Topology Qualifying Examination, May 1997

1.) Show that if X and Y are connected spaces then so is $X \times Y$.

2.) Prove: A retract of a Hausdorff space X is closed in X. What if X is not Hausdorff?

3.) Prove:
a) If $f : X \longrightarrow Y$ is continuous and proper with X, Y locally compact Hausdorff spaces then f is closed.
b) If $f : \mathbb{R} \longrightarrow X$ is a proper embedding with X a locally compact metric space then $f(\mathbb{R})$ is a retract of X.

4.) Show that $f\colon X \longrightarrow Y$ is continuous if $f|_{F_\alpha}$ is continuous $\forall \alpha \in \Lambda$ where $\{F_\alpha\}_{\alpha\in\Lambda}$ is a locally finite covering of X by closed sets.

5.) Prove or disprove:
a) If X and Y are homeomorphic metric spaces and X is complete then Y must be.
b) If F_α is a closed subspace of X_α $\forall \alpha \in \Lambda$ then $\prod\limits_{\alpha\in\Lambda} F_\alpha$ is closed in $\prod\limits_{\alpha\in\Lambda} X_\alpha$.
c) If X_α is a locally compact Hausdorff space $\forall \alpha \in \Lambda$ then so is $\prod\limits_{\alpha\in\Lambda} X_\alpha$.
d) $\mathbb{CP}^1$ is homeomorphic to the one-point compactification of $\mathbb{C}$.

6.) Find all 3-fold covers of the figure 8 (i. e. the underlying topological space of the symbol ∞) up to equivalence, indicating which are regular.

7.) Let G be a finite group acting freely on S^n, $n > 1$, and let S^n/G be the quotient space. Show that any continuous $f\colon S^n/G \longrightarrow S^1$ is homotopically trivial.

8.) For continuous $\sigma\colon I \longrightarrow S^1$, let $D(\sigma) = \tilde{\sigma}(1) - \tilde{\sigma}(0)$ where $\tilde{\sigma}\colon I \longrightarrow \mathbb{R}$ is a lift of σ. Prove:
a) D is well-defined.
b) If $f\colon S^1 \longrightarrow S^1$, $q(t) = e^{2\pi i t}$ then $D(fq)$ is an integer.
c) If $f, g\colon S^1 \longrightarrow S^1$ and f is homotopic to g then $D(fq) = D(gq)$.
d) If $f\colon S^1 \longrightarrow S^1$ then f is homotopic to the function that sends z to $z^{D(fq)}$.

9.) Let $U = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_0 \neq 0\}$ and $V = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid (z_1, z_2) \neq (0,0)\}$. Show:
a) U is homeomorphic to $\mathbb{C}^2$.
b) V is homotopy equivalent to $\mathbb{CP}^1$.
c) $\mathbb{CP}^2$ is simply connected.