Topology Qualifying Exam **Name:**
January 10th, 2014

Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail. Complete any **7** out of **8** for full credit.

1. Show that the closed interval $[0,1] \subseteq \mathbf{R}$ is compact in the usual Euclidean topology of $\mathbf{R}$ (where the open sets are generated by all open intervals). Do this by showing directly that every open cover has a finite sub-cover.

2. Let $\mathbf{T}^2 = S^1 \times S^1$ denote the 2-dimensional torus. Let $X = \mathbf{T}^2 \vee S^1$ denote the space that is the wedge of the torus and the circle. Describe all 3-fold covering spaces of X. Explain which of these are regular coverings.

3. Let $G = \mathrm{SL}_2(\mathbf{R})$ denote the group of 2×2 matrices with real entries and determinant 1, i.e.,

$$\mathrm{SL}_2(\mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{R}, \quad ad - bc = 1 \right\}$$

Let $X \subseteq \mathbf{C}$ denote the upper half place, i.e., $X = \{z = x + iy : y > 0\}$ with the induced topology. Define an action of G on X as follows: every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ yields a homeomorphism of X: $g \cdot z := \frac{az+b}{cz+d}$

(a) Is this action effective? Prove that it is effective or describe the kernel.

(b) Show that the action of G is transitive and find the isotropy group at $z = i$.

(c) Let $\Gamma \subseteq G$ be the subgroup, $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, i.e., the subgroup of matrices of G all of whose entries are integers. It is known (i.e., you don't have to prove) that Γ is generated by the elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. What is the geometric effect of S^k, T^m on X, where $k, m \in \mathbf{Z}$?

4. Let $\mathbf{R}^\omega$ denote the space that is product of countably many copies of $\mathbf{R}$ endowed with the product topology. Let $X \subseteq \mathbf{R}^\omega$ denote the subspace consisting of points have only finitely many non-zero coordinates. Show that X is path connected.

5. Let $\mathbf{D}^2 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leqslant 1\}$ denote the closed unit disk. Suppose X is a topological space such that there is a covering map $\rho : \mathbf{D}^2 \to X$. Identify the space X up to homeomorphism giving good reasons for your conclusion.

6. Let $X \subseteq \mathbf{R}^3$ denote the space that is the union,

$$\mathbf{S}^2 \cup (\{0\} \times \{0\} \times [-1,1]) = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\} \cup \{(0,0,z) \in \mathbf{R}^3 : -1 \leqslant z \leqslant 1\}$$

Compute the fundamental group of X.

7. Prove or construct a counterexample: Suppose X is a topological space such that $X = A \cup B$, where A is open and contractible, B is closed and contractible and $A \cap B$ is contractible. Then X is also contractible.

8. Suppose $g : \mathbf{S}^2 \to \mathbf{S}^2$ is a continuous map from the 2-sphere to itself such that $g(-x) \neq g(x)$ for all $x \in \mathbf{S}^2$. Show that g must be surjective. Is there a map $f : \mathbf{S}^2 \to \mathbf{S}^2$ that is surjective and such that $f(-x) = f(x)$ for some $x \in \mathbf{S}^2$? Explain your answer.