{\mathrm{new},\pm}(\Gamma_0^{(1)}(3))$ is given by

$$\begin{aligned} \chi_{10+} &= q - 36q^2 - 81q^3 + 784q^4 + \cdots, \\ \chi_{10-} &= q + 18q^2 + 81q^3 - 188q^4 + \cdots. \end{aligned}$$

So calculating the Hecke action on the λ part of P_i, we see that P_1, P_2, and P_3 are lifts from pairs $(\chi_{10-},\ \chi_{12})$, $(\chi_{10-},\ \Delta(\tau) + 3^6\Delta(3\tau))$, and $(\chi_{10+},\ \Delta(\tau) - 3^6\Delta(3\tau))$. So this gives evidence of Conjecture 3.2 (2) and (3).

4.3 The scalar-valued case for $p = 3$: comparison with Ihara's examples

In this section, we write the same sort of example as the one given in Sect. 4.1. Here we treat the case $p = 3$. Ihara calculated in [27] all the Hecke eigenforms of $\mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(3))$ for $k \leq 12$ and $k = 14$, giving eigenvalues for $T(2)$, $T(4)$, determining Euler 2 factors of the spinor L function. The smallest k where non-lift appears is $k = 11$. For $k = 11$, we have $\dim \mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3)) = 6$. He observed that in this space $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$, there are four lifts of Ihara type, coming from the three-dimensional space $S_{20}^{\mathrm{new}}(\Gamma_0^{(1)}(3))$ and one-dimensional space $S_{20}(SL_2(\mathbb{Z}))$. These cancel with $-(\dim S_{20}^{\mathrm{new}}(\Gamma_0^{(1)}(3)) + \dim S_{20}(SL_2(\mathbb{Z})))$ of the RHS of Theorem 3.1. The other two linearly independent eigenforms of $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$ are non-lifts, having the same Euler 2 factors given by

$$\begin{aligned}&(1 - 12(-9 + \sqrt{1489})2^{-s} + 2^{19-2s})(1 - 12(-9 - \sqrt{1489})2^{-s} + 2^{19-2s})\\ &\quad = 1 + 216 \cdot 2^{-s} + 845824 \cdot 2^{-2s} + 216 \cdot 2^{19-3s} + 2^{38} \cdot 2^{38-4s}.\end{aligned}$$

These two non-lifts come apparently from very different construction and it seems as if the multiplicity one breaks here, though this is not known. Anyway, we will compare these two non-lift eigenforms with one of Siegel Hecke eigenforms of weight 11 of $\Gamma_0'(3)$ and see that the Euler 2-factors coincide completely. So this suggests that two forms in $S_{11}(\Gamma_0'(3)) \cong S_{11}(\Gamma_0''(3))$ (one for each) correspond with two non-lifts of $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$. We explain the part for Siegel cusp forms here. We know that $\dim S_{11}(\Gamma_0(3)) = 0$, $\dim S_{11}(K(3)) = 1$, $\dim S_{11}(\Gamma_0'(3)) = 2$. (See [11,16] and the last section of this paper.) Here we have a lift by Gritsenko [9] from $J_{11,3}(SL_2(\mathbb{Z}))$ (one dimensional) to $S_{11}(K(3)) \subset S_{11}(\Gamma_0'(3))$. The spinor L function of this lift is (up to Euler 3 factor), equal to

$$\zeta(s-9)\zeta(s-10)L(s,f),$$

where $f \in S_{20}^{-}(\Gamma_0^{(1)}(3)) \cong J_{11,3}(SL_2(\mathbb{Z}))$, and the Euler 2 factor is

$$(1 - 2^{9-s})(1 - 2^{10-s})(1 + 1104 \cdot 2^{-s} + 2^{19-2s}).$$

These lifts cancel in $S_{11}(\Gamma_0'(3)) + S_{11}(\Gamma_0''(3)) - 2S_{11}(K(3))$ in the LHS and no contribution to the RHS of Theorem 3.1 (though there is the Ihara lift in $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$ corresponding to this Gritsenko lift). So if we take the other Hecke eigenforms in $S_{11}(\Gamma_0'(3))$ and $S_{11}(\Gamma_0''(3))$, we can expect that these correspond with two non-lifts in $\mathfrak{M}_{8,8}(\mathcal{U}_{pr}(3))$. We will see in this section that the Euler 2 factors of these forms coincide completely.

More concrete description of the non-lifted form in $S_{11}(\Gamma_0'(3))$ will be given in the following way. We denote by $\Gamma(3)$ the principal congruence subgroup of level 3, which is by definition

$$\Gamma(3) = \{\gamma \in Sp(2,\mathbb{Z}) \subset M_4(\mathbb{Z}) : \gamma \equiv 1_4 \bmod 3\}.$$

The graded ring $A(\Gamma(3))$ of Siegel modular forms of integral weight of $\Gamma(3)$ has been determined by Freitag and Salvati Manni [8]. They are generated by 5 forms A_i of weight 1, and 5 forms C_i of weight 3 with $1 \leq i \leq 5$ with 20 relations. The action of $Sp(2,\mathbb{Z})/\Gamma(3)$ on $(A_1, \ldots, A_5)$ and $(C_1, \ldots, C_5)$ are explicitly known (at least for generators of $Sp(2,\mathbb{Z})/\Gamma(3)$). Of course we have $\Gamma(3) \subset \Gamma_0'(3)$, so in principle we can obtain forms in $S_k(\Gamma_0'(3))$ by taking invariant forms in $S_k(\Gamma(3))$ by this action. Professor Hidetaka Kitayama kindly did this calculation for $k = 11$, responding to author's request, giving also the general dimension formula for $S_k(\Gamma_0'(3))$ for general k by calculating the invariant part. According to him, we have $S_{11}(\Gamma_0'(3)) = \mathbb{C}F_{11,a} + \mathbb{C}F_{11,b}$, where

$$\begin{aligned}
F_{11,a} = &(A_1^3A_3^3A_4^2C_4 - A_1^3A_3^3A_5^2C_5 - A_1^3A_3^2A_4^3C_3 + A_1^3A_3^2A_5^3C_3 + A_1^3A_4^3A_5^2C_5 \\
&- A_1^3A_4^2A_5^3C_4 - 3A_1^2A_2A_3^4A_4C_5 + 3A_1^2A_2A_3^4A_5C_4 + 3A_1^2A_2A_3A_4^4C_5 \\
&- 3A_1^2A_2A_3A_5^4C_4 - 3A_1^2A_2A_4^4A_5C_3 + 3A_1^2A_2A_4A_5^4C_3 - 4A_2^3A_3^3A_4^2C_4 \\
&+ 4A_2^3A_3^3A_5^2C_5 + 4A_2^3A_3^2A_4^3C_3 - 4A_2^3A_3^2A_5^3C_3 - 4A_2^3A_4^3A_5^2C_5 \\
&+ 4A_2^3A_4^2A_5^3C_4)/2592, \\
F_{11,b} = &(A_3^6A_4^2C_4 - A_3^6A_5^2C_5 + 2A_3^5A_4^3C_3 - 2A_3^5A_5^3C_3 - 2A_3^3A_4^5C_4 + 2A_3^3A_5^5C_5 \\
&- A_3^2A_4^6C_3 + A_3^2A_5^6C_3 + A_4^6A_5^2C_5 + 2A_4^5A_5^3C_4 - 2A_4^3A_5^5C_5 - A_4^2A_5^6C_4)/2592.
\end{aligned}$$

Here the forms A_i are defined by

$$A_i = \sum_{G \in M_2(\mathbb{Z})} \exp(\pi i Tr(S[G + P_i/3]Z)),$$

where $S = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $S[X] = {}^tXSX$ and

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ P_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ P_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\ P_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ P_5 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The definitions of C_i are complicated. If we use the linear relation in [8] Proposition 10, then C_i are given by a rational function of A_i and the denominator is $X_{10} = 2592\chi_{10} \in S_{10}(Sp(2,\mathbb{Z}))$ where χ_{10} is the unique cusp form of weight 10 with Fourier coefficient 1 at $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Apparently these C_i have a denominator in this expression, but actually they are all holomorphic functions. Since we do not need C_1 and C_2, we write down only C_3, C_4, C_5 for the completeness.

$$\begin{aligned}
X_{10}C_3 = &-48(-A_1^6A_2^3A_3^4 + 5A_1^3A_2^6A_3^4 - 4A_2^9A_3^4 + A_1^3A_2^3A_3^7 + 4A_2^6A_3^7 + 7A_1^6A_2^3A_3A_4^3 \\
&- 11A_1^3A_2^6A_3A_4^3 + 4A_2^9A_3A_4^3 - A_1^6A_3^4A_4^3 + 10A_1^3A_2^3A_3^4A_4^3 + 4A_2^6A_3^4A_4^3 + A_1^3A_3^7A_4^3 \\
&- 8A_2^3A_3^7A_4^3 - 11A_1^3A_2^3A_3A_4^6 - 8A_2^6A_3A_4^6 + 5A_1^3A_3^4A_4^6 + 4A_2^3A_3^4A_4^6 + 4A_3^7A_4^6 \\
&+ 4A_2^3A_3A_4^9 - 4A_3^4A_4^9 + A_1^8A_2A_3^2A_4A_5 - 11A_1^5A_2^4A_3^2A_4A_5 + 10A_1^2A_2^7A_3^2A_4A_5 \\
&+ A_1^5A_2A_3^5A_4A_5 - 34A_1^2A_2^4A_3^5A_4A_5 - 2A_1^2A_2A_3^8A_4A_5 - 11A_1^5A_2A_3^2A_4^4A_5 \\
&+ 50A_1^2A_2^4A_3^2A_4^4A_5 - 34A_1^2A_2A_3^5A_4^4A_5 + 10A_1^2A_2A_3^2A_4^7A_5 - 8A_1^7A_2^2A_4^2A_5^2 \\
&+ 16A_1^4A_2^5A_4^2A_5^2 - 8A_1A_2^8A_4^2A_5^2 + 25A_1^4A_2^2A_3^3A_4^2A_5^2 - 28A_1A_2^5A_3^3A_4^2A_5^2 \\
&+ 64A_1A_2^2A_3^6A_4^2A_5^2 + 16A_1^4A_2^2A_4^5A_5^2 + 8A_1A_2^5A_4^5A_5^2 - 28A_1A_2^2A_3^3A_4^5A_5^2 \\
&- 8A_1A_2^2A_4^8A_5^2 + 7A_1^6A_2^3A_3A_5^3 - 11A_1^3A_2^6A_3A_5^3 + 4A_2^9A_3A_5^3 - A_1^6A_3^4A_5^3 \\
&+ 10A_1^3A_2^3A_3^4A_5^3 + 4A_2^6A_3^4A_5^3 + A_1^3A_3^7A_5^3 - 8A_2^3A_3^7A_5^3 + 7A_1^6A_3A_4^3A_5^3 \\
&- 90A_1^3A_2^3A_3A_4^3A_5^3 + 10A_1^3A_3^4A_4^3A_5^3 - 8A_3^7A_4^3A_5^3 - 11A_1^3A_3A_4^6A_5^3 \\
&+ 4A_3^4A_4^6A_5^3 + 4A_3A_4^9A_5^3 - 11A_1^5A_2A_3^2A_4A_5^4 + 50A_1^2A_2^4A_3^2A_4A_5^4 - 34A_1^2A_2A_3^5A_4A_5^4 \\
&+ 50A_1^2A_2A_3^2A_4^4A_5^4 + 16A_1^4A_2^2A_4^2A_5^5 + 8A_1A_2^5A_4^2A_5^5 - 28A_1A_2^2A_3^3A_4^2A_5^5 \\
&+ 8A_1A_2^2A_4^5A_5^5 - 11A_1^3A_2^3A_3A_5^6 - 8A_2^6A_3A_5^6 + 5A_1^3A_3^4A_5^6 + 4A_2^3A_3^4A_5^6 + 4A_3^7A_5^6 \\
&- 11A_1^3A_3A_4^3A_5^6 + 4A_3^4A_4^3A_5^6 - 8A_3A_4^6A_5^6 \\
&+ 10A_1^2A_2A_3^2A_4A_5^7 - 8A_1A_2^2A_4^2A_5^8 + 4A_2^3A_3A_5^9 - 4A_3^4A_5^9 + 4A_3A_4^3A_5^9), \\
X_{10}C_4 = &-48(7A_1^6A_2^3A_3^3A_4 - 11A_1^3A_2^6A_3^3A_4 + 4A_2^9A_3^3A_4 - 11A_1^3A_2^3A_3^6A_4 - 8A_2^6A_3^6A_4 \\
&+ 4A_2^3A_3^9A_4 - A_1^6A_2^3A_4^4 + 5A_1^3A_2^6A_4^4 - 4A_2^9A_4^4 - A_1^6A_3^3A_4^4 \\
&+ 10A_1^3A_2^3A_3^3A_4^4 + 4A_2^6A_3^3A_4^4 + 5A_1^3A_3^6A_4^4
\end{aligned}$$

$$
\begin{aligned}
&+ 4A_2^3A_3^6A_4^4 - 4A_3^9A_4^4 + A_1^3A_2^3A_4^7 + 4A_2^6A_4^7 + A_1^3A_3^3A_4^7 - 8A_2^3A_3^3A_4^7 + 4A_3^6A_4^7\\
&+ A_1^8A_2A_3A_4^2A_5 - 11A_1^5A_2^4A_3A_4^2A_5 + 10A_1^2A_2^7A_3A_4^2A_5 - 11A_1^5A_2A_3^4A_4^2A_5\\
&+ 50A_1^2A_2^4A_3^4A_4^2A_5 + 10A_1^2A_2A_3^7A_4^2A_5 + A_1^5A_2A_3A_4^5A_5 - 34A_1^2A_2^4A_3A_4^5A_5\\
&- 34A_1^2A_2A_3^4A_4^5A_5 - 2A_1^2A_2A_3A_4^8A_5 - 8A_1^7A_2^2A_3^2A_5^2 + 16A_1^4A_2^5A_3^2A_5^2\\
&- 8A_1A_2^8A_3^2A_5^2 + 16A_1^4A_2^2A_3^5A_5^2 + 8A_1A_2^5A_3^5A_5^2 - 8A_1A_2^2A_3^8A_5^2\\
&+ 25A_1^4A_2^2A_3^2A_4^3A_5^2 - 28A_1A_2^5A_3^2A_4^3A_5^2 - 28A_1A_2^2A_3^5A_4^3A_5^2 + 64A_1A_2^2A_3^2A_4^6A_5^2\\
&+ 7A_1^6A_2^3A_4A_5^3 - 11A_1^3A_2^6A_4A_5^3 + 4A_2^9A_4A_5^3 + 7A_1^6A_3^3A_4A_5^3 - 90A_1^3A_2^3A_3^3A_4A_5^3\\
&- 11A_1^3A_3^6A_4A_5^3 + 4A_3^9A_4A_5^3 - A_1^6A_4^4A_5^3 + 10A_1^3A_2^3A_4^4A_5^3 + 4A_2^6A_4^4A_5^3\\
&+ 10A_1^3A_3^3A_4^4A_5^3 + 4A_3^6A_4^4A_5^3 + A_1^3A_4^7A_5^3 - 8A_2^3A_4^7A_5^3 - 8A_3^3A_4^7A_5^3 - 11A_1^5A_2A_3A_4^2A_5^4\\
&+ 50A_1^2A_2^4A_3A_4^2A_5^4 + 50A_1^2A_2A_3^4A_4^2A_5^4 - 34A_1^2A_2A_3A_4^5A_5^4 + 16A_1^4A_2^2A_3^2A_5^5\\
&+ 8A_1A_2^5A_3^2A_5^5 + 8A_1A_2^2A_3^5A_5^5 - 28A_1A_2^2A_3^2A_4^3A_5^5 - 11A_1^3A_2^3A_4A_5^6 - 8A_2^6A_4A_5^6\\
&- 11A_1^3A_3^3A_4A_5^6 - 8A_3^6A_4A_5^6 + 5A_1^3A_4^4A_5^6 + 4A_2^3A_4^4A_5^6 + 4A_3^3A_4^4A_5^6\\
&+ 4A_4^7A_5^6 + 10A_1^2A_2A_3A_4^2A_5^7 - 8A_1A_2^2A_3^2A_5^8 + 4A_2^3A_4A_5^9 + 4A_3^3A_4A_5^9 - 4A_4^4A_5^9),
\end{aligned}
$$

$$
\begin{aligned}
X_{10}C_5 = &- 48(-8A_1^7A_2^2A_3^2A_4^2 + 16A_1^4A_2^5A_3^2A_4^2\\
&- 8A_1A_2^8A_3^2A_4^2 + 16A_1^4A_2^2A_3^5A_4^2 + 8A_1A_2^5A_3^5A_4^2 - 8A_1A_2^2A_3^8A_4^2\\
&+ 16A_1^4A_2^2A_3^2A_4^5 + 8A_1A_2^5A_3^2A_4^5 + 8A_1A_2^2A_3^5A_4^5 - 8A_1A_2^2A_3^2A_4^8 + 7A_1^6A_2^3A_3^3A_5\\
&- 11A_1^3A_2^6A_3^3A_5 + 4A_2^9A_3^3A_5 - 11A_1^3A_2^3A_3^6A_5 - 8A_2^6A_3^6A_5 + 4A_2^3A_3^9A_5 + 7A_1^6A_2^3A_4^3A_5\\
&- 11A_1^3A_2^6A_4^3A_5 + 4A_2^9A_4^3A_5 + 7A_1^6A_3^3A_4^3A_5 - 90A_1^3A_2^3A_3^3A_4^3A_5 - 11A_1^3A_3^6A_4^3A_5\\
&+ 4A_3^9A_4^3A_5 - 11A_1^3A_2^3A_4^6A_5 - 8A_2^6A_4^6A_5 - 11A_1^3A_3^3A_4^6A_5 - 8A_3^6A_4^6A_5 + 4A_2^3A_4^9A_5\\
&+ 4A_3^3A_4^9A_5 + A_1^8A_2A_3A_4A_5^2 - 11A_1^5A_2^4A_3A_4A_5^2 + 10A_1^2A_2^7A_3A_4A_5^2 - 11A_1^5A_2A_3^4A_4A_5^2\\
&+ 50A_1^2A_2^4A_3^4A_4A_5^2 + 10A_1^2A_2A_3^7A_4A_5^2 - 11A_1^5A_2A_3A_4^4A_5^2 + 50A_1^2A_2^4A_3A_4^4A_5^2\\
&+ 50A_1^2A_2A_3^4A_4^4A_5^2 + 10A_1^2A_2A_3A_4^7A_5^2 + 25A_1^4A_2^2A_3^2A_4^2A_5^3 - 28A_1A_2^5A_3^2A_4^2A_5^3\\
&- 28A_1A_2^2A_3^5A_4^2A_5^3 - 28A_1A_2^2A_3^2A_4^5A_5^3 - A_1^6A_2^3A_5^4 + 5A_1^3A_2^6A_5^4 - 4A_2^9A_5^4 - A_1^6A_3^3A_5^4\\
&+ 10A_1^3A_2^3A_3^3A_5^4 + 4A_2^6A_3^3A_5^4 + 5A_1^3A_3^6A_5^4 + 4A_2^3A_3^6A_5^4 - 4A_3^9A_5^4 - A_1^6A_4^3A_5^4\\
&+ 10A_1^3A_2^3A_4^3A_5^4 + 4A_2^6A_4^3A_5^4 + 10A_1^3A_3^3A_4^3A_5^4 + 4A_3^6A_4^3A_5^4 + 5A_1^3A_4^6A_5^4\\
&+ 4A_2^3A_4^6A_5^4 + 4A_3^3A_4^6A_5^4 - 4A_4^9A_5^4 + A_1^5A_2A_3A_4A_5^5 - 34A_1^2A_2^4A_3A_4A_5^5\\
&- 34A_1^2A_2A_3^4A_4A_5^5 - 34A_1^2A_2A_3A_4^4A_5^5 + 64A_1A_2^2A_3^2A_4^2A_5^6 + A_1^3A_2^3A_5^7 + 4A_2^6A_5^7\\
&+ A_1^3A_3^3A_5^7 - 8A_2^3A_3^3A_5^7 + 4A_3^6A_5^7 + A_1^3A_4^3A_5^7 - 8A_2^3A_4^3A_5^7 - 8A_3^3A_4^3A_5^7 + 4A_4^6A_5^7\\
&- 2A_1^2A_2A_3A_4A_5^8).
\end{aligned}
$$

We denote by $f(Z) = \sum_T A(T)e^{2\pi i Tr(TZ)}$ the Fourier expansion of a Hecke eigenform $f \in S_k(\Gamma_0'(3))$ and for any positive integer n prime to 3, we define a double coset $T(n)$ by

$$T(n) = \bigcup_{g\in M_4(\mathbb{Z}),\ {}^t gJg=nJ} \Gamma_0'(3)g\Gamma_0'(3).$$

The action of $T(n)$ on Siegel modular forms is defined as usual (See [7] for example). We review how to calculate the action of $T(2)$ from the Fourier coefficients. For a half-integral matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ and a Siegel modular form f, we write Fourier coefficients of f at T by $A(T) = A(f; a, c, b) = A(a, c, b)$. We note here that $A(a, c, b)$ and $A(c, a, b)$ might be different. Then we have

$$
\begin{aligned}
A(T(2)f; (2, 1, 1)) &= A(4, 2, 2) + 2^{k-2}(A(1, 2, 1) + A(4, 2, -5))\\
&= A(4, 2, 2) + 2^{k-2}(A(1, 2, 1) + A(4, 1, 3)),\\
A(T(4)f; (2, 1, 1)) &= A(8, 4, 4) + 2^{k-2}(A(2, 4, 2) + A(8, 4, -10))
\end{aligned}
$$

$$\begin{aligned}
&\quad + 2^{2(k-2)}(A(2,4,-5)+A(11,4,13))\\
&= A(8,4,4)+2^{k-2}(A(2,4,2)+A(8,2,6))\\
&\quad + 2^{2(k-2)}(A(2,1,-1)+A(8,1,5))\\
&= A(8,4,4)+2^{k-2}(A(2,4,2)+A(8,2,6))\\
&\quad + 2^{2k-3}A(2,1,-1),\\
A(T(2)f;(3,1,1)) &= A(6,2,2),\\
A(T(4)f;(3,1,1)) &= A(12,4,4).
\end{aligned}$$

By computer calculation, we have the following table of the Fourier coefficients of $F_{11,a}$ and $F_{11,b}$.

(a,b,c)	$F_{11,a}/2592$	$F_{11,b}/2592$
(1, 2, 1)	0	0
(2, 1, 1)	− 314928	0
(2, 1, − 1)	314928	0
(2, 4, 2)	0	0
(4, 1, 3)	0	0
(4, 2, − 5)	0	0
(4, 2, 2)	2×34012224	0
(8, 1, 5)	314928	0
(8, 2, 6)	0	0
(8, 4, − 10)	0	0
(8, 4, 4)	537024×314928	0
(3, 1, 1)	− 3936600	− 1771470
(6, 2, 2)	-314928×7020	− 765275040
(3, 1, 2)	− 393660	− 177147
(6, 2, 4)	-314928×62	14171760

So we have

$$\begin{aligned}
&A(T(2)F_{11,a},(2,1,1)) = 170061120,\\
&A(T(2)F_{11,b},(2,1,1)) = 0,\\
&A(T(2)F_{11,a},(3,1,1)) = -314928\times 7020,\\
&A(T(2)F_{11,b},(3,1,1)) = -765275040,
\end{aligned}$$

and

$$\begin{pmatrix} T(2)F_{11,a}\\ T(2)F_{11,b}\end{pmatrix} = \begin{pmatrix} -216 & 1728\\ 0 & 432\end{pmatrix}\begin{pmatrix} F_{11,a}\\ F_{11,b}\end{pmatrix}.$$

So Hecke eigenforms are given by $f_{11,a} = -3F_{11,a} + 8F_{11,b}$ and $f_{11,b} = F_{11,b}$. We denote by $\lambda(f,n)$ the Hecke eigenvalue of f at $T(n)$ for $(n,3)=1$. By the above Fourier coefficients, we easily have

$$\begin{aligned}
&\lambda(f_{11,a},2) = -216,\\
&\lambda(f_{11.a},4) = -1061312,\\
&\lambda(f_{11,b},2) = 432,
\end{aligned}$$

Here $f_{11,b}$ corresponds with the Gritsenko lift from $J_{11,3}(SL_2(\mathbb{Z}))$. More concretely, the space $\oplus_{k=0}^{\infty} A_k(\Gamma_0^{(1)}(3))$ is generated by forms g_2, g_4, χ_6 of weight 2, 4, 6, respectively, where $g_2 = g_1^2$, $g_1 = \sum_{m,n\in\mathbb{Z}} q^{n^2+3nm+3m^2}$ with $q = e^{2\pi i\tau}$ ($\tau \in H_1$), $g_4 = E_4$ where E_k is the normalized Eisenstein series of weight k of $SL_2(\mathbb{Z})$ with constant term 1, and χ_6 is a cusp form defined by $\chi_6 = (2E_6 - 9g_2^3 + 7g_2E_4)/432$. The ideal of relations is generated by $1728\chi_6 = -9g_2^4 + 10g_2^2 g_4 - g_4^2$ (See [17]). The eigenform corresponding to $f_{11,b}$ is given by $-(g_4 - 5g_2^2)g_2^5\chi_6 + 1128g_2^2(g_4 - 5g_2^2)\chi_6^2$, where the eigenvalue at 2 is -1104. Here the signs of the Atkin-Lehner involution are all minus for g_2, $g_4 - 5g_2^2$ and χ_6. On the other hand, the Euler 2 factor $L_2(s, f_{11,a})$ of $f_{11,a}$ is given by

$$(1 - 12(-9 + \sqrt{1489})2^{-s} + 2^{19-2s})(1 - 12(-9 - \sqrt{1489})2^{-s} + 2^{19-2s}),$$

which is exactly the same as Ihara's example.

4.4 The case $p = 11$

The example in this subsection is essentially extracted from thesis [29] of Löschel, though a new observation on Atkin-Lehner parity and calculation are added here. We have $\dim \mathfrak{M}_{0,0}(\mathcal{U}_{pr}(11)) = 5$ and $\dim \mathfrak{M}_{1,1}(\mathcal{U}_{pr}(11)) = 1$ (See [12,13] p. 51). On the other hand, we have the following table of dimensions of Siegel cusp forms ([20]).

k	$\dim S_4(\Gamma_0(11))$	$\dim S_4(\Gamma_0'(11))$	$\dim S_4(\Gamma_0''(11))$	$\dim S_4(K(11))$
3	0	0	0	0
4	7	1	1	1

So for $k = 3$, the LHS of Theorem 3.1 is 0. For weight $0 = k - 3$, we have exceptionally the constant function in $\mathfrak{M}_{0,0}(\mathcal{U}_{pr}(11))$, which corresponds with the lift from Eisenstein series E_4 of weight 4. The remaining four forms are all Ihara lifts, two of which are of Yoshita type from

$$S_2^{\mathrm{new}}(\Gamma_0^{(1)}(11)) \times S_4^{\mathrm{new}}(\Gamma_0^{(1)}(11)) = S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) \times S_4^{\mathrm{new},+}(\Gamma_0^{(1)}(11)),$$

and two of which are of Saito–Kurokawa type from $S_4^{\mathrm{new}}(\Gamma_0^{(1)}(11))$. This can be seen by using Löschel's description of explicit lattices and ideal classes in [29] and using the theory of Ihara lifts in [22,27]. More precisely, this can be shown as follows. The definite quaternion algebra $D_{11,\infty}$ of discriminant 11 is given by $\mathbb{Q}+\mathbb{Q}\alpha+\mathbb{Q}\beta+\alpha\beta$ with $\alpha^2 = -11$, $\beta^2 = -1$, $\alpha\beta = -\beta\alpha$. The class number and the type number of $D_{11,\infty}$ are two. One of maximal orders is given by

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\beta + \mathbb{Z}\frac{1+\alpha}{2} + \mathbb{Z}\frac{(1+\alpha)\beta}{2},$$

and a right $\mathcal{O}$ ideal class different from $\mathcal{O}$ is represented by

$$J = 2\mathbb{Z} + \mathbb{Z}(1+\beta) + \mathbb{Z}\frac{1+\alpha\beta}{2} + \mathbb{Z}\frac{1+3\beta+\alpha+\alpha\beta}{4}.$$

(See [29]). Representatives of classes in the principal genus in this case are also given in [29] by

$$L_\kappa = O^2 h_\kappa, \qquad h_\kappa \in GL_2(D) \text{ for } 1 \le \kappa \le 5,$$

where $H_\kappa = h_\kappa\, {}^t\overline{h_\kappa}$ are given by

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$H_2 = \begin{pmatrix} 2 & (1+\alpha)/2 \\ (1-\alpha)/2 & 2 \end{pmatrix},$$
$$H_3 = \begin{pmatrix} 3 & (1+3\beta+\alpha+\alpha\beta)/2 \\ (1-3\beta-\alpha-\alpha\beta)/2 & 3 \end{pmatrix},$$
$$H_4 = \begin{pmatrix} 7 & 1+2\beta+\alpha+\alpha\beta \\ 1-2\beta-\alpha-\alpha\beta & 4 \end{pmatrix},$$
$$H_5 = \begin{pmatrix} 5 & 2+2\beta+\alpha \\ 2-2\beta-\alpha & 4 \end{pmatrix}.$$

We also need more lattices to use in the theory of Ihara lifts, and for $1 \le \kappa \le 5$, we define

$$L_\kappa^J = J^2 h_\kappa.$$

We define theta functions associated with these lattices by

$$\vartheta_\kappa(\tau) = \sum_{(x,y)\in L_\kappa} e^{2\pi i(n(x)+n(y))\tau}, \quad \vartheta_\kappa^J(\tau) = \sum_{(x,y)\in L_\kappa^J} e^{2\pi i(n(x)+n(y))\tau}.$$

By easy computer calculation, we see

$$\vartheta_1(\tau) = 1 + 8q + 24q^2 + 48q^3 + 120q^4 + \cdots,$$
$$\vartheta_2(\tau) = 1 + 24q^2 + 24q^3 + 168q^4 + \cdots,$$
$$\vartheta_3(\tau) = 1 + 12q^2 + 72q^3 + 144q^4 + \cdots,$$
$$\vartheta_4(\tau) = 1 + 4q + 16q^2 + 68q^3 + 128q^4 + \cdots,$$
$$\vartheta_5(\tau) = 1 + 24q^2 + 24q^3 + 168q^4 + \cdots,$$
$$\vartheta_1^J(\tau) = 1 + 24q^2 + 24q^3 + 168q^4 + \cdots,$$
$$\vartheta_2^J(\tau) = 1 + 12q^2 + 72q^3 + 144q^4, + \cdots$$
$$\vartheta_3^J(\tau) = 1 + 18q^2 + 48q^3 + 156q^4 + \cdots,$$
$$\vartheta_4^J(\tau) = 1 + 6q + 12q^2 + 90q^3 + 108q^4 + \cdots,$$
$$\vartheta_5^J(\tau) = 1 + 12q + 36q^2 + 12q^3 + 120q^4 + \cdots.$$

These theta functions are associated with $10 = 2 \times 5$ dimensional space of automorphic forms in the product of forms on $D_A^\times$ of weight 0 and $\mathfrak{M}_{0,0}(\mathcal{U}_{pr}(11))$. The space spanned by $\vartheta_i(\tau)/4 + \vartheta_i^J(\tau)/6$ is three-dimensional space spanned by $E_4(\tau) + 11^3 E_4(11\tau)$ and $S_4(\Gamma_0^{(1)}(11))$, and the space spanned by $\vartheta_i(\tau) - \vartheta_i^J(\tau)$ is equal to $S_4(\Gamma_0^{(1)}(11))$. Besides, the first lattice combinations come from constant function of $D_A^\times$ and the second lattice combinations come from a form on $D_A^\times$ which corresponds with a cusp form of weight 2 of level 11 in the Eichler correspondence. Here we have $\dim S_4(\Gamma_0^{(1)}(11)) = \dim S_4^{\mathrm{new},+}(\Gamma_0^{(1)}(11)) = 2$ and $\dim S_2(\Gamma_0^{(1)}(11)) = \dim S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) = 1$. By the lifting theory of Ihara, the above facts mean that the space $\mathfrak{M}_{0,0}(\mathcal{U}_{pr}(11))$ consists of Saito Kurokawa type lifts from the Eisenstein series of weight 4 and $S_4(\Gamma_0^{(1)}(11))$, and Yoshida type lifts from $S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) \times S_4^{\mathrm{new},+}(\Gamma_0^{(1)}(11))$. This fits Conjecture 3.2 completely.

For $k = 4$, the LHS of Theorem 3.1 in this case is $2-7-2=-7$. Here we have $\dim S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) = \dim S_2^{\mathrm{new}}(\Gamma_0^{(1)}(11)) = 1$, $\dim S_6(\Gamma_0^{(1)}(11)) = 4$, $\dim S_6^{\mathrm{new},+}(\Gamma_0^{(1)}(11)) = 1$. So the above -7 in LHS is explained by forms in $S_4(\Gamma_0(11))$ obtained by the Saito–Kurokawa lift from $S_6(\Gamma_0^{(1)}(11))$ (4 dimensional) and the Yoshida lift from $S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) \times S_6^{\mathrm{new},-}(\Gamma_0^{(1)}(11))$ (three dimensional). On the other hand, by [29] p. 77 and 78 (where there are several typos, in particular about the weights), we can prove that one-dimensional space $\mathfrak{M}_{1,1}(\mathcal{U}_{pr}(11))$ is spanned by a lift from $S_2^{\mathrm{new},-}(\Gamma_0^{(1)}(11)) \times S_6^{\mathrm{new},+}(\Gamma_0^{(1)}(11))$ as predicted by Conjecture 3.2.

5 Proof of Theorem 3.1

The dimension formula for each term in Theorem 3.1 is mostly known by the trace formula. For readers' convenience, we first explain what is known in which reference and what is unknown. The dimension $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ is known for any $\nu_1 \geq \nu_2 \geq 0$ for $\nu_1 \equiv \nu_2 \bmod 2$ for any prime p in [12] I. The dimension $\dim S_{k,j}(K(p))$ is known for all primes p if $k \geq 3$ and $j=0$, and if $k \geq 5$ and $j \geq 2$ ([16] for $k \geq 5$ and $j=0$ and [20] for $k=3,4$ for $j=0$, [19,23] for $k \geq 5$ and $j \geq 2$). The dimension $\dim S_{k,j}(\Gamma_0(p))$ is known for all primes p for $k \geq 5, j=0$ by [11], for $k=3,4$ for $j=0$ in [20], for $j \geq 2$ with $k \geq 5$ in [41]). Dimensions of $S_{k,j}(\Gamma_0'(p)) \cong S_{k,j}(\Gamma_0''(p))$ were given in [13], [20] for $k \geq 3, j=0$ and [41] for $k \geq 5, j \geq 2$, but we must assume that $p \neq 2,3$ as far as we give them by trace formula as in [13] since some local calculations at 2 and 3 necessary for the trace formula has never been done. When $j=0$, we have another method to calculate dimensions for $p=2,3$, which will be explained in §5.3. In §5.1 and 5.2, we prove Theorem 3.1 under the assumptions that $p \neq 2,3$, and that $k \geq 3$ for $j=0$, and that $k \geq 5$ for $j \geq 2$.

5.1 Review on the compact twist and characters

Here for readers' convenience, we quote the formula for $\dim \mathfrak{M}_{k+j-3.k-3}(\mathcal{U}_{pr}(p))$ from [12](I) p. 591–591, restricting to the case that the level is a prime p. We define polynomials $f_i(x)$ for $1 \leq i \leq 12$ by

$$
\begin{aligned}
&f_1(x)=(x-1)^4, && f_2(x)=(x-1)^2(x+1)^2,\\
&f_3(x)=(x-1)^2(x^2+1), && f_4(x)=(x-1)^2(x^2+x+1),\\
&f_5(x)=(x-1)^2(x^2-x+1), && f_6(x)=(x^2+1)^2,\\
&f_7(x)=(x^2+x+1)^2, && f_8(x)=(x^2+1)(x^2+x+1),\\
&f_9(x)=(x^2+x+1)(x^2-x+1), && f_{10}(x)=x^4+x^3+x^2+x+1,\\
&f_{11}(x)=x^4+1, && f_{12}(x)=x^4-x^2+1.
\end{aligned}
$$

We fix an integer $k \geq 3$ and an even integer $j \geq 0$ and we denote by $\tau = \tau_{k+j-3,k-3}$ the irreducible representation of G_∞^1 corresponding to the Young diagram parameter $(k+j-3, k-3)$. For each i ($1 \leq i \leq 12$), we fix an element $g_i \in G_\infty^1$ whose principal polynomial is $f_i(x)$ or $f_i(-x)$. Since we assumed that j is even, we have $\tau(g_i) = \tau(-g_i)$ and we also have $Tr(\tau(g_i)) = Tr(\tau(-g_i))$, where Tr denotes the trace of matrices. The traces $Tr(\tau(g_i))$ are easily obtained by the well-known character formula in [42]. For any integer d and a prime p, we denote by $\left(\frac{-d}{p}\right)$ the Kronecker symbol for $\mathbb{Q}(\sqrt{-d})$, that is, if $p \neq 2$, then this is the Legendre symbol and if $p=2$, then this is 1 for $d \equiv 7 \bmod 8$, -1 for $d \equiv 3 \bmod 8$, and 0 otherwise.

Theorem 5.1 ([12](I)) *For any prime p, and any integer $k \geq 3$, any even integer $j \geq 0$, we have*

$$\dim \mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p)) = \sum_{i=1}^{12}\Big(H_i^{cpt} \times Tr(\tau(g_i))\Big),$$

where H_i^{cpt} are given as follows.

$$\begin{aligned}
H_1^{cpt} &= \frac{1}{2^6\cdot 3^2\cdot 5}(p-1)(p^2+1),\\
H_2^{cpt} &= \frac{1}{2^6\cdot 3^2}(p-1)^2 \times \begin{cases} 7 & if\, p \neq 2,\\ 13 & if\, p = 2,\end{cases}\\
H_3^{cpt} &= \frac{1}{2^4\cdot 3}(p-1)\left(1-\left(\frac{-1}{p}\right)\right),\\
H_4^{cpt} &= H_5^{cpt} = \frac{1}{2^3\cdot 3^2}(p-1)\left(1-\left(\frac{-3}{p}\right)\right),\\
H_6^{cpt} &= \frac{1}{2^5}\left(1-\left(\frac{-1}{p}\right)\right)(1-\delta_{p2}) + \frac{5}{2^5\cdot 3}(p-1),\\
H_7^{cpt} &= \frac{1}{2^2\cdot 3^2}\left(1-\left(\frac{-3}{p}\right)\right) + \frac{1}{2\cdot 3^2}(p-1)(1-\delta_{p3}),\\
H_8^{cpt} &= \frac{1}{2^2\cdot 3}\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right),\\
H_9^{cpt} &= \begin{cases} \dfrac{1}{3^2}\left(1-\left(\dfrac{-3}{p}\right)\right)^2 & if\, p\neq 2,\\ \dfrac{5}{18} & if\, p = 2,\end{cases}\\
H_{10}^{cpt} &= \frac{1}{5}\times\begin{cases} 1 & if\, p = 5,\\ 4 & if\, p \equiv 4 \bmod 5,\\ 0 & otherwise,\end{cases}\\
H_{11}^{cpt} &= \frac{1}{2^3}\times\begin{cases} 1 & if\, p = 2,\\ 0 & if\, p\equiv 1 \bmod 8,\\ 2 & if\, p \equiv 3, 5 \bmod 8,\\ 4 & if\, p\equiv 7 \bmod 8,\end{cases}\\
H_{12}^{cpt} &= \frac{1}{2^2\cdot 3}\left(1-\left(\frac{-3}{p}\right)\right).
\end{aligned}$$

In order to compare this with the split group, we need the explicit shape of the characters. We write this down here in the way suitable for the comparison. We define that the notation $[a_0,\ldots,a_{m-1};m]_n$ means a_i if $n \equiv i \bmod m$. For $1 \leq i \leq 12$ and any integers k and j, we define notation $C_i(k,j)$ and $C_{i,1}(k,j)$, $C_{i,2}(k,j)$ as follows.

$$\begin{aligned}
C_1(k,j) &= (j+1)(k-2)(j+k-1)(j+2k-3),\\
C_2(k,j) &= (-1)^k(k-2)(j+k-1),\\
C_3(k,j) &= [(k-2)(-1)^{j/2}, -(j+k-1), -(k-2)(-1)^{j/2}, j+k-1; 4]_k,\\
C_4(k,j) &= (j+k-1)[1,-1,0;3]_k + (k-2)[1,0,-1;3]_{j+k},\\
C_5(k,j) &= (j+k-1)[-1,-1,0,1,1,0;6]_k + (k-2)[1,0,-1,-1,0,1;6]_{j+k},\\
C_{6,1}(k,j) &= (-1)^{j/2}(2k+j-3),
\end{aligned}$$

$$C_{6,2}(k,j) = (-1)^{j/2+k}(j+1),$$
$$C_{7,1}(k,j) = (2k+j-3)[1,-1,0;3]_j,$$
$$C_{7,2}(k,j) = (j+1)[0,1,-1;3]_{j+2k},$$
$$C_8(k,j) = \begin{cases} [1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k \cdots \text{ if } j \equiv 0 \bmod 12, \\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k \cdots \text{ if } j \equiv 2 \bmod 12, \\ [1,-1,0,0,-1,1,-1,1,0,0,1,-1;12]_k \cdots \text{ if } j \equiv 4 \bmod 12, \\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k \cdots \text{ if } j \equiv 6 \bmod 12, \\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k \cdots \text{ if } j \equiv 8 \bmod 12, \\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k \cdots \text{ if } j \equiv 10 \bmod 12, \end{cases}$$
$$C_9(k,j) = \begin{cases} [1,0,0,-1,0,0;6]_k \cdots \text{ if } j \equiv 0 \bmod 6, \\ [-1,1,0,1,-1,0;6]_k \cdots \text{ if } j \equiv 2 \bmod 6, \\ [0,-1,0,0,1,0;6]_k \cdots \text{ if } j \equiv 4 \bmod 6, \end{cases}$$
$$C_{10}(k,j) = \begin{cases} [1,0,0,-1,0;5]_k \cdots \text{ if } j \equiv 0 \bmod 10, \\ [-1,1,0,0,0;5]_k \cdots \text{ if } j \equiv 2 \bmod 10, \\ 0 \cdots \text{ if } j \equiv 4 \bmod 10, \\ [0,0,0,1,-1;5]_k \cdots \text{ if } j \equiv 6 \bmod 10, \\ [0,-1,0,0,1;5]_k \cdots \text{ if } j \equiv 8 \bmod 10, \end{cases}$$
$$C_{11}(k,j) = \begin{cases} [1,0,0,-1;4]_k \cdots \text{ if } j \equiv 0 \bmod 8, \\ [-1,1,0,0;4]_k \cdots \text{ if } j \equiv 2 \bmod 8, \\ [-1,0,0,1;4]_k \cdots \text{ if } j \equiv 4 \bmod 8, \\ [1,-1,0,0;4]_k \cdots \text{ if } j \equiv 6 \bmod 8, \end{cases}$$
$$C_{12,1}(k,j) = (-1)^{k+j/2}[1,-1,0;3]_j,$$
$$C_{12,2}(k,j) = -(-1)^{j/2}[0,1,-1;3]_{j+2k}.$$

Lemma 5.2 *We have the following relations.*

$$Tr(\tau(g_1)) = C_1(k,j)/6,$$
$$Tr(\tau(g_2)) = -C_2(k,j)/2,$$
$$Tr(\tau(g_3)) = C_3(k,j)/2,$$
$$Tr(\tau(g_4)) = C_4(k,j)/3,$$
$$Tr(\tau(g_5)) = C_5(k,j),$$
$$Tr(\tau(g_6)) = (C_{6,1}(k,j) - C_{6,2}(k,j))/4,$$
$$Tr(\tau(g_7)) = (C_{7,1}(k,j) - C_{7,2}(k,j))/3,$$
$$Tr(\tau(g_8)) = -C_8(k,j),$$
$$Tr(\tau(g_9)) = -C_9(k,j),$$
$$Tr(\tau(g_{10})) = -C_{10}(k,j),$$
$$Tr(\tau(g_{11})) = -C_{11}(k,j),$$
$$Tr(\tau(g_{12})) = -(C_{12,1}(k,j) + C_{12,2}(k,j)).$$

5.2 Comparison with split case

We review the dimension formula of Siegel cusp forms. For any discrete subgroup $\Gamma \subset Sp(2,\mathbb{R})$, we may write the dimensions as a sum of contributions of Γ conjugacy classes in Γ, using the Selberg trace formula. We denote by $H_{i,\Gamma}$ $(1 \le i \le 12)$ the contribution of semi-simple elements whose principal polynomials are $f_i(\pm x)$ and $H_{i,\Gamma}^{qu}$ $(1 \le i \le 7)$ the contribution of non-semi-simple elements (unipotent and quasi unipotent elements) whose principal polynomials are also $f_i(\pm x)$. Instead of the subscript Γ of $H_{i,\Gamma}$ and $H_{i,\Gamma}^{qu}$, we write $H_{i,0}$ and $H_{i,0}^{qu}$ for $\Gamma_0(p)$, $H_{i,d}$ and $H_{i,d}^{qu}$ for $\Gamma_0'(p)$, $H_{i,K}$ and $H_{i,K}^{qu}$ for $K(p)$. Since we should compare $2\dim S_{k,j}(\Gamma_0'(p)) - \dim S_{k,j}(\Gamma_0(p)) - 2\dim S_{k,j}(K(p))$ with $\mathfrak{M}_{k+j-3,k-3}(\mathcal{U}_{pr}(p))$, and every element is semi-simple for compact twist, we define

$$H_i^* = 2H_{i,d} - H_{i,0} - 2H_{i,K} - H_i^{cpt}, \qquad H_i^{qu,*} = 2H_{i,d}^{qu} - H_{i,0}^{qu} - 2H_{i,K}^{qu}.$$

We review here known formulas and give values of H_i^* and $H_i^{qu,*}$. When $j = 0$, the formula for $H_{i,0}$ and $H_{i,0}^{qu}$ are known in [11] for all primes p, for $H_{i,K}$ and $H_{i,K}^{qu}$ in [16,19] for all primes, for $H_{i,d}$ and $H_{i,d}^{qu}$ in [13] for $p \ne 2, 3$. Originally these are given for $k \ge 5$ but the results for $k = 3$ and 4 are in [20]. For the case $j > 0$ and $k \ge 5$, see [41].

In order to write down the contribution of non-semi-simple elements, we prepare some more notation. We put

$$\begin{aligned}
\chi_1 &= \frac{1}{2^4 \cdot 3^2}(j+1)(2k+j-3),\\
\chi_{2,1} &= \frac{1}{2^4}(-1)^k,\\
\chi_{2,2} &= (-1)^k(2k+j-3),\\
\chi_3 &= [(-1)^{j/2}, -1, -(-1)^{j/2}, 1; 4]_k,\\
\chi_{4,1} &= [1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k},\\
\chi_{4,2} &= [1, 0, 1; 3]_k + [0, -1, -1; 3]_{j+k},\\
\chi_5 &= [-1, -1, 0, 1, 1, 0; 6]_k + [1, 0, -1, -1, 0, 1; 6]_{j+k}.
\end{aligned}$$

Proposition 5.3 *We assume that $p \ne 2, 3$. Then we have*

$$\begin{aligned}
&H_1^* = H_2^* = H_3^* = H_4^* = H_5^* = H_8^* = H_9^* = H_{10}^* = H_{11}^* = 0,\\
&H_3^{qu,*} = H_5^{qu,*} = 0.
\end{aligned}$$

Proof The proof is a straight forward calculation, so here we give only data to use. For $p \ne 2$ and 3, we have

$$\begin{aligned}
H_{1,0} = H_{1,d} &= \frac{1}{2^7 \cdot 3^3 \cdot 5}(p+1)(p^2+1)C_1(k,j),\\
H_{1,K} &= \frac{1}{2^7 \cdot 3^3 \cdot 5}(p^2+1)C_1(k,j),\\
H_{2,0} &= \frac{7}{2^7 \cdot 3^2}(p+1)^2 C_2(k,j),\\
H_{2,d} &= \frac{7}{2^6 \cdot 3^2}(p+1)C_2(k,j),\\
H_{2,K} &= \frac{7}{2^6 \cdot 3^2}C_2(k,j),\\
H_{3,0} &= \frac{1}{2^5 \cdot 3}(p+1)\left(1 + \left(\frac{-1}{p}\right)\right)C_3(k,j),
\end{aligned}$$

$$
\begin{aligned}
H_{3,d} &= \frac{1}{2^5\cdot 3}\left(p+2+\left(\frac{-1}{p}\right)\right)C_3(k,j),\\
H_{3,K} &= \frac{1}{2^4\cdot 3}C_3(k,j),\\
H_{4,0} &= \frac{1}{2^3\cdot 3^3}(p+1)\left(1+\left(\frac{-3}{p}\right)\right)C_4(k,j),\\
H_{4,d} &= \frac{1}{2^3\cdot 3^3}\left(p+2+\left(\frac{-3}{p}\right)\right)C_4(k,j),\\
H_{4,K} &= \frac{1}{2^2\cdot 3^3}C_4(k,j),\\
H_{5,0} &= \frac{1}{2^3\cdot 3^2}(p+1)\left(1+\left(\frac{-3}{p}\right)\right)C_5(k,j),\\
H_{5,d} &= \frac{1}{2^3\cdot 3^2}\left(p+2+\left(\frac{-3}{p}\right)\right)C_5(k,j),\\
H_{5,K} &= \frac{1}{2^2\cdot 3^2}C_5(k,j),\\
H_{8,0} &= \frac{1}{2^2\cdot 3}\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)C_8(k,j),\\
H_{8,d} &= \frac{1}{2^2\cdot 3}\left(2+\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)\right)C_8(k,j),\\
H_{8,K} &= \frac{1}{2\cdot 3}C_8(k,j),\\
H_{9,0} &= \frac{1}{3^2}\left(1+\left(\frac{-3}{p}\right)\right)^2C_9(k,j),\\
H_{9,d} &= \frac{2}{3^2}\left(1+\left(\frac{-3}{p}\right)\right)C_9(k,j),\\
H_{9,K} &= \frac{2}{3^2}C_9(k,j),\\
H_{10,0} &= 5^{-1}C_{10}(k,j)[1,4,0,0,0;5]_p,\\
H_{10,d} &= 5^{-1}C_{10}(k,j)[1,4,0,0,0;5]_p,\\
H_{10,K} &= 5^{-1}C_{10}(k,j)[1,2,0,0,2;5]_p,\\
H_{11.0} &= \frac{1}{2^3}C_{11}(k,j)\times\begin{cases}4 & \text{if } p\equiv 1 \bmod 8,\\ 2 & \text{if } p\equiv 3,5 \bmod 8,\\ 0 & \text{if } p\equiv 7 \bmod 8,\end{cases}\\
H_{11,d} &= \frac{1}{2}C_{11}(k,j)\times\begin{cases}1 & \text{if } p\equiv 1 \bmod 8,\\ 0 & \text{otherwise},\end{cases}\\
H_{11,K} &= \frac{1}{2^2}C_{11}(k,j)\times\begin{cases}1 & \text{if } p\equiv \pm 1 \bmod 8,\\ 0 & \textit{otherwise}.\end{cases}
\end{aligned}
$$

It is also known that

$$
\begin{aligned}
H_{3,0}^{qu} &= -\frac{1}{2^2}\left(1+\left(\frac{-1}{p}\right)\right)\chi_3,\\
H_{3,d}^{qu} &= -\frac{1}{2^3}\left(3+\left(\frac{-1}{p}\right)\right)\chi_3,\\
H_{3,K}^{qu} &= -\frac{1}{2^2}\chi_3,
\end{aligned}
$$

$$
\begin{aligned}
H_{5,0}^{qu} &= -\frac{1}{2\cdot 3}\left(1+\left(\frac{-3}{p}\right)\right)\chi_5,\\
H_{5,d}^{qu} &= -\frac{1}{2^2\cdot 3}\left(3+\left(\frac{-3}{p}\right)\right)\chi_5,\\
H_{5,K}^{qu} &= -\frac{1}{2\cdot 3}\chi_5.
\end{aligned}
$$

So the proposition is obtained by a direct calculation in each case.

Proposition 5.4 *We have*

$$
\begin{aligned}
H_6^* &= \frac{p}{2^4\cdot 3}\left(\left(\frac{-1}{p}\right)-1\right)C_{6,1}(k,j)+\frac{p-1}{2^4\cdot 3}\left(\frac{-1}{p}\right)C_{6,2}(k,j),\\
H_7^* &= \frac{p}{2^2\cdot 3^2}\left(\left(\frac{-3}{p}\right)-1\right)C_{7,1}(k,j)+\frac{p-1}{2^2\cdot 3^2}\left(\frac{-3}{p}\right)C_{7,2}(k,j),\\
H_{12}^* &= -\frac{1}{2^2\cdot 3}\left(\left(\frac{-3}{p}\right)-1\right)\left(\frac{-1}{p}\right)(-1)^{k+j/2}[1,-1,0;3]_j\\
&\quad -\frac{1}{2^2\cdot 3}\left(\left(\frac{-1}{p}\right)-1\right)\left(\frac{-3}{p}\right)(-1)^{j/2}[0,1,-1;3]_{j+2k},\\
H_1^{qu,*} &= \frac{p-1}{2^3\cdot 3}(j+1)-\frac{p(p-1)}{2^4\cdot 3^2}(j+1)(2k+j-3),\\
H_2^{qu,*} &= -\frac{(-1)^k}{2^4}\left(1-\left(\frac{-1}{p}\right)\right),\\
H_4^{qu,*} &= -\frac{1}{3^2}[1,-1,0;3]_j\times[0,1,-1;3]_{j+2k}\left(1-\left(\frac{-3}{p}\right)\right),\\
H_6^{qu,*} &= -\frac{1}{2^3}(-1)^{j/2}\left(\left(\frac{-1}{p}\right)-1\right),\\
H_7^{qu,*} &= -\frac{1}{2\cdot 3}\left(\left(\frac{-3}{p}\right)-1\right)\times[1,-1,0;3]_j.
\end{aligned}
$$

Proof We put

$$
\begin{aligned}
H_{6,1} &= \frac{1}{2^5\cdot 3}\left(p+\left(\frac{-1}{p}\right)\right)+\frac{1}{2^7\cdot 3}\left(p\left(\frac{-1}{p}\right)+1\right),\\
H_{6,2} &= \frac{1}{2^5\cdot 3}\left(p+\left(\frac{-1}{p}\right)\right)-\frac{1}{2^7\cdot 3}\left(p\left(\frac{-1}{p}\right)+1\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
H_{6,0} &= \frac{1}{2^7}\left(p+2+\left(\frac{-1}{p}\right)\right)C_{6,2}(k,j)+\frac{5}{2^7\cdot 3}\left(p+2+\left(\frac{-1}{p}\right)\right)C_{6,1}(k,j),\\
H_{6,d} &= \frac{1}{2^7}(p+1)\left(1+\left(\frac{-1}{p}\right)\right)C_{6,2}(k,j)+\frac{5}{2^7\cdot 3}(p+1)\left(1+\left(\frac{-1}{p}\right)\right)C_{6,1}(k,j),\\
H_{6,K} &= H_{6,1}C_{6,1}(k,j)+H_{6,2}C_{6,2}(k,j),\\
H_6^{cpt} &= \left(\frac{1}{2^7}\left(1-\left(\frac{-1}{p}\right)\right)+\frac{5}{2^7\cdot 3}(p-1)\right)\\
&\quad\times(C_{6,1}(k,j)-C_{6,2}(k,j)).
\end{aligned}
$$

So we easily see the result for H_6^*. We put

$$
\begin{aligned}
H_{7,1} &= \frac{1}{2^3\cdot 3^2}\left(p+\left(\frac{-3}{p}\right)\right)+\frac{1}{2^3\cdot 3^3}\left(p\left(\frac{-3}{p}\right)+1\right),\\
H_{7,2} &= \frac{1}{2^3\cdot 3^2}\left(p+\left(\frac{-3}{p}\right)\right)-\frac{1}{2^3\cdot 3^3}\left(p\left(\frac{-3}{p}\right)+1\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
H_{7,0} &= \left(p+2+\left(\frac{-3}{p}\right)\right)\left(\frac{1}{2\cdot 3^3}C_{7,1}(k,j)+\frac{1}{2^2\cdot 3^3}C_{7,2}(k,j)\right),\\
H_{7,d} &= (p+1)\left(1+\left(\frac{-3}{p}\right)\right)\left(\frac{1}{2\cdot 3^3}C_{7,1}(k,j)+\frac{1}{2^2\cdot 3^3}C_{7,2}(k,j)\right),\\
H_{7,K} &= H_{7,1}C_{7,1}(k,j)+H_{7,2}C_{7,2}(k,j),\\
H_7^{cpt} &= \left(\frac{1}{2^2\cdot 3^3}\left(1-\left(\frac{-3}{p}\right)\right)+\frac{1}{2\cdot 3^3}(p-1)\right)\times(C_{7,1}(k,j)-C_{7,2}(k,j)).
\end{aligned}
$$

So we have the result for H_7^*. We have

$$
\begin{aligned}
H_{12,0} &= \frac{1}{2^2\cdot 3}\left(2+\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)\right)C_{12,2}(k,j),\\
H_{12,d} &= \frac{1}{2^2\cdot 3}\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)C_{12,2}(k,j),\\
H_{12,K} &= \frac{1}{2^3\cdot 3}\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right)C_{12,1}(k,j)\\
&\quad+\frac{1}{2^3\cdot 3}\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)C_{12,2}(k,j),\\
H_{12}^{cpt} &= -\frac{1}{2^2\cdot 3}\left(1+\left(\frac{-3}{p}\right)\right)(C_{12,1}(k,j)+C_{12,2}(k,j)).
\end{aligned}
$$

So we obtain H_{12}^*.

Next we consider the contribution of non-semi-simple elements. It is known that

$$
\begin{aligned}
H_{1,0}^{qu} &= \frac{1}{2^3\cdot 3}(p+3)(j+1)-\frac{1}{2^3\cdot 3}(p+1)-(p+1)\chi_1,\\
H_{1,d}^{qu} &= \frac{1}{2^2\cdot 3}(p+1)(j+1)-\frac{1}{2^4\cdot 3}(p+3)-\frac{(p+1)^2}{2}\chi_1,\\
H_{1,K}^{qu} &= \frac{1}{2^3\cdot 3}(p+1)(j+1)-\frac{1}{2^3\cdot 3}-p\chi_1.
\end{aligned}
$$

So we have

$$
H_1^{qu,*} = \frac{1}{2^3\cdot 3}(j+1)(p-1)-p(p-1)\chi_1.
$$

It is known that

$$
\begin{aligned}
H_{2,0}^{qu} &= \left(7 - \left(\frac{-1}{p}\right)\right)\chi_{2,1} - \frac{1}{2^3\cdot 3}(p+1)\chi_{2,2},\\
H_{2,d}^{qu} &= \left(7 - \left(\frac{-1}{p}\right)\right)\chi_{2,1} - \frac{1}{2^4\cdot 3}(p+3)\chi_{2,2},\\
H_{2,K}^{qu} &= \left(4 - \left(\frac{-1}{p}\right)\right)\chi_{2,1} - \frac{1}{2^3\cdot 3}\chi_{2,2}.
\end{aligned}
$$

So we have

$$
H_2^{qu,*} = -\left(1 - \left(\frac{-1}{p}\right)\right)\chi_{2,1}.
$$

It is known that

$$
\begin{aligned}
H_{4,0}^{qu} &= -\frac{1}{2\cdot 3^2}\left(1 + \left(\frac{-3}{p}\right)\right)\chi_{4,1} - \frac{2}{3^2}\left(1 + \left(\frac{-3}{p}\right)\right)\chi_{4,2},\\
H_{4,dash}^{qu} &= \frac{1}{2^2\cdot 3^2}\left(-5 + \left(\frac{-3}{p}\right)\right)\chi_{4,1} - \frac{2}{3^2}\left(1 + \left(\frac{-3}{p}\right)\right)\chi_{4,2},\\
H_{4,K}^{qu} &= -\frac{1}{2\cdot 3^2}\left(3 - 2\left(\frac{-3}{p}\right)\right)\chi_{4,1} - \frac{2}{3^2}\left(\frac{-3}{p}\right)\chi_{4,2}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
H_4^{qu,*} &= \frac{1}{3^2}\left(1 - \left(\frac{-3}{p}\right)\right)(\chi_{4,1} - 2\chi_{4,2})\\
&= -\frac{1}{3^2}[1,-1,0;3]_j \times [0,1,-1;3]_{j+2k}\left(1 - \left(\frac{-3}{p}\right)\right).
\end{aligned}
$$

It is known that

$$
\begin{aligned}
H_{6,0}^{qu} &= -\frac{1}{2^3}\left(3 + \left(\frac{-1}{p}\right)\right)(-1)^{j/2},\\
H_{6,d}^{qu} &= -\frac{1}{2^2}\left(1 + \left(\frac{-1}{p}\right)\right)(-1)^{j/2},\\
H_{6,K}^{qu} &= -\frac{1}{2^3}\left(1 + \left(\frac{-1}{p}\right)\right)(-1)^{j/2}.
\end{aligned}
$$

So we have

$$
H_6^{qu,*} = -\frac{1}{2^3}\left(\left(\frac{-1}{p}\right) - 1\right)(-1)^{j/2}.
$$

It is known that

$$
\begin{aligned}
H_{7,0}^{qu} &= -\frac{1}{2\cdot 3}\left(3 + \left(\frac{-3}{p}\right)\right)[1,-1,0;3]_j,\\
H_{7,d}^{qu} &= -\frac{1}{3}\left(1 + \left(\frac{-3}{p}\right)\right)[1,-1,0;3]_j,\\
H_{7,K}^{qu} &= -\frac{1}{2\cdot 3}\left(1 + \left(\frac{-3}{p}\right)\right)[1,-1,0;3]_j.
\end{aligned}
$$

So we have

$$
H_7^{qu,*} = -\frac{1}{2\cdot 3}\left(\left(\frac{-3}{p}\right) - 1\right)[1,-1,0;3]_j.
$$

Proof of Theorem 3.1 To complete the proof of Theorem 3.1, we note that for even $j \geq 0$, we have

$$\dim S_{j+2}^{\mathrm{new}}(\Gamma_0(p)) + \delta_{j0} = \frac{p-1}{12}(j+1)$$
$$-\frac{1}{4}(-1)^{j/2}\left(\left(\frac{-1}{p}\right)-1\right) - \frac{1}{3}[1,-1,0;3]_j\left(\left(\frac{-3}{p}\right)-1\right),$$

and for $k \geq 3$, we have

$$\dim S_{2k+j-2}^{\mathrm{new}}(\Gamma_0(p)) + \dim S_{2k+j-2}(SL_2(\mathbb{Z}))$$
$$= \frac{p(2k+j-3)}{12} - \frac{1}{4}(-1)^{k+j/2}\left(\frac{-1}{p}\right) - \frac{1}{3}[0,1,-1;3]_{2k+j}\left(\frac{-3}{p}\right) - \frac{1}{2}.$$

By the way, here each term in RHS of the above two equalities is the contribution of $\pm 1_2$, of elements of order 4, of elements of order 3 and 6, and of unipotent elements (which is zero for the first equality) in this order. Then we see easily that the sum of the values given in Proposition 5.4 coincides with (-1) times the product of the above two dimensions, and we have Theorem 3.1. More precisely, we see by direct calculation that $-H_1^{qu,*}$ is the product of the contribution of $\pm 1_2$ of the RHS of the first equality above and the contribution of $\pm 1_2$ and unipotent elements of the second equality above. We see $-H_4^{qu,*}$ is the product of the contribution of order 3 and 6 of RHS of the first and the second. We see $-H_7^{qu,*}$ is the product of the contribution of order 3, 6 of the first and that of unipotent elements of the second. Similarly, $-H_6^{qu,*}$ is the product of order 4 and unipotent, $-H_2^{qu,*}$ is the product of order 4 and order 4. Also $-H_{12}^{*}$ is the sum of the product of order 3, 6 for the first and order 4 for the second, and order 4 for the first and order 3, 6 for the second. Similarly, $-H_6^{*}$ is the sum of the product of order 4 for the first and $\pm 1_2$ for the second, and the product of $\pm 1_2$ for the first and order 4 for the second. Similarly $-H_7^{*}$ is the sum of the product of order 3, 6 for the first and $\pm 1_2$ for the second, and $\pm 1_2$ for the first and order 3, 6 for the second. □

5.3 The case $p = 2$ and 3 with $j = 0$

Here we give a proof of Theorem 3.1 for $j = 0$ in case $p = 2, 3$, since these are given by very different method from the one used in other cases.

We denote by $\Gamma(N) = \{g \in Sp(2,\mathbb{Z}); g \equiv 1_4 \bmod N\}$ the principal congruence subgroup of level N of $Sp(2,\mathbb{Z})$. When $p = 2$ and 3, the actions of $Sp(2,\mathbb{F}_2) \cong Sp(2,\mathbb{Z})/\Gamma(2)$ and $Sp(2,\mathbb{F}_3)/\{\pm 1_4\}$ on $A_k(\Gamma(p))$ for $p = 2$ and 3 are known, respectively, (See [26] for $p = 2$ and [8] for $p = 3$.) So for any group Γ with $\Gamma(p) \subset \Gamma \subset Sp(2,\mathbb{Z})$ for $p = 2$ and $p = 3$, we can calculate the dimensions $\dim A_k(\Gamma)$. The dimensions of $\dim S_k(\Gamma)$ is obtained for $k \geq 6$ by the surjectivity of the Siegel Φ operator by [34] and for $k \leq 4$ by checking each case directly. Although $K(p)$ is not contained in $Sp(2,\mathbb{Z})$, the dimension of $S_k(K(p))$ is known for all k for $p = 2, 3$ (See [2,4,16,20,24]). On the other hand, the dimensions of $\mathfrak{M}_{\nu_1,\nu_2}(\mathcal{U}_{pr}(p))$ have been given in [12] (I) for all p, ν_1, ν_2 with $\nu_1 \equiv \nu_2 \bmod 2$. The dimension formula for $\Gamma_0^{(1)}(p)$ is classical. So gathering all these, we obtain the result. Indeed, the generating functions of dimensions of cusp forms for $p = 2, 3$ are given as follows.

$$
\begin{aligned}
\sum_{k=0}^{\infty} \dim A_k(\Gamma_0(2))t^k &= \frac{(1+t^{19})}{(1-t^2)(1-t^4)^2(1-t^6)},\\
\sum_{k=0}^{\infty} \dim S_k(\Gamma_0(2))t^k &= \frac{t^6+t^8-t^{14}}{(1-t^2)(1-t^4)^2(1-t^6)},\\
\sum_{k=0}^{\infty} \dim A_k(\Gamma_0'(2))t^k &= \frac{(1+t^{11})(1+t^6+t^8+t^{10}+t^{12}+t^{18})}{(1-t^4)^2(1-t^6)(1-t^{12})},\\
\sum_{k=0}^{\infty} \dim S_k(\Gamma_0'(2))t^k &= \frac{t^8+2t^{10}+2t^{12}+t^{16}+2t^{18}-t^{22}-t^{28}}{(1-t^4)^2(1-t^6)(1-t^{12})}\\
&\quad + \frac{t^{11}(1+t^6+t^8+t^{10}+t^{12}+t^{18})}{(1-t^4)^2(1-t^6)(1-t^{12})},\\
\sum_{k=0}^{\infty} \dim A_k(K(2))t^k &= \frac{(1+t^{10})(1+t^{12})(1+t^{11})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},\\
\sum_{k=0}^{\infty} \dim S_k(K(2))t^k &= \frac{t^8+t^{10}+t^{12}+t^{22}+t^{24}-t^{32}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})}\\
&\quad + \frac{t^{11}(1+t^{10})(1+t^{12})}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},\\
\sum_{k=3}^{\infty} \dim \mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(2))t^k &= \frac{t^3(1+t^{13})(1-t^4+t^8)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},\\
&= \frac{t^3(1+t^{13})(1+t^{12})}{(1-t^2)(1-t^6)(1-t^8)(1-t^{12})},\\
\sum_{k=2}^{\infty} \dim S_{2k-2}(\Gamma_0^{(1)}(2))t^k &= \frac{t^5}{(1-t)(1-t^2)} = \frac{t^5+t^6}{(1-t^2)^2},\\
\sum_{k=2}^{\infty} \dim S_{2k-2}(SL_2(\mathbb{Z}))t^k &= \frac{t^7}{(1-t^2)(1-t^3)} = \frac{t^7+t^{10}}{(1-t^2)(1-t^6)}.
\end{aligned}
$$

Gathering all these together, we have

$$
\begin{aligned}
&2\dim S_k(\Gamma_0'(2)) - \dim S_k(\Gamma_0(2)) - 2\dim S_k(K(2))\\
&\quad = \dim \mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(2)) - \delta_{k3} - \dim S_{2k-2}(\Gamma_0^{(1)}(2)) + \dim S_{2k-2}(SL_2(\mathbb{Z})).
\end{aligned}
$$

Since $S_2(\Gamma_0(2)) = S_4(SL_2(\mathbb{Z})) = 0$ and

$$
\dim S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(2)) = \dim S_{2k-2}(\Gamma_0^{(1)}(2)) - 2\dim S_{2k-2}(SL_2(\mathbb{Z})),
$$

this is exactly the statement in Theorem 3.1 for $j=0$ and $p=2$.

Next we consider the case $p=3$. We have

$$
\begin{aligned}
\sum_{k=0}^{\infty} \dim A_k(\Gamma_0'(3))t^k &= \frac{f(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},\\
\sum_{k=0}^{\infty} \dim A_k(\Gamma_0(3))t^k &= \frac{1+2t^4+t^6+t^{15}(1+2t^2+t^6)}{(1-t^2)(1-t^4)(1-t^6)^2},\\
\sum_{k=0}^{\infty} \dim A_k(K(3))t^k &= \frac{(1+t^{12})(1+t^8+t^9+t^{10}+t^{11}+t^{19})}{(1-t^4)(1-t^6)^2(1-t^{12})},
\end{aligned}
$$

where we put

$$f(t) = 1 + 3t^6 + 2t^8 + t^{10} + 4t^{12} + 2t^{14} + 2t^{18} + t^{20} + t^9(1 + 2t^2 + 2t^6 + 4t^8 + t^{10} + 2t^{12} + 3t^{14} + t^{20}).$$

(Calculation for $S_k(\Gamma_0'(3))$ was done by H. Kitayama by using [8]. The other cases has been known as explained before.) By these, we can show that

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0'(3))t^k = \frac{f_{\text{cusp}}(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$
$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0(3))t^k = \frac{(1+2t^4+t^6)(t^4+t^6-t^{10})+t^{15}(1+2t^2+t^6)}{(1-t^2)(1-t^4)(1-t^6)^2},$$
$$\sum_{k=0}^{\infty} \dim S_k(K(3))t^k = \frac{f^K_{\text{cusp}}(t)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},$$

where

$$f_{\text{cusp}}(t) = t^6 + 2t^8 + t^9 + 3t^{10} + 2t^{11} + 5t^{12} + 2t^{14} + 2t^{15} + t^{16} + 4t^{17} + 4t^{18} + t^{19} + t^{20} + 2t^{21} - 2t^{22} + 3t^{23} - t^{28} + t^{29},$$
$$f^K_{\text{cusp}}(t) = t^6 + t^9 + t^{12} - t^{13} - t^{14} + t^{15} + t^{18} + t^{21} + t^{24} - t^{25} - t^{26} + t^{27}.$$

We also have

$$\sum_{k=2}^{\infty} \dim S^{\text{new}}_{2k-2}(\Gamma_0^{(1)}(3)) = \frac{t^4(1+t+t^2-t^3)}{(1-t^2)(1-t^3)},$$
$$\sum_{k=2}^{\infty} \dim S_{2k-2}(SL_2(\mathbb{Z})) = \frac{t^7+t^{10}}{(1-t^2)(1-t^6)},$$

and

$$\sum_{k=3}^{\infty} \mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(3)) = \frac{t^3(1+2t^8+t^9+t^{12}+2t^{13}+t^{21})}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}.$$

So we have

$$2\dim S_k(\Gamma_0'(3)) - \dim S_k(\Gamma_0(3)) - 2\dim S_k(K(3)) = \dim \mathfrak{M}_{k-3,k-3}(\mathcal{U}_{pr}(3)) - \delta_{k3} - \dim S^{\text{new}}_{2k-2}(\Gamma_0^{(1)}(3)) - \dim S_{2k-2}(SL_2(\mathbb{Z})).$$

Since $S_2(\Gamma_0^{(1)}(3)) = 0$, we have the assertion of Theorem 3.1 for $p = 3$ and $j = 0$.

Correction:

In [12] I p. 592 l. 8, "if $6 \nmid D(B)$" should read "if $2 \nmid D(B)$ and $3 \nmid D(B)$".

In [13] p. 43 l. 6, in the explanation for $f_7(x)$, add "γ_3".

In [13] p. 44, the right-hand side of $t(\hat{\hat{\delta}}_3, k) + t(\hat{\hat{\delta}}_4, k) = \left(3 - \left(\frac{-1}{p}\right)\right)/2^3$ should read $(-1)^k\left(3 - \left(\frac{-1}{p}\right)\right)/2^3$.

In [13] p. 46, the right-hand side of $t(\hat{\hat{\delta}}_3, k) + t(\hat{\hat{\delta}}_4, k) = \left(3 - \left(\frac{-1}{p}\right)\right)/2^4$ should read $(-1)^k\left(3 - \left(\frac{-1}{p}\right)\right)/2^4$.

In [13], p. 47, in the formula for T_6, the denominator should read $2^5 \cdot 3$ instead of $2^6 \cdot 3$. In [14] p. 590 l. 8, $\lambda(q)q^{2m-4}T$ should read $\lambda(q)q^{2m-3}T$.
In [22] p. 321 l. 3, $GSp(n-1,\mathbb{Z}_p)$ should read $GSp(n-1,\mathbb{Q}_p)$ and $n_p(\eta)\mathbb{Z}_p^{\pm}$ should read $n_p(\eta)\mathbb{Z}_p^{\times}$. Also in [22] p. 321 l. 14, $GSp(n-1,\mathbb{Z}_p)$ should read $GSp(n-1,\mathbb{Q}_p)\cap M_{2n-2}(\mathbb{Z}_p)$.

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References

1. Aoki, H.: On Siegel paramodular forms of degree 2 with small levels. Int. J. Math. **27**, 1650011 (2016). 20 pp
2. Aoki, H., Ibukiyama, T.: Simple graded rings of Siegel modular forms, differential operators and Borcherds products. Int. J. Math. **16**, 249–279 (2005)
3. Boecherer, S., Schulze-Pillot, R.: Siegel modular forms and theta series attached to quaternion algebras II. Nagoya Math. J. **147**, 71–106 (1997)
4. Dern, T.: Paramodular forms of degree 2 and level 3. Comment. Math. Univ. St. Paul. **51**, 157–194 (2002)
5. Eichler, M.: Über die Darstellbarkeit von Modulformen durch Thetareihen. J. Reine Angew. Math. **195**, 156–171 (1955)
6. Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in Mathematics **55**, pp. v+148. Birkhäuser Boston, Inc., Boston, MA, (1985)
7. Evdokimov, S.A.: Euler products for congruence subgroups of the Siegel group of genus 2. Math. USSR Sbornik **28** (1976)
8. Freitag, E., Salvati Manni, R.: The Burkhardt group and modular forms. Transform. Groups **9**, 25–45 (2004)
9. Gritsenko, V.: Arithmetical lifting and its applications, Number theory (Paris, 1992–1993), London Math. Soc. Lecture Note Series **215**, pp. 103–126. Cambridge Univ. Press, Cambridge (1995)
10. Hashimoto, K.: On Brandt matrices associated with the positive definite quaternion Hermitian forms. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27**, 227–245 (1980)
11. Hashimoto, K.: The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two I. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30**, 403–488 (1983)
12. Hashimoto, K., Ibukiyama, T.: On class numbers of positive definite binary quaternion harmitian forms. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27**, 549–601 (1980); **28**, 695–699 (1982); **30**, 393–401 (1983)
13. Hashimoto, K., Ibukiyama, T.: On relations of dimensions of automorphic forms of $Sp(2,\mathbb{R})$ and its compact twist $Sp(2)$ II, Automorphic forms and number theory (Sendai 1983). Adv. Stud. Pure Math. **7**, 31–102 (1985)
14. Ibukiyama, T.: On symplectic Euler factors of genus two. J. Fac. Sci. Univ. Tokyo Sec. IA Math. **30**, 587–614 (1984)
15. Ibukiyama, T.: On automorphic forms of $Sp(2,\mathbb{R})$ and its compact form $Sp(2)$. Séminaire de Théorie des nomble de Paris, 1982–1983 . Progr. Math. **51**, pp. 125–134. Birkhäuser Boston Inc. (1984)
16. Ibukiyama, T.: On relations of dimensions of automorphic forms of $Sp(2,\mathbb{R})$ and its compact twist $Sp(2)$ I, Automorphic forms and number theory (Sendai, 1983). Adv. Stud. Pure Math. **7**, 7–29 (1985)
17. Ibukiyama, T.: On Siegel modular varieties of level three. Int. J. Math. **2**, 17–35 (1991)
18. Ibukiyama, T.: On some alternating sums of dimemsions of Siegel modular forms of general degree and cusp configurations. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **40**, 245–283 (1993)
19. Ibukiyama, T.: Paramodular forms and compact twist. Automorphic Forms on $GSp(4)$. In: Furusawa, M. (ed.)Proceedings of the 9-th Autumn Workshop on Number Theory, pp. 37–48 (2007)
20. Ibukiyama, T.: Dimension formulas of Siegel modular forms of weight 3 and supersingular abelian surfaces. In: Proceedings of the 4-th Spring Conference. Abelian Varieties and Siegel Modular Forms, pp. 39–60 (2007)
21. Ibukiyama, T.: Saito-Kurokawa liftings of level N and practical construction of Jacobi forms. Kyoto J. Math. **52**, 141–178 (2012)
22. Ibukiyama, T., Ihara, Y.: On automorphic forms on the unitary symplectic group $Sp(n)$ and $SL_2(\mathbb{R})$. Math. Ann. **278**, 307–327 (1987)
23. Ibukiyama, T., Kitayama, H.: Dimension formulas of paramodular forms of squarefree level and comparison with inner twist. J. Math. Soc. Jpn. **69**, 597–671 (2017)
24. Ibukiyama, T., Onodera, F.: On the graded ring of modular forms of the Siegel paramodular group of level 2. Abh. Math. Semin. Univ. Hamburg. **67**, 297–305 (1997)
25. Ibukiyama, T., Zagier, D.: Higher Spherical Polynomials/ MPIM preprint series 2014–16, p. 97 (2014)
26. Igusa, J.: On Siegel modular forms genus two II. Am. J. Math. **86**, 392–412 (1964)
27. Ihara, T.: On certain arithmetical Dirichlet series. J. Math. Soc. Jpn. **16**, 214–225 (1964)

28. Langlands, R.P.: Problems in the theory of automorphic forms. In: Lectures in Modern Analysis and Applications III, pp. 18–61. Lecture Notes in Math. **170**, Springer, Berlin (1970)
29. Löschel, R.: Thetakorrespondenz automorpher Formen. Inaugural Dissertation zur Erlangung des Doktorgrades der Mathematisch- Naturwissenshaftlichen Fakultät der Universität zu Köln., pp. ii+102 (1997)
30. Matsuda, I.: Dirichlet series corresponding to Siegel modular forms of degree 2, level *N*. Sci. Pap. Coll. Gen. Ed. Univ. Tokyo **28**, 21–49 (1978)
31. Matsuda Makino, I.: On the meromorphy of Dirichlet series corresponding to Siegel cusp form of degree 2 with respect to $\Gamma_0(N)$. Comment. Math. Univ. St. Pauli **34**, 105–134 (1985)
32. Poor, C., Schmidt, R., Yuen, D.S.: Paramodular forms of level 8 and weights 10 and 12. Int. J. Number Theory **14**, 417–467 (2018)
33. Roberts, B., Schmidt, R.: Local new forms for *GSp*(4). In: Lecture Notes in Mathematics, vol. 1918, pp. viii+307. Springer, Berlin (2007)
34. Satake, I.: Surjectivité globale de l'opérateur Φ. Séminaire H. Cartan 1957/58, Fonctions Automorphes Exposé 16, École Normale Supérieure (1958)
35. Schmidt, R.: Iwahori-spherical representations of *GSp*(4) and Siegel modular forms of degree 2 with square-free level. J. Math. Soc. Jpn. **57**, 259–293 (2005)
36. Schmidt, R.: On classical Saito–Kurokawa liftings. J. Reine Angew. Math. **604**, 211–236 (2007)
37. Saha, A., Schmidt, R.: Yoshida lifts and simultaneous non-vanishing of dihedral twists of modular *L*-functions. J. Lond. Math. Soc. **88**(2), 251–270 (2013)
38. Shimizu, H.: On zeta functions of quaternion algebras. Ann. Math. **81**(2), 166–193 (1965)
39. Shimura, G.: Arithmetic of alternating forms and quaternion hermitian forms. J. Math. Soc. Jpn. **15**, 33–65 (1963)
40. Skoruppa, N.-P., Zagier, D.: Jacobi forms and a certain space of modular forms. Invent. Math. **94**, 113–146 (1988)
41. Wakatsuki, S.: Dimension formulas for spaces of vector-valued Siegel cusp forms of degree two. J. Number Theory **132**, 200–253 (2012)
42. Weyl, H.: The Classical Groups. Their Invariants and Representations, p.), xiv+320. Princeton University Press, Princeton (1946)
43. Yamauchi, T.: On the traces of Hecke operators for a normalizer of $\Gamma_0(N)$. J. Math. Kyoto Univ. **13**, 403–411 (1973)
44. Yoshida, H.: Siegel's modular forms and the arithmetic of quadratic forms. Invent. Math. **60**, 193–248 (1980)