19 AUGUST 1991

ALGEBRA QUALIFYING EXAM

There are four parts to this exam: Be sure you do at least one problem from each part, and you should try to finish seven problems altogether from among the choices offered in each section. Every problem has equal weight.

Group Theory: *Do 2 of the following 3 problems.*

1. Prove that an abelian group is simple if and only if it is finite and of prime order.
2. Show that if H is a subgroup of index n in a finite group G, then the number of conjugates of H in G divides n.
3. Prove that there are no simple groups of order $495 = 3^2 \cdot 5 \cdot 11$.

Ring Theory: *All rings have an identity, and all ring homomorphisms map the identity of the domain to the identity of the target. Do 2 of the following 3 problems.*

4. Prove that an ideal M in a commutative ring R is maximal if and only if for every $r \in R - M$ there is some $x \in R$ such that $rx - 1 \in M$.
5. Prove that
   (a) $\mathbf{Z}[X]/(2,\, X^2)$ is a 4-element ring which is not a field, whereas
   (b) $\mathbf{Z}[X]/(2,\, X^2 + X + 1)$ is a 4-element field.
6. Prove that if R is a ring which contains a field K, and R has no zero divisors, and R is finite-dimensional as a vector space over K, then R is a division ring.

Module Theory: *All rings have an identity. Do 1 of the following 2 problems.*

7. Let A, B and C be modules over a commutative ring R. Prove that
$$\mathrm{Hom}_R(A \otimes B,\, C) \simeq \mathrm{Bilin}_R(A, B;\, C),$$
where $\mathrm{Bilin}_R(A, B;\, C)$ denotes the group of all R-bilinear homomorphisms from $A \times B$ to C.
8. Prove that $A = \mathbf{Z}[X]/(X - 2)$ and $B = \mathbf{Z}[X]/(X - 3)$ are isomorphic as $\mathbf{Z}$-modules (i.e., as abelian groups), but not isomorphic as $\mathbf{Z}[X]$-modules. (*Hint: It is true in general, and may be as easy to prove, that whenever I and J are distinct ideals in a commutative ring R then R/I and R/J are not isomorphic as R-modules.*)

Field Theory: *Do 2 of the following 3 problems.*

9. Let K be a field and let f be an irreducible polynomial in $K[X]$ of prime degree p. Let F be an extension of K over which f is no longer irreducible. Show that p divides $[F : K]$.
10. Let F be a finite field of characteristic p. Describe completely the structure of both the additive group $(F, +)$ and the multiplicative group $F^\times$ of F (as abelian groups). (*Hint: F is a vector space over its prime field.*)
11. Describe the splitting field and identify the Galois group of $x^6 - 5$ over $\mathbf{Q}$.