ALGEBRA QUALIFYING EXAM

August 1993

0. INSTRUCTIONS

There are four parts to the exam. You are to work two, and **only two**, problems from each part. Each of the eight problems attempted will have equal weight and partial credit will be given.

I. GROUPS

Work two of the following problems.

1. Let G be a simple group whose order is strictly greater than two. If $\phi : G \to S_n$ is a homomorphism of G into the symmetric group of degree n, prove that $\mathrm{I}m\phi$ is contained in the alternating group A_n. (*Hint: It is the normality, not the simplicity, of A_n which is relevant to this problem*).

2. Let k be a field, $G = GL_2(k)$. The sets $S = \{a \in GL_2(k) | \det(a) = 1\}$ and $\Delta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} | 0 \neq \delta \in k \right\}$ are subgroups of $GL_2(k)$.
   (a) Define $\theta : \Delta \to \mathrm{A}ut(S)$ by $\theta(d) = \theta_d$, where $\theta_d(a) = dad^{-1}$. Prove that θ is a group homomorphism.
   (b) Prove that the semidirect product $S \rtimes_\theta \Delta$ is isomorphic to $GL_2(k)$.

3. Let p and q be distinct prime numbers. Prove that any group of order pq is solvable.

II. RINGS

Work two of the following problems.

1. Let R_1 and R_2 be rings, $R = R_1 \times R_2$ their direct product. Prove that every ideal of the ring R has the form $I = \{(a, b) | a \in I_1, b \in I_2\}$, where I_1 is an ideal of R_1 and I_2 is an ideal of R_2.

2. In the polynomial ring $\mathbb{Z}[X]$ let $f = X^3 - X + 2$.
   (a) Prove that $P = f\mathbb{Z}[X]$ is a prime ideal of $\mathbb{Z}[X]$.
   (b) Prove that P is **not** a maximal ideal of $\mathbb{Z}[X]$.

3. Let R be a commutative domain, I an ideal of R. As is easily verified, the set $S = \{1 + a | a \in I\}$ is a multiplicatively closed subset of R. Prove that the extended ideal $S^{-1}I$ is contained in every maximal ideal of the localization $S^{-1}R$.

III. FIELDS

Work two of the following problems.

1. Let f be a nonzero irreducible element in the polynomial ring $\mathbb{C}[x, y]$. If F is the field of fractions of $\mathbb{C}[x, y]/(f)$, prove that the transcendence degree of F over $\mathbb{C}$ is exactly one.

2. Let E/k be a finite Galois extension, $G = \mathrm{A}ut(E/k)$ the Galois group of the extension.
   (a) For each $\alpha \in E$, let $S_\alpha = \{g \in G | g(\alpha) = \alpha\}$. Verify that S_α is a subgroup of G.
   (b) Now let H be an arbitrary subgroup of G. Prove that $H = S_\alpha$ for some $\alpha \in E$.

3. Compute the Galois group over $\mathbb{Q}$ of the polynomial $X^6 - 3$.

IV. MODULES

Work two of the following problems.

1. Let R be a principal ideal domain, F a free R-module of finite rank. Let $\phi : F \to F$ be an R-endomorphism of F, $K = \mathrm{K}er\phi$. Prove that there exists an R-submodule L of F such that $K \oplus L = F$. (*Hint: Be careful; "most" submodules of F are not summands of F*).

2. Let R be a commutative ring, I an ideal of R, $L = \{a \in R | aI = 0\}$.
   (a) Prove that each $a \in L$ induces an R-homomorphism $\overline{\lambda}_a : R/I \to R$.
   (b) Using (a), prove that the R-modules L and $\mathrm{H}om_R(R/I, R)$ are isomorphic.

3. Let R be a commutative Noetherian ring, $R[X]$ the ring of polynomials over R, I an ideal of $R[X]$. Prove that if $R[X]/I$ is a finitely generated R-module, then I contains a monic polynomial.