Topology Qualifying Exam, January 9, 2015

Instructions: Provide justification for each of your answers and make your arguments clear, but try to avoid excessive detail. Please do five problems from each part.

Part 1.

1. Let (X, d) be a compact metric space, and suppose $f \colon X \to X$ is an *isometry*:

$$d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Prove that f is a homeomorphism.

2. Prove that every compact Hausdorff space is regular, and that every compact Hausdorff space is normal.

3. Let X be a space with a countable basis.
(a) Prove that every subset $A \subset X$ has a countable dense subset.
(b) Prove that if $A \subset X$ is uncountable then uncountably many points of A are limit points of A.

4. Let $p \colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact for every $y \in Y$.
(a) Show that if X is Hausdorff, then so is Y.
(b) Show that if X is locally compact, then so is Y.

5. Consider the topologist's sine curve, which is the subset of the plane

$$S = \{x \times \sin(1/x) \mid 0 < x \leq 1\}.$$

Its closure $\overline{S}$ is the set $S \cup \{0 \times y \mid -1 \leq y \leq 1\}$, which is connected, because S is connected. Prove that $\overline{S}$ is not path connected.

6. Let $\{X_\alpha \mid \alpha \in J\}$ be an indexed collection of topological spaces X_α.
(a) Define the product topology on $X = \prod_{\alpha \in J} X_\alpha$.
(b) Prove that the projection maps $p_\alpha \colon X \to X_\alpha$ are continuous.
(c) Given a function $f \colon Z \to X$, state a criterion, in terms of the projections p_α, for f to be continuous. (You do not need to prove your assertion.)
(d) Prove that if each space X_α is path connected, then X is path connected.

Part 2.

7. Recall that the projective plane RP^2 is the space obtained by attaching a 2–cell to the circle by an attaching map $S^1 \to S^1$ with winding number (or degree) 2.
(a) What is the fundamental group of RP^2? Explain how you know this.
(b) Prove that every continuous map $f \colon RP^2 \to S^1$ is homotopic to a constant map.

8. Let $\mathcal{C}(\mathbb{R}, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the compact-open topology. For each n let $f_n \colon \mathbb{R} \to \mathbb{R}$ be the constant function defined by $f_n(x) = n$ for all x. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{C}(\mathbb{R}, \mathbb{R})$ has no convergent subsequence.

9. Recall that the Möbius band M is the space obtained as the quotient of the unit square $[0, 1] \times [0, 1]$ by the relation which identifies the points $(0, y)$ and $(1, 1 - y)$ for each $y \in [0, 1]$.

(a) Find the fundamental group of M, perhaps by relating it to another space. State carefully any theorems that you use.
(b) Describe all of the connected, finite-sheeted covering spaces of M, up to equivalence. Explain why you have found all of them.
(c) How many homeomorphism types are there among the covering spaces you found in part (b)?

10. Define what it means for a map $f\colon X \to Y$ to be a *homotopy equivalence*. Then prove that if f is a homotopy equivalence, and $f(x_0) = y_0$, then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Be sure to state clearly any properties of π_1 you are using.

11. Consider the figure-eight space $X = S^1 \vee S^1$ whose fundamental group $\pi_1(X, x_0)$ is the free group $F(a, b)$ with free generating set $\{a, b\}$. Draw pictures of covering spaces $p\colon \widetilde{X} \to X$ (with basepoint $\tilde{x}_0$) with the following properties:
(a) p is infinite-sheeted and regular
(b) p is infinite-sheeted but not regular
(c) p is finite-sheeted and the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ contains the element a^3 but not the element aba
(d) $\pi_1(\widetilde{X}, \tilde{x}_0)$ is a free group of rank 6.

Be sure your pictures have enough information in them that the covering maps, as well as the covering spaces, are determined.

12. Give the proof that the fundamental group of the circle is infinite cyclic. You may state and use (without proof) the usual path lifting and homotopy lifting properties.