TOPOLOGY QUALIFYING EXAM

August 1994

Work as many problems as you can. Give complete explanations, but try not to waste time verifying obvious details.

1. Suppose $f : X \to Y$ is a bijection, and $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$. Show that f is a homeomorphism.

2. Suppose X is compact and A is an infinite subset. Show directly from the definitions that A must have a limit point.

3. Let $\mathbb{R}^2_+ = \{(x, y) : y > 0\}$, and set

$$A = \{(x, y) \in \mathbb{R}^2_+ : (x, y) \text{ lies on a line of irrational slope passing through } (0, 0)\}$$
$$B = \{(x, y) \in \mathbb{R}^2_+ : (x, y) \text{ lies on a line of irrational slope passing through } (1, 0)\}$$

   a. Show $A \cup B$ is connected.
   b. Prove that every component of $\mathbb{R}^2_+ \setminus (A \cup B)$ is a single point.

4. Let X be a Hausdorff space. Prove that X is normal if and only if given any closed subset F and any open set $\mathcal{O}$ such that $F \subset \mathcal{O}$, there exists an open set $\mathcal{U}$ such that $F \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{O}$.

5. Let $X = \prod_{i=1}^{\infty} [0, 1]$, and let $Y = \{x \in X \mid \pi_i(x) = 0 \text{ except for finitely many } i\}$.
   a. Determine whether Y is compact when X is given the product topology.
   b. Determine whether Y is compact when X is given the box topology.

6. Suppose X is normal, and $A \subset X$ is closed.
   a. Prove that if $f : A \to \mathbb{R}^n$, then f extends continuously to X.
   b. Now suppose A is simply connected. Prove that if $f : A \to S^1$, then f extends continuously to X. (*Hint:* What is the universal cover of S^1?)

7. Let $X \subset \mathbb{R}^2$ be the set of vertical lines with integer x-intercepts.
   a. Describe the construction of $\widehat{X}$, the one-point compactification of X, and explain its topology.
   b. Determine whether $\widehat{X}$ is homeomorphic to the Hawaiian earring, that is, the union of all circles in $\mathbb{R}^2$ with center $(1/n, 0)$ and radius $1/n$ where n is a positive integer.
   c. Determine whether $\widehat{X}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$, that is, $\mathbb{R}$ with $\mathbb{Z}$ identified to a point.

Do **two** of the following three problems.

8. Let X denote the space resulting from the identification of the edges of a two-dimensional hexagon as indicated in the diagram below. Use Van Kampen's theorem to compute $\pi_1(X)$.

9. Let $C(Y, Z)$ be the space of continuous maps from Y to Z. Given $f : X \times Y \to Z$, define $\tilde{f} : X \to C(Y, Z)$ by $\tilde{f}(x)(y) = f(x, y)$. Show that if f is continuous then $\tilde{f}$ is continuous.

10. Let Y be locally path connected. Suppose $f : X \to Y$ is a local homeomorphism and that f has the path lifting property. Show that f is a covering map.