Homework #12: The Eigentheory
– due Friday December 11th

(1) 7.1.8 but add (e) \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \).

(2) 7.1.18

(3) 7.2.10

(4) 7.2.19 plus find a matrix \( B \) (possibly with complex entries) so that \( B^2 = A \).

(5) (a) Let \( a_1, a_2, a_3 \) be numbers. Find the characteristic polynomial of

\[
A = \begin{pmatrix} 0 & 0 & a_1 \\ 1 & 0 & a_2 \\ 0 & 1 & a_3 \end{pmatrix}.
\]

(b) Find a matrix with integer entries which has \( \sqrt[3]{2} \) as an eigenvalue. (Use the previous part. What is a polynomial with \( \sqrt[3]{2} \) as a root?)

(c) Let \( a_1, a_2, \ldots, a_n \) be numbers. Find the characteristic polynomial of

\[
A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}.
\]

(6) Let \( a, b, c \) be distinct numbers, and \( r, s, t \in \mathbb{R} \). Let’s consider the problem of finding a quadratic polynomial \( p(x) = d_0 + d_1 x + d_2 x^2 \) so that \( p(a) = r, p(b) = s, \) and \( p(c) = t \).

(a) Consider the function \( L : P_2 \to \mathbb{R}^3 \) given by \( L(p) = (p(a), p(b), p(c)) \). Show that \( L \) is a linear transformation.

(b) Write down the matrix associated to \( L \) using the ordered basis \( \{1, x, x^2\} \) of \( P_2 \) and the standard basis of \( \mathbb{R}^3 \).

(c) Find the nullspace of \( L \). What is its dimension? (Hint: How many roots can a quadratic or linear polynomial have?) What, then, is the rank of \( L \)?

(d) To double check the previous problem: Can the determinant of the associated matrix be 0? (Recall problem 3.2.22.)

(e) So, is there necessarily a quadratic polynomial \( p(x) = d_0 + d_1 x + d_2 x^2 \) so that \( p(a) = r, p(b) = s, \) and \( p(c) = t \)? (Think about what the rank of \( L \) means.)