REAL ANALYSIS FALL 1995

M.A. Comprehensive/Ph.D. Qualifying Exam

In order to get full credit, you must provide complete justifications of your answers. You have three hours to complete as many questions as you can.

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1. State and prove the Schroeder-Bernstein Theorem. Use it to prove that, if X is an infinite set, then X has a proper subset of the same cardinality.

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2. Describe the construction of the classical Cantor set C. Explain (with full justifications) how to construct a continuous monotone function L on $[0,1]$ with $L(0) = 0$, $L(1) = 1$ and which is constant on the complementary intervals of C.

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3. Let S be an arbitrary set or real numbers. Define the outer measure $m^*(S)$. Prove that, if $\{A_n\}_{n=1}^{\infty}$ is a countable collection of subsets of $\mathbb{R}$, then

$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leqslant \sum_{n=1}^{\infty} m^*(A_n).$$

Explain what it means for the set $E \subset \mathbb{R}$ to be measurable. Prove that, if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of disjoint measurable subsets of $\mathbb{R}$, then

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m^*(E_n).$$

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4. Find the values of the following limits, making clear and justified uses of appropriate convergence theorems

$$\lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1+(nx)^2}\, dx \qquad\qquad \lim_{n\to\infty} \int_0^{\infty} \Big(1+\frac{x}{n}\Big)^{-n} \sin\frac{x}{n}\, dx.$$

5. State and prove Fatou's Lemma. Give an example to show that strict inequality can occur.

6. Let the functions f_α be defined by

$$f_\alpha(x) = \begin{cases} x \sin \dfrac{1}{x^\alpha}, & x > 0 \\ 0, & x = 0 \end{cases}$$

Find all the values of $\alpha \geqslant 0$ such that

a) f_α is continuous,
b) f_α is of bounded variation on the interval $[0, 1]$,
c) f_α is absolutely continuous on $[0, 1]$.