Algebra Qualifying Exam

August 2007, University of Oklahoma

1. Let $\zeta = e^{2\pi i/10}$. Analyze the representation $\mathbb{Q}(\zeta)/\mathbb{Q}$: Show that it is a Galois extension, find the degree, find the minimal polynomial of ζ over $\mathbb{Q}$, determine the Galois group, determine the number of intermediate fields, and describe all intermediate fields explicitly.

2. Let E/F be a Galois extension of degree 100. Show that there is a unique intermediate field M of degree 4 over F, and that M is Galois over F.

3. For a prime number p let $\mathbb{F}_{p^n}$ be the field with p^n elements.

a) List all intermediate fields of the extension $\mathbb{F}_{p^{12}}/\mathbb{F}_p$. Draw a diagram illustrating all inclusions between these fields.

b) Determine the number of elements α of $\mathbb{F}_{p^{12}}$ such that $\mathbb{F}_{p^{12}} = \mathbb{F}_p(\alpha)$.

4. Consider the polynomial ring $\mathbb{Z}[X]$.

a) Let $I = \{a_0 + a_1X + \ldots + a_nX^n \in \mathbb{Z}[X] : \sum_{i=0}^{n}(-1)^i a_i = 0\}$. Show that I is an ideal. Is it a prime ideal? A maximal ideal?

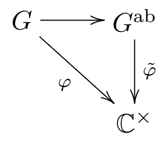
b) For a prime number p let $J = \{a_0 + a_1X + \ldots + a_nX^n \in \mathbb{Z}[X] : \sum_{i=0}^{n}(-1)^i a_i \in p\mathbb{Z}\}$. Show that J is an ideal. Is it a prime ideal? A maximal ideal?

5. Let G be any group. A *character* of G is a homomorphism $\varphi : G \to \mathbb{C}^\times$. Let $\hat{G}$ be the set of all characters of G.

a) Define, in a natural way, on $\hat{G}$ a composition law that makes $\hat{G}$ into a group. Verify the group axioms.

b) Determine the group of characters for $G = \mathbb{Z}$.

c) Let H be the commutator subgroup of G, i.e., the subgroup generated by all elements of the form $xyx^{-1}y^{-1}$, $x, y \in G$. Let $G^{\mathrm{ab}} = G/H$. Show that for a given character φ of G there exists a character $\tilde{\varphi}$ of G^{ab} such that the diagram



is commutative.

6. A matrix $M \in M(n \times n, \mathbb{C})$ is called *nilpotent* if $M^k = 0$ for some $k \geq 0$. Let S be the set of all nilpotent matrices in $M(n \times n, \mathbb{C})$.

a) Show that the group $G = \mathrm{GL}(n, \mathbb{C})$ acts on S via conjugation.

b) In the case $n = 5$, determine the number of orbits for this action and list one representative from each orbit.

7. Let $\mathbb{F}_p$ be the field with p elements.

a) Determine the number of elements of $\mathrm{GL}(2, \mathbb{F}_p)$.

b) Determine the number of elements of the subgroup P of $\mathrm{GL}(4, \mathbb{F}_p)$ consisting of all matrices of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

c) Show that the subgroup U of $\mathrm{GL}(4, \mathbb{F}_p)$ consisting of all matrices of the form

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a Sylow p–subgroup of P.

8. Let G be a finite group and $P < G$ a Sylow p-subgroup. Let $N_G(P)$ be the normalizer of P in G. Let $H < G$ be a subgroup containing $N_G(P)$. Prove that $N_G(H) = H$.