Research Statement

1. Background. My area of research is the representation theory of reductive p-adic groups. Specifically I am interested in how Bushnell-Kutzko's theory of types and covers can be used to obtain explicit results on the reducibility of parabolically induced representations.

Representation theory is a central area of modern mathematics. In particular, the representation theory of reductive p-adic groups is central to the web of conjectures known as the Langlands program whose influence pervades current research in automorphic forms and number theory. The origins of representation theory go back to Frobenius and others in the 1890s in the study of finite groups. A key tool introduced by Frobenius is the method of induction, a way of building representations of a group from representations of subgroups. To make this an effective means of constructing or classifying representations, one needs to be wise in the choice of both the subgroups one induces from and the representations one induces.

In the case of reductive real or p-adic groups, it is natural to induce from what are called parabolic subgroups. For general linear groups, these are (up to conjugacy) the subgroups of invertible block upper-triangular matrices. A proper parabolic subgroup is not reductive but admits a canonical reductive quotient. For example, in the case of block upper-triangular matrices, this reductive quotient is isomorphic to the corresponding group of block diagonal matrices. One takes a representation of this reductive quotient, views it as a representation of the parabolic subgroup and then induces. The whole process is called parabolic induction.

In contrast to reductive real groups, a central feature of reductive p-adic groups is that there are irreducible representations that never occur as subrepresentations of parabolically induced representations. These are the *supercuspidal* representations. They serve as fundamental building blocks. Indeed, suppose Π is an irreducible representation of a reductive p-adic group G. By insights of Harish-Chandra and others it is known that there is a unique parabolic subgroup of G (up to conjugacy) and a unique irreducible supercuspidal representation π of its reductive quotient (up to conjugacy) such that Π occurs in the representation obtained from π via parabolic induction. Moreover, only finitely many Π are related to the supercuspidal representation π in this way.

Thus, a core problem in p-adic representation theory is to understand when and how parabolically induced representations decompose, especially when the inducing representation is supercuspidal. This is the problem I study in my thesis in a very special situation.

In the case of finite groups, Mackey theory provides an efficient way of decomposing induced representations through the action of certain intertwining operators. The same operators can be used to study parabolically induced representations for p-adic groups. Their construction, however, is considerably more subtle and involves a process of analytic continuation. By work of Langlands and Shahidi [12], one knows that properties of these intertwining operators give rise to local L-functions and that these L-functions are the key to many reducibility questions. In [13] Shahidi studies certain reducibility questions in this way for the split classical groups.

Bushnell-Kutzko's method of types and covers [3] provides another way of studying reducibility questions. It relies on detailed knowledge of the internal structure of the inducing representation and certain related constructions. In circumstances where this is available, the method can lead to strikingly explicit results. Indeed, in [4] Kutzko and Morris use the method to reconsider a special case of the situation studied by Shahidi in [13] and obtain a sharper form of his results. In my thesis, I study a situation that is analogous to the one considered by Kutzko and Morris. But I consider a case of certain non-split classical groups. Pursuing the same basic strategy, I obtain an explicit reducibility result as described in more detail below.

2. Question. Let F be a non-Archimedean local field of characteristic not equal to 2 with uniformizer ϖ_F and finite residue field k_F of order q. Let D be the unique quaternionic division algebra of degree 4 over F. Then the residue field k_D of D has order q^2 and we have the ring of integers $\mathcal{O}_D$ with unique maximal ideal $\mathfrak{p}_D$. We then have the usual involution on D: $x \mapsto \bar{x}$. For $\varepsilon = \pm 1$ we let

$$J_\varepsilon = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}$$

and define the group

$$G_\varepsilon = \{g \in \mathrm{GL}_{2n}(D) \mid g^* J_\varepsilon g = J_\varepsilon\}$$

where for $g = (g_{ij})$, $g^* = (\overline{g_{ij}})^\top = (\overline{g_{ji}})$. Then G_1 is an inner form of $\mathrm{Sp}_{4n}(F)$ and G_{-1} is an inner form of $\mathrm{SO}_{4n}(F)$. Because the final result will be independent of ε, we write G for G_ε in what follows. However, we point out that the two cases $\varepsilon = \pm 1$ are treated separately in many proofs.

Let P be the Siegel parabolic subgroup of G with Levi-factor $L \simeq \mathrm{GL}_n(D)$ and unipotent radical U. Let π be an irreducible unitary supercuspidal representation of L of *depth zero*. That π has depth zero means that π contains an irreducible representation ρ_L of the compact open subgroup $K = \mathrm{GL}_n(\mathcal{O}_D)$ of L where ρ_L is inflated from an irreducible cuspidal representaion of the finite quotient $\mathrm{GL}_n(k_D)$ such that

$$\pi = \mathrm{ind}_{\widetilde{K}}^{L} \widetilde{\rho_L}.$$

Here $\widetilde{K}$ is some compact-mod-center extension of K and $\widetilde{\rho_L}$ is a suitable extension of ρ_L to $\widetilde{K}$. We can now form the normalized parabolically induced representation $\iota_P^G(\pi)$ and ask the question:

When is $\iota_P^G(\pi)$ reducible?

3. Answer. With ω_π being the central character of π the answer is

Theorem 3.1. *The induced representation $\iota_P^G(\pi)$ is reducible if and only if π is self-dual and $\omega_\pi(\varpi_F) = -1$.*

In [8] Muić and Savin have criteria for reducibility of $\iota_P^G(\pi)$ without the restriction that π has depth zero. Their approach uses *global methods* and the result is expressed in terms of the Jacquet-Langlands correspondence. Specific information about this correspondence is required to make their result explicit. This is not hard to do in our depth zero case, but it is also not trivial. I have adopted a *local approach* answering the question using Bushnell and Kutzko's method of types and covers described in [3]. The method translates the problem into one about certain Hecke algebra modules.

With $\mathfrak{R}(G)$ being the category of all smooth representations of G, we consider a certain subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$. The different $\mathfrak{R}^{\mathfrak{s}}(G)$ are the factors in the Bernstein decomposition of $\mathfrak{R}(G)$. We likewise get a category $\mathfrak{R}^{\mathfrak{s}_L}(L)$ of certain smooth representations of the fixed Levi-factor L. With this notation we have normalized parabolic induction

$$\iota_P^G : \mathfrak{R}^{\mathfrak{s}_L}(L) \to \mathfrak{R}^{s}(G).$$

We now have the Siegel parahoric

$$\mathcal{P} = \begin{pmatrix} \mathcal{O}_D & \mathcal{O}_D \\ \mathfrak{p}_D & \mathfrak{p}_D \end{pmatrix} \cap G.$$

With U^- being the opposite of U we use the Iwahori decomposition

$$\mathcal{P} = (\mathcal{P} \cap U^-)(\mathcal{P} \cap L)(\mathcal{P} \cap U)$$

to extend ρ_L from $K \simeq \mathcal{P} \cap L$ to a representation ρ of $\mathcal{P}$ by defining it to be trivial on $\mathcal{P} \cap U^-$ and $\mathcal{P} \cap U$.

Part of the theory of types and covers are functors

$$m_G : \mathfrak{R}^{\mathfrak{s}}(G) \longrightarrow \mathcal{H}(G,\rho) - \mathrm{Mod} \qquad \text{and}$$
$$m_L : \mathfrak{R}^{\mathfrak{s}_L}(L) \longrightarrow \mathcal{H}(L,\rho_L) - \mathrm{Mod}.$$

Here the Hecke algebra $\mathcal{H}(G,\rho)$ is defined by

$$\mathcal{H}(G,\rho) = \{f : G \to \mathrm{End}_{\mathbb{C}}(\rho) \mid \mathrm{supp}(f) \text{ is compact and } f(p_1 g p_2) = \rho(p_1) f(g) \rho(p_2)\ \forall p_i \in \mathcal{P},\ g \in G\},$$

and $\mathcal{H}(L,\rho_L)$ is defined likewise. We are suppressing the technical details, but from them follow a key component of the theory of types and covers, namely that both of these functors are equivalences of categories. This all means that we get the following commutative diagram which allows us to transfer the question at hand to a question about modules over Hecke algebras. We have

$$\begin{tikzcd}
\mathfrak{R}^{\mathfrak{s}}(G) \arrow[r] & \mathcal{H}(G,\rho) - \mathrm{Mod} \\
\mathfrak{R}^{\mathfrak{s}_L}(L) \arrow[r] \arrow[u, "\iota_P^G"] & \mathcal{H}(L,\rho_L) - \mathrm{Mod} \arrow[u, "(t_P)_*"']
\end{tikzcd}$$

where $(t_P)_*$ is the map induced by an embedding of $\mathbb{C}$-algebras $t_P : \mathcal{H}(L,\rho_L) \hookrightarrow \mathcal{H}(G,\rho)$. All of this comes from Bushnell and Kutzko's theory. The real work is trying to find the Hecke algebras.

Relying on work by Lusztig (see [5]), we prove:

Theorem 3.2. *The Hecke algebra $\mathcal{H}(G,\rho)$ has the presentation*

$$\mathcal{H}(G,\rho) \simeq \mathcal{H} = \langle s_1, s_2 \mid s_i^2 = 1 + (q^n - q^{-n}) s_i \rangle.$$

In addition to this, it is readily known that $\mathcal{H}(L,\rho_L) \simeq \mathcal{D} = \mathbb{C}[d,d^{-1}]$ where d is an indeterminate. Composing maps we then get equivalences of categories $M_G : \mathfrak{R}^{\mathfrak{s}}(G) \longrightarrow \mathcal{H} - \mathrm{Mod}$ and $M_L : \mathfrak{R}^{\mathfrak{s}_L}(L) \longrightarrow \mathcal{D} - \mathrm{Mod}$. And so induction corresponds to a map

$$\iota_* : \mathcal{D} - \mathrm{Mod} \to \mathcal{H} - \mathrm{Mod}.$$

The irreducible representations in $\mathfrak{R}^{\mathfrak{s}_L}(L)$ are

$$\mathrm{Irr}\, \mathfrak{R}^{\mathfrak{s}_L}(L) = \{\pi_\chi = \pi \otimes \chi \mid \chi \text{ unramified character of } L\}.$$

With $\mathcal{D}$ being commutative, we have that irreducible $\mathcal{D}-$modules are characters $\psi : \mathcal{D} \to \mathbb{C}$. And each such character is determined by its value at d. For ψ unitary, it is well-known that $\iota_*(\psi)$ is reducible if and only if $\psi(d) = -1$.

Lemma 3.3.

$$M_L(\pi_\chi)(d) = \omega_{\pi_1}(\varpi_F)\chi(\varpi_F).$$

Noting that $\pi_1 = \pi$ we finally have that $\iota_P^G(\pi)$ is reducible if and only if $\iota_*(M_L(\pi))$ is reducible if and only if

$$M_L(\pi)(d) = \omega_\pi(\varpi_F) = -1.$$

From [8] and [10] we get the necessary condition that π be self-dual for $\iota_P^G(\pi)$ to be reducible. Using this we can state the above slightly differently: $\iota_P^G(\pi)$ is reducible if and only if

$$\pi \text{ is self-dual and } \omega_\pi \neq 1.$$

4. Future questions. I plan to use the method of types and covers to look at other reducibility questions for depth zero representations. By work of Morris [6], the relevant types are known. Moreover Morris in [7] has also described the corresponding Hecke algebras. To use these results to study reducibility problems one would need to make parts of this description explicit. In particular, one would need to know the precise parameters that control the quadratic relations in the Hecke algebras. These can be extracted from Lusztig's work. There are subtleties, however, in using this work in the p-adic setting (as in the critical sign ε_G of [4]).

A natural case to examine is that of p-adic unitary groups. These are (outer) forms of general linear groups. I plan to look at reducibility questions for these groups where the inducing representation is supercuspidal of depth zero but without the restriction that the inducing parabolic be (proper) maximal.

The arguments in my thesis rely heavily on the hypothesis that the inducing representation π has depth zero. I am interested in trying to remove this assumption. Sécherre and Stevens have constructed types for the supercuspidal representations of general linear groups over division algebras [11]. I would need to construct covers for these types in the setting of quaternionic hermitian and anti-hermitian groups and would then need to describe the corresponding Hecke algebras. I have no direct experience, however, in the construction of types. As a first step in this problem, I plan to look at the 2×2 case. Many of the complications of the general case are absent here as the Siegel Levi is $\mathrm{GL}_1(D)$ whose irreducible representations can be described in a straightforward manner (see chapter 13 in [1]).

I am also interested in using the method of types and covers to obtain explicit Plancherel formulas for depth zero representations. This is an interesting question in its own right but is also closely related to reducibility questions. In [2], Bushnell, Henniart and Kutzko establish a framework that allows the transfer of Plancherel measure from the Hecke algebra of a type to the corresponding p-adic group. Opdam and others have done extensive work on general Plancherel formulas for affine Hecke algebras (see [9]). By combining some of this work with explicit descriptions of the Hecke algebras in [6], my hope is to in certain cases give a full Plancherel formula for the depth zero spectrum. In particular, I plan to study this problem for unitary groups.

References.

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