Topology Qualifying Review Exam
May 14, 2001

• Each problem is worth 10 points, except for the first one. Total 120 points.

1 [30 points]. Carefully define each term. Do as the example.
Example: **topology of point-wise convergence**
Answer: Let X be a *set*, Y a *topological space*. For $x \in X$ and an open set $U \subset Y$, let $S(x, U) = \{f : f \in Y^X$ and $f(x) \in U\}$. The topology on Y^X generated by the *subbasis* $\{S(x, U) : x \in X, U \subset Y$ open$\}$ is called the topology of point-wise convergence on Y^X. It is identical to the product topology on Y^X.

(1) product topology on $\Pi_{\alpha\in\Lambda} X_\alpha$
(2) locally finite
(3) continuous (most general case, no metric)
(4) compact-open topology
(5) totally bounded
(6) partition of unity on X dominated by an open covering $\mathcal{U}$
(7) *f is homotopic to g relative to A*
(8) deformation retract
(9) covering space
(10) properly discontinuous action

2. Find examples. Explain your examples briefly.
2A. A locally compact space which is not compact.
2B. A bijective, continuous map which is not a homeomorphism
2C. A connected, path-connected space which is not locally path-connected.

3. Prove/Disprove: Let X be a topological space, $A \subset X$. Suppose $a \in \overline{A}$ (closure of A). Then there exists a sequence of points $\{a_n \in A : n \in \mathbb{Z}^+\}$ which converges to a.

4. Prove/Disprove: A complete, bounded metric space is compact.

5. Let X be a locally compact Hausdorff space which is not compact. Prove that X has a one-point compactification.

6. Let $p : X \to Y$ be a quotient map. Let Z be a space and let $f : Y \to Z$ be a map. If $g = f \circ p : X \xrightarrow{p} Y \xrightarrow{f} Z$ is continuous, then f is continuous.

7. Let X be a locally compact Hausdorff space; let $\mathcal{C}(X, Y)$ be the space of all continuous maps from X to Y with the compact-open topology. Prove that the evaluation map $e : X \times \mathcal{C}(X, Y) \to Y$ defined by

$$e(x, f) = f(x)$$

is continuous.

8. (All spaces are connected, path-connected and locally path-connected). A continuous map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism of groups $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Describe the map f_*, and show it is a group homomorphism.

9. (All spaces are connected, path-connected and locally path-connected). Let $p : E \to B$ be a universal covering map (i.e., E is simply connected). Choose base points $e_0 \in E$, $b_0 = p(e_0) \in B$. Show that there is a bijective map $\pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$.

10. Calculate the fundamental group of $(T\#P) \vee S$, where T= torus; P= projective plane; and S = 2-sphere; $\#$ denotes the connected sum, and $\vee$ denotes the wedge.