Ph. D. Qualifying Examination in Topology

August 1998

Instructions: Work as many problems as you can. Justify your answers with clear and concise arguments.

1. Suppose that X has a subspace D which is uncountable, and such that the subspace topology on D is the discrete topology. Prove that X is not second countable.

2. Let X be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the compact-open topology. For $n \geq 1$, define $f_n\colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \cos(x/n)$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ converges.

3. Let U be the open covering $\{(n-1, n+1) \mid n \in \mathbb{Z}\}$ of the real line. Find an explicit partition of unity subordinate to U.

4. Let $f\colon \overline{S_\Omega} \to \mathbb{R}$ be continuous. Prove that there exists $x \in S_\Omega$ such that the restriction of f to $[x, \Omega]$ is constant.

5. Let $f\colon S^1 \to T$ be an imbedding from the unit circle S^1 into the 3-dimensional torus $T = S^1 \times S^1 \times S^1$. Let C be the image of f. Let $g\colon S^1 \to \mathbb{R}^3$ be an imbedding, whose image is a trefoil knot K. Define $h\colon C \to K$ by $h(x) = gf^{-1}(x)$. Prove that there exists a continuous map $H\colon T \to \mathbb{R}^3$ so that the restriction of H to C equals h.

6. Let X be a space which is not connected, and let Y be $\prod_{i=1}^{\infty} X$, with the product topology. Let $y = (x_i)_{i=1}^{\infty}$ be a point in Y. Prove that Y is not locally connected at y.

7. Let $X \subset \mathbb{R}$ be the subspace of rational numbers. Prove that there is no complete metric inducing the topology on X.

8. Let $X \subset \mathbb{R} \times \mathbb{R}$ be the subspace $\{\frac{1}{n} \mid n = 1, 2, 3, \ldots\} \times (-2, 2)$. For $n \geq 1$ let $x_n = (\frac{1}{n}, (-1)^n)$. Let $\widehat{X}$ be the 1-point compactification of X. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to ∞ in $\widehat{X}$.

9. Let X_1 and X_2 be two disjoint copies of the 2-sphere, and for $i = 1, 2$, let $x_{i,N}$ be the north pole of X_i and let $x_{i,S}$ be the south pole of X_i. Form a quotient space Y from the disjoint union $X_1 \coprod X_2$ by identifying $x_{1,N}$ with $x_{2,N}$, and $x_{1,S}$ with $x_{2,S}$. Prove that Y is not simply-connected. (Hint: Retract Y to a subspace which is not simply-connected.)

10. Let $X = \prod_{i=1}^{\infty} \mathbb{Z}$ with the product topology. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X, and denote the i^{th} coordinate of x_n by $x_{n,i}$. Let $S \subseteq \mathbb{Z}$ be the set of integers that appear as the coordinate of some x_n, that is, $S = \{m \in \mathbb{Z} \mid \text{for some } n \text{ and some } i,\ m = x_{n,i}\}$. Let $f\colon X \to \mathbb{R}$ be a continuous function. Prove that if the set of values $\{f(x_n) \mid n \geq 1\}$ is not a bounded subset of $\mathbb{R}$, then S is not a bounded subset of $\mathbb{Z}$.