Algebra PhD Qualifying Examination – January 2018

Instructions:

- Please write a neat, clear, thoughtful, and hopefully correct solution to each of the following problems. Please show *all* relevant work.
- You should do as many problems as the time allows. You are not expected to answer all parts of all questions in order to pass the exam. In particular, even if you haven't solved the earlier parts of a problem, you are free to assume they are true and use them to solve the later parts.
- Each problem is worth the same. Partial credit will be given, but a complete solution of one problem is worth more than partial work on two problems.
- Good luck.

Problems:

1. Let G be a finite group and A a nonempty G-set (ie. a nonempty set with an action by G) where we write $g.a$ for the action of $g \in G$ on $a \in A$. Also assume the action of G on A is *transitive* (that is, for every $a, b \in A$, there exists $g \in G$ such that $g.a = b$).
   (a) Recall $G_a = \{h \in G \mid h.a = a\}$. Prove that G_a is a subgroup of G for any $a \in A$.
   (b) If $a, b \in A$ and $g \in G$ such that $g.a = b$, then prove $G_b = gG_ag^{-1}$.
   (c) Let $C = \cap_{a \in A} G_a$. Prove that C is a normal subgroup of G.
   (d) We call the action of G on A *faithful* if the corresponding group homomorphism $\rho : G \to \mathrm{Perm}(A)$ is one-to-one. Prove that the action of G on A is faithful if and only if $C = \{e_G\}$.
2. Let G be a group with $|G| = 105$.
   (a) Prove that G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.
   (b) Prove that if G has a normal Sylow 3-subgroup, then G is abelian.
   (c) Please give a complete, irredundant list of the isomorphism types all abelian groups with order 105. Be sure to explain why you know your list is complete and has no repeats.
3. Let p and q be prime numbers and let $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ be $\mathbb{Z}$-modules via left multiplication. Please determine the abelian group $\mathbb{Z}/p\mathbb{Z} \otimes_Z \mathbb{Z}/q\mathbb{Z}$ for all p and q. Please do this explicitly. Do not just cite a theorem!
4. A ring R with 1 is called *simple* if the only two-sided ideals are $\{0\}$ and R.
   (a) Show that a commutative ring is simple if and only if it is a field.
   (b) Show that $R = \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is a simple ring. Show R has zero-divisors. Hence, not all elements are units and so the non-commutative version of the first part fails to be true.

5. Let F be a field and let $R = F[x]$.

   (a) If V is a finite-dimensional F-vector space and $f : V \to V$ is a linear map, then explain how to make V into a left R-module via $x.v = f(x)$ for any $v \in V$. Be sure to verify the axioms required to verify that V is indeed a left R-module. We will write V_f for this left R-module.

   (b) Let F be algebraically closed. A left R-module V is called *simple* if the only submodules are $\{0\}$ and V. Show that if V has dimension 2 or more, then it cannot be simple.

   (c) Let $F = \mathbb{R}$ and let V be a 2-dimensional $\mathbb{R}$-vector space. Give an explicit $f : V \to V$ such that V_f is a simple R-module. Of course, be sure to prove that your V_f is indeed simple. This shows being algebraically closed in the previous part of the problem is required.

6. (a) Let K denote the splitting field of $x^7 - 2$ over $\mathbb{Q}$.

   (b) Determine K.

   (c) Determine the Galois group $G = \text{Gal}(K/\mathbb{Q})$.

   (d) Determine all intermediate fields between the splitting field K and $\mathbb{Q}$.