Qualifying Exam in Topology

August 1993

Instructions: Work as many problems as you can. Justify your answers with clear and concise arguments.

1) Prove that compact regular spaces are normal.

2) If a subspace A of $\mathbb{R}^n$ has the property that every continuous $f : A \to \mathbb{R}$ has a maximum then show that A must be compact.

3) Let X be a complete metric space and assume $f : X \to X$ satisfies $d\,(f(x), f(y)) < t\,d(x, y)$ for some $0 < t < 1$ and for all x, y in X. Prove that f has a unique fixed point.

4) Prove that a retract of a Hausdorff space must be a closed subspace.

5) Let X be an uncountable set with the discrete topology.
   a) Determine all compact subsets of X.
   b) Prove that the one-point compactification $\widehat{X}$ cannot be imbedded into the plane $\mathbb{R} \times \mathbb{R}$.

6) Let C be the standard Cantor set contained in the real line $\mathbb{R}$, and let X be a subspace of the 2-sphere S^2. Suppose there exists a homeomorphism $f\colon X \to C$. Prove that there exists a continuous map $F\colon S^2 \to \mathbb{R}$ whose restriction to X equals f.

7) Define a metric on a countably infinite product of $[0, 1]$'s inducing the product topology (verify that it induces the product topology).

8) Prove
   a) If $f, g : X \to S^n$ are continuous and $f(x) \neq -g(x)$ for all x in X, then f is homotopic to g.
   b) A continuous $f\colon S^n \to S^n$ either has a fixed point or is homotopic to the antipodal map.

9) Prove or disprove:
   a) If $f : X \to Y$ is continuous and injective, then $f_\# : \pi_1(X, x) \to \pi_1\,(Y, f(x))$ is injective.
   b) If $f : X \to Y$ is continuous and surjective, then $f_\# : \pi_1(X, x) \to \pi_1\,(Y, f(x))$ is surjective.
   c) If $c\colon A \subseteq X$ is the inclusion and $r\colon X \to A$ is a retraction then $c_\#$ is injective and $r_\#$ is surjective.

10) Show that
    a) S^n is simply connected for $n \geq 2$.
    b) $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$ for $n \geq 3$.