Fall 2017

TOPOLOGY QUALIFY EXAM

Instructions: You have three hours to finish the exam. There are three parts to the exam. For the proofs, provide justifications and make your arguments clear, giving an appropriate level of detail.

I

Definitions and statements of theorems (solve ALL problems)

1. State van Kampen's theorem.
2. State Urysohn's Lemma.
3. Give the definition of a topological space being connected.
4. Give the definition of null-homotopic.
5. State the definition of a manifold and give two examples, one compact and one non-compact.

II

Solve FOUR of these problems.

1. Show that the lower limit topology on $\mathbb{R}$ is not homeomorphic to the standard topology. Hint: study a basic open set in the lower limit topology.
2. Prove that every compact Hausdorff space is normal.
3. Consider $\mathbb{R}$ with the finite complement topology. Show that every subset is compact.
4. Construct an example of a connected space which is not locally connected at any of its points.
5. In the subspace topology, is $\mathbb{R} - \mathbb{Q}$ a manifold? If not, is there a topology under which it is?

III

Solve FOUR of these problems.

1. Let $X \to S^2$ be a covering map. Determine all possibilities for X up to homeomorphism.
2. Show that the Möbius strip does not retract onto its boundary. Hint: consider the Möbius strip as a square with certain edge identifications.
3. Let $p : \tilde{X} \to X$ be a covering map such that $p^{-1}(x)$ is non-empty and finite for every $x \in X$. Show that X is compact and Hausdorff if and only if $\tilde{X}$ is compact and Hausdorff.
4. Construct a space whose fundamental group is $\langle a, b \mid b^2 \rangle$. Then describe and sketch the universal cover of this space.
5. Let $X = S^2 \cup S^2 / \sim$ where $p \sim q$ if p, q are either 1) both north poles of the spheres or 2) both south poles of the spheres. Compute the fundamental group of X.