Homework #10: In Deep with Determinants
–due Wednesday November 11th

(1) (a) The grader hasn’t been to class, and is surely unfamiliar with the language of “slices” and "slice products", etc. Please write out our definition of the determinant from class. Use complete sentences, as if you were writing a book. Your target audience is your classmates two weeks ago.

(b) Suppose that $A$ is a square matrix, and that $B$ is obtained from $A$ by multiplying one of the columns of $A$ by a number $c$. Explain why $\det(B) = c \det(A)$, in terms of the definition you gave above.

(2) 3.2.2 (a-e), but you may also use the slice product method or the Laplace (cofactor) expansion if you prefer. Just make clear what you’re doing.

(3) 3.2.3, 3.2.4

(4) 3.2.8, 3.2.9, 3.2.10. (These should be pretty easy; you can use Theorem 3.9 in the text for all of them.)

(5) (a) 3.2.22. Suggestion: First take the transpose. Then use row operations to clear out the first column under the leading one. Next, use the Laplace expansion applied to the first row. You’re essentially left with a $2 \times 2$ determinant, which should be easy to compute. Try to factor things out as much as possible, so that you wind up with the answer in the book, rather than expanding everything out.

(b) Let $a_1, a_2, b_1, b_2$ be numbers. Compute and factor

$$
\det \begin{pmatrix} 1 & 1 & 1 \\
1 & a_1 & a_1 \\
b_1 & b_2 & a_2 
\end{pmatrix}.
$$

Hint: $(a_1 - b_1)$ is one of the factors.

(c) Let $a_3, b_3$ be more numbers. Compute and factor

$$
\det \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & a_1 & a_1 & a_1 \\
b_1 & b_2 & a_2 & a_2 \\
b_1 & b_2 & b_3 & a_3 
\end{pmatrix}.
$$
Suggestion: Use row operations to clear out the first column under the leading one. Next, use the Laplace expansion applied to the first row. You’re essentially left with a $3 \times 3$ determinant. For one of the rows of this $3 \times 3$ matrix, there is an obvious common factor. After you factor it out, apply part (b) above.

(6) In class we derived the Laplace expansion by organizing the permutations into groups according to which number comes first. This allows you, for example, to express a $4 \times 4$ determinant as a linear combination of four $3 \times 3$ determinants. In this problem we will derive an expression for the determinant of a $4 \times 4$ matrix as a combination of certain $2 \times 2$ determinants multiplied by other $2 \times 2$ determinants. The approach is to organize the permutations of 1234 according to which two numbers come first. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. $$

There are two slices made from choosing $a_{11}$ and $a_{22}$ as the first two entries. These correspond to the permutations of 1234 starting with 12.

(a) What are the two slices?

Note that the contribution to the determinant of $A$ from these two slices is $a_{11}a_{22} \det(M_{(1,2)\times(1,2)})$. By $M_{(i_1,i_2)\times(j_1,j_2)}$ we generally mean the $2 \times 2$ submatrix of $A$ formed by deleting both the $i_1$ and $i_2$ rows, as well as both the $j_1$ and $j_2$ columns. Thus here for instance,

$$M_{(1,2)\times(1,2)} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}. $$

Now consider the two slices corresponding to the permutations of 1234 starting with 21.

(b) What are they? What is their contribution to the determinant of $A$? Try to involve the factor $\det(M_{(1,2)\times(1,2)})$ in your expression.
If you did this correctly, then when you add the contributions from permutations starting with 12 to the ones starting with 21 you’ll get

$$\text{det}(M_{(3,4)\times(3,4)}) \cdot \text{det}(M_{(1,2)\times(1,2)}) \cdot \text{det}(M_{(3,4)\times(3,4)}) \cdot \text{det}(M_{(1,2)\times(1,2)})$$.

There will be 5 other terms like this. The other terms are also products of two “complementary” $2 \times 2$ submatrices...but with signs $\pm$ that you’ll need to figure out.

(c) Figure out the sum of the contributions of the permutations starting with 24 and 42. Write it as a product of $2 \times 2$ minors as above, but with the correct sign.

(d) Now finish the problem by expressing $\text{det}(A)$ as a sum of six products of two $2 \times 2$ minors, with the correct signs.

(e) Use your formula from the previous part to compute the determinant of the matrix of Example 2 on page 159 of the text. That is, illustrate your formula with

$$A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{pmatrix}$$.

Remark: This method generalizes to any size. If $A$ is an $n \times n$ matrix, one can write $\text{det}(A)$ as a (signed) sum of \binom{n}{2} products of $2 \times 2$ minors with their complementary $(n-2) \times (n-2)$ minors.