Topology Qualifying Exam **Name:**
August 13, 2012

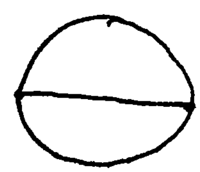
Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail unless they are specifically called for in the problem. Work at least ten problems.

1. Let X be a topological space and let A be a subspace of X. Show that the derived set A' of A is a subset of the closure $\overline{A}$ of A but that A' need not equal $\overline{A}$.

2. Let $f : X \to Y$ be a continuous function between topological spaces. Let α and β be paths in X. Show that if α and β are path homotopic then so are $f\alpha$ and $f\beta$.

3. Let $\mathbb{R}_\ell$ denote the real line with the lower limit topology.
(a) Show that the collection of sets $\mathcal{B} = \{[a, b) \cup (c, d) \mid a, b, c, d \in \mathbb{R} \text{ and } a < b < c < d\}$ forms a basis for this topology.
(b) Does the collection of sets $\mathcal{S} = \{[a, b) \cup (c, d] \mid a, b, c, d \in \mathbb{R} \text{ and } a < b < c < d\}$ form a subbasis for $\mathbb{R}_\ell$? Explain.
(c) Discuss convergence of the sequence $(1/n)_{n \in \mathbb{Z}_+}$ in the Euclidean topology, in $\mathbb{R}_\ell$ and in $\mathcal{T}_\mathcal{S}$.

4. (a) Define what it means for two spaces to have the same homotopy type.
(b) Sketch a proof that the 'theta space' (this is the subspace of $\mathbb{R}^2$ pictured below) has the same homotopy type as the wedge of two circles.
(c) If X and Y have the same homotopy type and X is path connected must Y also be path connected? Explain.
(d) If X and Y have the same homotopy type and X is compact must Y also be compact? Explain.

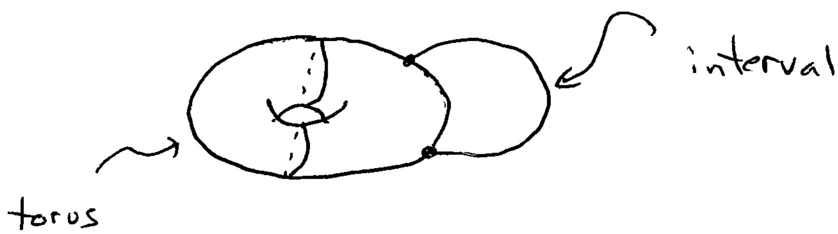
Theta Space $X =$ 

5. (a) Give an outline of a proof of the Extreme Value Theorem: *If X is a nonempty compact space and $f : X \to \mathbb{R}$ is continuous then $f(X)$ has a maximal element.*
(b) Part (a) presumes that $\mathbb{R}$ has the Euclidean topology. Is the statement also true if $\mathbb{R}$ has the cofinite topology?

6. Let X, Y and Z be topological spaces and let $f : X \to Y$ be a quotient map.
(a) Prove that a function $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Z$ is continuous.
(b) Prove that a continuous closed surjection is a quotient map.
(c) Show that every continuous surjection $f : [0, 1] \to [0, 1]$ is a quotient map.

7. The **cone** CX of a topological space X is the quotient space $CX = X \times [0, 1]/\sim$ where $(x, t) \sim (y, s)$ if and only if either $(x, t) = (y, s)$ or $s = t = 1$.
(a) Show that CX is contractible for every space X.
(b) Let A be the image of $X \times \{0\}$ under the projection from $X \times [0, 1]$ to C. Must A be a retract of CX?
(c) Give an example of a space X for which the subspace A from part (b) is a deformation retract of CX.

8. Let X be the union of a torus $T = S^1 \times S^1$ and an interval that meets the torus in two distinct points as shown in the figure below.
(a) Use Van Kampen's Theorem to determine a presentation for the fundamental group of X.
(b) How many distinct non-equivalent 3-sheeted covering spaces $p : \widetilde{X} \to X$ are there? Draw pictures depicting two of them.
(c) How many different homeomorphism types are there among the 3-sheeted covering spaces $\widetilde{X}$ you found in (b)?



9. Prove that a compact Hausdorff space is regular.

10. Let $X = \prod_{\alpha \in J} X_\alpha$ where $\{X_\alpha \mid \alpha \in J\}$ is a collection of topological spaces and let $\pi_\beta : X \to X_\beta$ denote the projection onto X_β
(a) Show that a function f from a space Y to X is continuous if and only if $\pi_\beta \circ f$ is continuous for each $\beta \in J$.
(b) Show that X is path connected if X_α is path connected for each $\alpha \in J$.

11. Let X be a simply connected space. Show that every continuous function $f : S^1 \to X$ extends to a continuous function $F : B^2 \to X$.
(note: $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subset of $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. To say that '$f$ extends to F' means that f is the restriction of F to S^1.)

12. Consider the wedge of two circles $X = S^1 \vee S^1$ whose fundamental group $\pi_1(X, x_0)$ is the free group $F(a, b)$ on $\{a, b\}$ as shown in the figure below.
Draw pictures of a covering space $p : \widetilde{X} \to X$ for which
(a) p is infinite-sheeted and regular.
(b) p is infinite-sheeted but not regular.
(c) $im(p_*)$ is the normal subgroup generated by $\{a^3, b\}$.
(d) $im(p_*)$ is the normal subgroup generated by $\{a^3, a^2ba^{-2}, ab, ab^{-1}\}$.
(e) $im(p_*)$ is the subgroup generated by $\{a^3, a^2ba^{-2}, ab, ab^{-1}\}$.