ANALYSIS

Qualifying Examination

May 2000

In the following m denotes the Lebesque measure on $\mathbb{R}$. We write m_x to indicate that the measure is related to the variable x, if necessary.

1) Show that the sets of accumulation points of the Cantor set is the Cantor set itself.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that the set $\{x \in \mathbb{R} \,|\, f \text{ is continuous at } x\}$ is a G_δ.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $\mathcal{A}$ be the collection of sets $\mathcal{A}$ such that $f^{-1}(A)$ is measurable. Show that $\mathcal{A}$ is a σ-algebra.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be lower semicontinuous, show that f is Lebesgue measurable. (A function $f : D \to \mathbb{R}$ is called lower semicontinuous at x if for given $\epsilon > 0$ there is a $\delta > 0$ such that $f(x) - \epsilon \leq f(y)$, for all $y \in D$ with $|x - y| < \delta$. A function $f : D \to \mathbb{R}$ is called lower semicontinuous on D, if it is lower semicontinuous at all points of D).

5) Let $(X, \mathcal{B})$ be a measurable space and let μ, ν be two signed measures on $(X, \mathcal{B})$. Prove or disprove the following statements:

i) $\mu + \nu$ is a signed measure

ii) For a real number λ we have $\lambda\mu$ is a signed measure.

6) Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ a integrable function. Show

$$\lim_{t \to 0} \int_{\mathbb{R}} (f(x) - f(x + t))g(x)dm = 0.$$

7) Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function and $E = \{x \in \mathbb{R} \,|\, f'(x) = 0\}$. Show that $m(f(E)) = 0$.

8) Let $(X, \mathcal{B})$ be a measurable space and let μ, ν be two signed measures on $(X, \mathcal{B})$. Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

9) Let $(X, \mathcal{B}, \mu)$ be a finite measure space, let f be a function in $L^1(\mu)$ and let f_n be a sequence of integrable functions converging to $f \mu - a.e.$. Show that $f_n \to f$ in $L^1(\mu)$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in N$ and all measurable sets A with $\mu(A) < \delta$, we have

$$\int_A f_n d\mu < \epsilon.$$

10) Let μ be an invariant measure on the homogeneous space (X, G) and f a μ–integrable function on X. Then for all $g \in G$ we have

$$\int_X f \circ g d\mu = \int_X f d\mu.$$