MA/PhD qualifying examination in Topology. May, 1992

Instructions: Work as many problems as you can. Give clear and concise arguments; do not waste time by giving excessive detail. Apply major theorems when possible.

I. Let $\prod X_\alpha$ be a product of a collection of nonempty spaces. Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be the projection. Prove that π_β takes open sets to open sets.

II. For $r \in \mathbb{R}$, define A_r to be $\{r+n \mid n \in \mathbb{Z}\} \subseteq \mathbb{R}$. Let $U_r = \mathbb{R} - A_r$. Find an explicit locally finite open cover of $\mathbb{R}$ that is a refinement of $\{U_r \mid r \in \mathbb{R}\}$.

III. Prove that if X is a regular space, $x \in X$, and U is a neighborhood of x, then there exists a neighborhood V of x such that $\overline{V} \subseteq U$.

IV. Let $I = [0,1]$ and let $X = \mathcal{C}(I, I)$, the space of continuous maps from I to I. Prove that X is not equicontinuous.

V. Recall that a subspace A of X is called a *retract* of X if there exists a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Let $i : [0,1] \to \mathbb{R}^n$ be an imbedding. Prove that $i([0,1])$ is a retract of $\mathbb{R}^n$.

VI. Let X be a Möbuis band. Let $h : S^1 \to X$ be an imbedding which carries S^1 homeomorphically onto the boundary of X. Let $s_0 \in S^1$. What homomorphism is $h_* : \pi_1(S^1, s_0) \to \pi_1(X, h(s_0))$? Explain your answer.

VII. Let A be a closed subset of a metric space X and let X^* be the quotient space obtained from X by collapsing A to a point. Prove that X^* is Hausdorff.

VIII. Let $Y = \prod_{i=1}^\infty \mathbb{R}$ (with the product topology).

(a) Let $X_1 = \{(x_n) \in Y \mid \sum x_n$ converges and $\sum x_n = 1\}$. Prove or disprove that X_1 is compact.

(b) Let $X_2 = \{(x_n) \in Y \mid \sum x_n^2$ converges and $\sum x_n^2 = 1\}$. Prove or disprove that X_2 is compact.

IX. Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the natural numbers, let Y be a metric space, and let $f : X \to Y$ be a function. Prove that f is continuous if and only if the sequence $\{f(n)\}$ converges to $f(\infty)$.

X. Prove that if $f_0, f_1 : X \to Y$ are homotopic maps, and $g_0, g_1 : Y \to Z$ are homotopic maps, then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic maps. Prove that if X is a space and Y is a contractible space, then any two maps from X to Y are homotopic.