Algebra PhD Qualifying Examination – January 2013

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show *all* relevant work.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam.
- Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- Good luck.

Problems:

1. (a) Let G be a group and N a normal subgroup of G of index n. Show that $g^n \in N$ for every $g \in G$.

   (b) Let G and H be finite groups such that $(|G|, |H|) = 1$. Show that if $\phi : G \to H$ is a homomorphism, then $\phi(g) = 1_H$ for all $g \in G$ (where 1_H is the identity element of H).

2. Let A and B be subgroups of the additive group of rationals $\mathbb{Q}$. If A is isomorphic to B and $f : A \to B$ is an isomorphism, then show that there is a $q \in \mathbb{Q}$ such that $f(x) = qx$ for all $x \in A$.

3. Let G be a non-trivial finite group and let p be the smallest prime number dividing the order of G. Let H be a subgroup of G of index p. Show that H is normal.

4. Suppose that any element x of a commutative ring A with 1 satisfies $x^n = x$ for some $n > 1$ (depending on x). Prove that every prime ideal of A is maximal.

5. Let R be a commutative ring with identity and let I and J be ideals of R.

   (a) Define
   $$(I : J) = \{r \in R \mid rx \in I \text{ for all } x \in J\}$$
   Show that $(I : J)$ is an ideal of R containing I.

   (b) Show that if P is a prime ideal of R and $x \notin P$, then $(P : (x)) = P$. Here (x) denotes the ideal generated by x.

6. Let R be a commutative ring with identity.

   (a) We say that R has the Descending Chain Condition on Ideals (DCC) if for any chain of ideals in R,
   $$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots ,$$
   there exists an $n \geq 1$ (depending on the chain) for which $I_k = I_n$ for all $k \geq n$. Please show that $R = \mathbb{R}[x]$ *does not* have the DCC.

   (b) We say that R has the Ascending Chain Condition on Ideals (ACC) if for any chain of ideals in R,
   $$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots ,$$
   there exists an $n \geq 1$ (depending on the chain) for which $I_k = I_n$ for all $k \geq n$. Please show that $R = \mathbb{R}[x]$ *does* have the ACC.

   (c) Prove that if R has the ACC, then every ideal is finitely generated (as an ideal). Is the converse true?

7. Let $f(x) = x^4 - 11 \in \mathbb{Q}[x]$.

(a) Explicitly determine the Galois group of $f(x)$ over $\mathbb{Q}$.

(b) Explicitly determine the lattice of intermediate fields for the splitting field of $f(x)$ over $\mathbb{Q}$.

8. Let K/F be a Galois extension of fields such that $\text{Gal}(K/F) \cong A_4$, the alternating group on 4 letters.

   (a) Prove that there is a unique field E such that $F \subseteq E \subseteq K$ and $[E : F] = 3$.

   (b) Prove that there is no intermediate field E such that $F \subseteq E \subseteq K$ and $[E : F] = 2$.