Department of Mathematics
M.A. Comprehensive/Ph.D. Qualifying Exam in Analysis
Summer, 1990

Directions: Work any eight problems. You have three hours.

1. Let $(X, \mathcal{N}, \mu)$ be a measure space.
   (a) State the following theorems.
   (i) Monotone Convergence Theorem
   (ii) Fatou's Lemma
   (iii) Dominated Convergence Theorem
   (b) Prove that (i) implies (ii).

2. Suppose $g \in L_1(\mathbb{R})$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subseteq \mathbb{R}$ is measurable and $\lambda(A) < \delta$, then $|\int_A g d\lambda| < \epsilon$.

3. Let $\{f_k\}$ and f be non-negative measurable functions on $\mathbb{R}$. Suppose that for all k, $f_k(x) \leq f(x)$ a.e. on $\mathbb{R}$. Prove that $\lim_{k\to\infty} \int_{-\infty}^{\infty} f_k d\lambda = \int_{-\infty}^{\infty} f d\lambda$.

4. (a) Prove that if a sequence $\{f_k\}$ in $L_p(\mathbb{R})$ converges to $f \in L_p(\mathbb{R})$ in the L_p norm, then $\{f_k\}$ converges to f in measure.
   (b) Give an example of a sequence $\{f_k\}$ in $L_p(\mathbb{R})$ such that $\{f_k\}$ converges to $f \in L_p$ in the L_p norm, but $\{f_k\}$ does not converge pointwise to f.

5. Let $1 \leq p < \infty$, and suppose $\{f_k\}$ is a sequence in $L_p([1, \infty))$ which converges to $f \in L_p([1, \infty))$ in the L_p norm. Prove that

$$\lim_{k\to\infty} \int_1^{\infty} \frac{f_k(x)}{x} d\lambda = \int_1^{\infty} \frac{f(x)}{x} d\lambda\,.$$

6. (a) State the Baire Category Theorem.
   (b) Prove there exists a set E of first category in R such that $\lambda(E) > 0$.

7. (a) Show that if f and g are absolutely continuous (AC) on $[a, b]$, then $f \cdot g$ is AC on $[a, b]$.

(b) Show that if f and g are AC on $[a, b]$ then

$$\int_a^b fg' d\lambda = f(b)g(b) - f(a)g(a) - \int_a^b f' g d\lambda .$$

8. (a) Suppose $(X, \mathcal{N}, \mu)$ is a measure space. Show that if E_k is a sequence of sets in $\mathcal{N}$ such that $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < \infty$, then

$$\mu(\cap_{k=1}^{\infty} E_k) = \lim_{k \to \infty} \mu(E_k) .$$

(b) Show by example that the statement in (a) is false if $\mu(E_1) = \infty$.

9. Suppose f and g are positive measurable functions on $\mathbb{R}$, and define $h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ for $x \in \mathbb{R}$. Show, justifying each step, that

$$\int_{-\infty}^{\infty} e^x h(x)dx = \big(\int_{-\infty}^{\infty} e^x f(x)dx\big)\big(\int_{-\infty}^{\infty} e^x g(x)dx\big) .$$

10. (a) Let μ and ξ be measures on a σ-algebra $\mathcal{N}$ of subsets of X. Prove that if $\xi << \mu$ and $\xi \perp \mu$, then $\xi(E) = 0$ for all $E \in \mathcal{N}$.
(b) State the Lebesgue Decomposition Theorem for measures.