MA/PhD qualifying examination in Topology. August, 1992

Instructions: Work as many problems as you can. Give clear and concise arguments; do not waste time by giving excessive detail. Apply major theorems when possible.

I. Let X be a space and let A and B be connected subsets of X. Prove that if $A \cap \overline{B}$ is nonempty, then $A \cup B$ is connected.

II. Let X be a Hausdorff space. Prove that if every open subspace of X is paracompact, then every subspace of X is paracompact.

III. Let z_n be a sequence of points in a product space $\prod_{\alpha\in\mathcal{A}} X_\alpha$, and let $\pi_\beta\colon \prod_{\alpha\in\mathcal{A}} X_\alpha \to X_\beta$ denote the projection map. Prove that the sequence z_n converges if and only if for every $\beta \in \mathcal{A}$, $\pi_\beta(z_n)$ converges in X_β.

IV. Let X be a normal space and let $A = \{a_1, a_2, \ldots\}$ be a countable subset such that $a_i \neq a_j$ for $i \neq j$ and such that A has no limit points in X.

(a) Prove that the subspace topology on A equals the discrete topology on A.
(b) Prove that there exists an unbounded continuous function from X to $\mathbb{R}$.

V. Let $f\colon [0,1] \to S^1$ be defined by $f(t) = e^{6\pi i t}$. Find explicitly the lift of f to the standard covering space $p\colon \mathbb{R} \to S^1$ with initial point 2.

VI. Let X be the quotient space obtained from the real line by collapsing the open interval $(0,1)$ to a point. Prove that X is not Hausdorff.

VII. Let X be the one-point compactification of the integers. Construct an imbedding of X into the real line $\mathbb{R}$.

VIII. Let X be the reals with the lower limit topology. Prove that X is separable but not second countable.

IX. Let (X,d) be a metric space whose diameter equals 4. Let $\mathcal{U}$ be the open cover $\{X - \{x\} \mid x \in X\}$. Prove that 3 is a Lebesgue number for $\mathcal{U}$.