Qualifying Exam in Topology

May, 1993

Instructions: Work as many problems as you can. Justify your answers with clear and concise arguments.

1. State the following theorems:

a) Baire Category

b) Brouwer Fixed-point

c) Tychonoff Compactness

d) Tietze Extension

e) Urysohn Metrization

2. Let X be a metric space. Prove that:

a) For any $A \subseteq X$, $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ defines a continuous function from X to $\mathbb{R}$.

b) $\overline{A} = \{x \in X \mid d(x, A) = 0\}$

c) X is a normal topological space.

3. Given a function $f : X \to Y$, show that f is continuous if any of the following hold:

a) For every x in X there is an open $U \subseteq X$, containing x, so that $f|_U$ is continuous.

b) $X = \bigcup_{i=1}^{n} A_i$ where A_i is closed and $f|_{A_i}$ is continuous for $i = 1, 2, \ldots, n$.

c) $X = \bigcup_{\alpha \in \Lambda} A_\alpha$ where A_α is closed, $\{A_\alpha\}_{\alpha \in \Lambda}$ is a locally finite collection of closed subsets and $f|_{A_\alpha}$ in continuous for each $\alpha \in \Lambda$.

4. Prove:

a) A subspace of a normal space is completely regular.

b) A locally compact Hausdorff space is completely regular.

(*Hint: Consider the one-point compactification.*)

5. Let A be a subspace of a regular space X. Show that X/A is Hausdorff if and only if A is closed.

6. For each of the following pairs of spaces, prove or disprove that the two spaces are homeomorphic:
a) $(0,1)$ and $(0,\infty)$
b) $[0,1)$ and $[0,1]$
c) $(0,1)$ and $[0,1)$
d) $[0,1]$ and $[0,1]\times[0,1]$
e) $[0,1)\times[0,1]$ and $[0,1)\times[0,1)$.
(*If the formula needed to describe a homeomorphism is complicated, a clearly explained sequence of pictures is acceptable.*)

7. Let D be the metric on $\prod_{i=1}^{\infty}[0,\frac{1}{3^i}]$ defined by $D\left((x_i),(y_i)\right)=\sup\{|x_i-y_i|\}$ (do not prove that D is a metric).
a) Prove that D induces the product topology.
b) Prove that the restriction of the function $f:\prod_{i=1}^{\infty}[0,\frac{1}{3^i}]\to[0,1]$ defined by $f(x_1,x_2,\dots)=2\sum_{i=1}^{\infty}x_i$ to the subset $\prod_{i=1}^{\infty}\{0,\frac{1}{3^i}\}$ is a homeomorphism onto the Standard Cantor set.

8. Prove that a homotopy equivalence $f:X\to Y$ induces a one-to-one correspondence between the path components of X and Y.

9. Show that for $n>1$ there are no essential maps:
a) S^n to S^1,
b) S^1 to S^n.
(*You may use the fact that S^n is simply connected for $n>1$.*)

10. Prove the following weak version of the Seifert-van Kampen theorem: If $X=U\cup V$ where U,V are open, $U\cap V$ is path connected and x is in $U\cap V$ then $\pi_1(X,x)$ is generated by the images of $\pi_1(U,x)$ and $\pi_1(V,x)$ in $\pi_1(X,x)$.