Instructons:

1. Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show *all* relevent work.
2. Please write your solutions on one side only of the provided paper. Please put your last name at the top of each page. Before handing in your solutions, please number the pages consecutively.
3. You should do as many problems as the time allows. You are not expected to answer all parts of all questions. *However you should do at least two problems from each of the four sections.*
4. Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
5. Good luck.

Standing Assumptions:

1. All rings are assumed to have 1.
2. All modules are assumed to be unital left modules.

1 Groups

1. (a) If G is a finite non-cyclic abelian group then there is a positive integer $k < |G|$ such that for every $g \in G$, $g^k = e_G$, where e_G is the identity of G.
   (b) Classify all abelian groups of order $2^4 \cdot 5^2 \cdot 11^3$.
2. Let p be a prime and let G be a finite group of order p^n for some $n \geq 1$. Please prove that the center of G is nontrivial.
3. Let p be a prime number and let G be a group of order $3p^k$ where $k \geq 1$. Please show that G is not simple (be sure to consider all possible cases!).
4. Classify all groups of order 10 up to isomorphism. Be sure to fully justify your answer.
5. Let G be a finite group. Suppose that G has a normal subgroup N whose order $|N|$ and index $[G : N]$ are relatively prime. Show that N is the only subgroup of G of order $|N|$.

2 Rings

1. Let P be a prime ideal in a commutative ring R with 1. Let $S = \{r \in R \mid r \notin P\}$.
   (a) Show S is a multiplicatively closed subset of R.
   (b) Show that $S^{-1}P = \{a/b \in S^{-1}R \mid a \in P, b \in S\}$ is the unique maximal ideal in $S^{-1}R$.
2. Let R be a ring in which $x^2 = x$ for all $x \in R$.
   (a) Prove that R is commutative.
   (b) Let P be a prime ideal in R. Prove that $R/P \cong \mathbb{Z}/2\mathbb{Z}$, the finite field with two elements.
3. If R is an integral domain which satisfies the descending chain condition on ideals, then please show R is a field.
4. Let P be a prime ideal of $\mathbb{Z}[x]$ and suppose $P \cap \mathbb{Z} = \{0\}$. Prove that P is principal.

3 Modules

1. Prove that $A = \mathbb{Z}[x]/(x+12)$ and $B = \mathbb{Z}[x]/(x-13)$ are isomorphic as $\mathbb{Z}$-modules, but not isomorphic as $\mathbb{Z}[x]$-modules.

2. Let $p, q \in \mathbb{Z}$ be primes. Please show that if $p \neq q$, then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = 0$. If $p = q$ then give an explicit $\mathbb{Z}$-module isomorphism $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$.

3. Let R be a commutative ring. Show that R is a Noetherian ring if and only if $M_2(R)$ is a Noetherian ring. Note that the same arguments will prove the analagous statement when 2 is replaced by any $n \geq 1$.

4. Let R be a ring and $f : M \to N$ and $g : N \to M$ be R-module homomorphisms such that $g \circ f = \mathrm{Id}_M$. Show that $N \cong \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$.

4 Fields

1. Let K be a field and let $f(x)$ be an irreducible polynomial in $K[x]$ of prime degree p. Let F be an extension of K over which $f(x)$ is no longer irreducible. Show that p divides $[F : K]$.

2. Let E be the splitting field of $x^4 + 7$ over $\mathbb{Q}$.

   (a) Please describe the splitting field by giving generators as a field extension of $\mathbb{Q}$.

   (b) Please describe the Galois group of the extension $\mathbb{Q} \subseteq Q$ by describing its elements and their action on E. Plese also determine to which common group the Galois group is isomorphic.

   (c) Find all intermediate fields K between E and $\mathbb{Q}$.

3. Let $\mathbb{F}_p$ denote the field with p elements. Determine all primes p for which $x^2 + 1$ is irreducible in $\mathbb{F}_p[x]$.

4. Let E be a Galois extension of F with $[E : F] = 2$ and characteristic of F not equal to 2. Prove that there exists $x \in E$ with $x^2 \in F$.