REAL ANALYSIS

Qualifying Examination

May 1999

Let m be the Legesgue measure, and $< E_i >$ be a sequence of measurable sets.

1a. Prove that

$$m(\cup_{i=1}^{\infty} E_1) \leq \sum_{i=1}^{\infty} mE_i$$

1b. If the sets E_n are pairwise disjoint, prove that

$$m(\cup_{i=1}^{\infty} E_1) = \sum_{i=1}^{\infty} mE_i$$

2a. State Hahn-Banach Theorem.

2b. Use the Hahn-Banach Theorem to prove that the Riesz Representation Theorem does not hold for bounded linear fuctional on $L^{\infty}[0,1]$, or the dual space of $L^{\infty}[0,1]$ is not $L^1[0,1]$.

3a. Prove that if $< f_n >$ is a sequence of mappings of a countable set D into a metric space Y such that for each $x \in D$ the closure of the set $\{f_n(x) : 0 \leq n < \infty\}$ is compact, then there is a subsequence that converges for each $x \in D$.

3b. What is a "diagonal process"? or "Cantor's diagonal method"?

3c. State Ascoli-Arzelá Theorem.

4. State and Prove Riesz representation theorem in Hilbert space.

5. Prove that every bounded sequence in a separable Hilbert space contains a weakly convergent subsequence. (Recall: Let S be a normed linear space. A sequence $< x_n >$ is said to converge to $x \in X$ if $f(x_n) \to f(x)$ for all f in the dual space X^*.)

6a. State the Radon-Nikodym Theorem.

6b. Let (X, M) be a measurable space and let μ, ν, λ be finite measures defined on M. If $\nu \ll \mu$ and $\mu \ll \lambda$ show that $\nu \ll \lambda$ and that their corresponding Radon-Nikodym derivatives satisfy the following:

$$\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right]\left[\frac{d\mu}{d\lambda}\right].$$

Let (X, B, μ) be a measure space.

7a. State the following theorems:
(i) Lebesgue Dominated Convergence Theorem
(ii) Fatou's Lemma
(iii) Monotone Convergence Theorem

7b. Prove that (ii) implies (i).

8a. Give an example of a function $f : (0,1) \to \mathbb{R}$ such that $f \in L^p(0,1)$ but f does not belong to $L^q(0,1)$ for some real number p and q.

8b. Let f be a bounded measurable function on $[0,1]$. Prove that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

9a. Define a (real-valued) function to be absolutely continuous on $[a, b]$.

9b. Define a (real-valued) function of bounded variation over $[a, b]$.

9c. Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0, f(0) = 0$. Prove or disaprove that f is of bounded variation in $[0,1]$.

9d. Prove that if f is a Lipschitz on $(0,1)$ (i.e. there exists a constant K such that $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for every x_1 and x_2 in $(0,1)$.), then it is differentiable almost everywhere.

10. Prove the following generalization of Fatou's Lemma: if $< f_n >$ is a sequence of non-negative functions, then

$$\int \underline{\lim} f_n \leq \underline{\lim} \int f_n.$$