Department of Mathematics

PhD Qualifying Exam in Analysis (MATH 5453-5463)

August 2010

Directions: Work as many problems as you can; there are 100 points total. If you are asked to "state" a theorem, then no proof is expected unless it is asked for explicitly. You have three hours to complete this exam.

Notation: Unless specified otherwise, $L^p(\mu)$ (for $1 \le p \le \infty$) denotes the space of L^p functions on an abstract measure space $(X, \mathcal{M}, \mu)$. We use notations such as $L^p([a,b])$, $L^p([a,\infty)$, and $L^p(\mathbb{R}))$ to denote the space of L^p functions on the indicated interval with respect to the standard Lebesgue measure, which we denote by $\mathfrak{m}$. Integrals with respect to the standard Lebesgue measure may be denoted by either $\int_{[a,b]} f \, d\mathfrak{m}$ or $\int_a^b f(t) \, d\mathfrak{m}(t)$. We also use $C([a,b])$ to denote the set of continuous real-valued functions on the interval $[a,b]$.

(1) (a) [7pts] Let (X,d) and (Y,ρ) be metric spaces and let $f : X \to Y$ be uniformly continuous on X. Prove that if $(x_n) \subseteq X$ is a Cauchy sequence in X, then $\big(f(x_n)\big)$ is a Cauchy sequence in Y.

(b) [3pts] Does the conclusion of part (a) continue to hold if we only assume that f is continuous on X? Explain.

(2) (a) [3pts] State the Baire Category Theorem.

(b) [7pts] Let (f_n) be a sequence in $C([0,1])$ and let $f \in C([0,1])$ be a function with the property that for every $x \in [0,1]$ there exists $n \in \mathbb{N}$ such that $f_n(x) = f(x)$. Prove that that for some $n \in \mathbb{N}$ the function f_n must agree with f on some open subinterval of $[0,1]$ having positive length.

(3) (a) [3pts] Define what it means for a collection $\mathcal{M}$ of subsets of a set X to be a σ-algebra on X.

(b) [5pts] Let $f : X \to Y$ be a function from a set X into a set Y. If $\mathcal{M}$ is a σ-algebra on X, then prove that $\{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$ is a σ-algebra on Y (be clear about any properties of the inverse image mapping that you use, but you don't have to prove any such properties).

(4) (a) [3pts] State the Monotone Convergence Theorem for the Lebesgue integral.

(b) [6pts] Evaluate the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n}{n + nt + t^4} \, d\mathfrak{m}(t),$$

and justify all steps in your computation.

(5) [8pts] Prove that if $f : [a,b] \to \mathbb{R}$ is increasing and absolutely continuous on $[a,b]$, then f maps Lebesgue-measure-zero sets to Lebesgue-measure-zero sets (that is, prove if $Z \subseteq [a,b]$ is Lebesgue measurable and satisfies $\mathfrak{m}(Z) = 0$, then $\mathfrak{m}(f(Z)) = 0$).

(6) (a) [8pts] Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 \le p < \infty$, let $f : X \to \mathbb{R}$ and $\{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ be functions in $L^p(\mu)$. Prove that if $f_n \to f$ in the L^p norm, then $f_n \to f$ in measure.

(b) [3pts] Give an example to show that the converse of the implication in part (a) does not hold.

(7) [6pts] Let $1 < p < \infty$ and consider the Lebesgue space $L^p([1,\infty))$. For which value(s) of $\alpha \in \mathbb{R}$ is the following assertion true?

$$f \in L^p([1,\infty)) \quad \Rightarrow \quad \int_1^\infty |f(t)| t^{-\alpha} \, d\mathfrak{m}(t) < \infty$$

(note that your answer may depend on p).

(8) [8pts] Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $\mathfrak{m}(E) > 0$. Prove that for every $0 < \delta < 1$ there exists an open interval J such that $\mathfrak{m}(E \cap J) \ge \delta \mathfrak{m}(J)$.

(9) [5pts] Let λ and μ be positive measures defined on a measurable space $(X, \mathcal{M})$ and suppose that $\lambda \ll \mu$ and $\lambda \perp \mu$. Prove that $\lambda(E) = 0$ for every $E \in \mathcal{M}$.

(10) Let $X = [0,1]$, let $\boldsymbol{\mathcal{M}}$ denote the σ-algebra of Lebesgue measurable subsets of $[0,1]$, let $\mathfrak{m} : \boldsymbol{\mathcal{M}} \to [0,1]$ be the Lebesgue measure on $\boldsymbol{\mathcal{M}}$, and let $\mu : \boldsymbol{\mathcal{M}} \to [0,\infty]$ be the counting measure.

(a) [2pts] Show that $\mathfrak{m} \ll \mu$.

(b) [7pts] Prove that there exists no Lebesgue measurable function $f : [0,1] \to [0,\infty]$ such that $\mathfrak{m}(E) = \int_E f\, d\mu \ \forall\, E \in \boldsymbol{\mathcal{M}}$.

(c) [2pts] Explain the relevance of part (b) to the Radon-Nikodým theorem.

(11) (a) [3pts] State Fubini's theorem for functions defined on the product of two measure spaces.

(b) [3pts] Consider the measure $(\mathbb{N}, \boldsymbol{\mathcal{P}}(\mathbb{N}), \mu)$, where $\boldsymbol{\mathcal{P}}(\mathbb{N})$ is the power set of $\mathbb{N}$ and μ is the counting measure. Describe the product measure space of $(\mathbb{N}, \boldsymbol{\mathcal{P}}(\mathbb{N}), \mu)$ with itself (that is, say without proof what the σ-algebra on $\mathbb{N} \times \mathbb{N}$ and the product measure are).

(c) [8pts] Let $F : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be the function defined by

$$F(m,n) = \begin{cases} 2 - \frac{1}{2^m} & \text{if } m = n, \\ -2 + \frac{1}{2^m} & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Explain why F is a measurable function on the product measure space in part (b) and evaluate both of the repeated integrals

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} F(m,n)\, d\mu(m) \right] d\mu(n) \quad \text{and} \quad \int_{\mathbb{N}} \left[\int_{\mathbb{N}} F(m,n)\, d\mu(n) \right] d\mu(m).$$

Explain any discrepancy between your computation and the statement of Fubini's theorem in part (a).