# Real Analysis Qualifying Exam – August 2018

---

Name: ______________________________ ID number: ______________

**Instructions:**

- **USE A SEPARATE SHEET OF PAPER FOR EACH PROBLEM**. Make sure to **write your ID number** on the top right corner of each sheet of paper, but **do not write your name**. Also, **clearly identify the problem** being solved on each sheet.
- **USE YOUR TIME WISELY.** You will have 3 hours for the exam. If you get stuck on a problem, move on and come back to it later.
- **DO NOT ERASE.** Just cross out the stuff you no longer want. You may later realize that what you had was actually useful!
- **JUSTIFY YOUR ANSWERS THOROUGHLY.** For any theorem that you wish to cite, you should either give its name or a statement of the theorem.
- **YOU ONLY NEED TO SOLVE 5 OF THE PROBLEMS. If you attempt more than 5 problems, clearly indicate which problems you are choosing to be graded.** Each problem is worth 20 points.
- In order to receive full credit, you must show appropriate, legible work.
- **NOTATION**: $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Unless otherwise specified, the standard metrics/topologies/measures are always assumed for all spaces involved. $m$ denotes Lebesgue measure on the real line. As customary, the integral of $f$ with respect to Lebesgue measure may be written as $\int f(x)dx$ instead of $\int f dm$. $L^p(X, \mathscr{M}, \mu)$ denotes the space of $\mathscr{M}$-measurable functions $f$ such that $|f|^p$ is $\mu$-integrable (with the necessary modification in the case $p = \infty$), and oftentimes we write just $L^p(X)$ when the measure and the $\sigma$-algebra are clear from context.

| Problem | Points | Problem | Points |
|---|---|---|---|
| 1 | | 6 | |
| 2 | | 7 | |
| 3 | | 8 | |
| 4 | | 9 | |
| 5 | | 10 | |

(1) Let $f \in L^1([0,1])$ be nonnegative. Show that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}dx = m\big(\{x \in [0,1] \,:\, f(x) > 0\}\big).$$

(2) Let $\{f_k\}$ be a sequence of increasing functions on $[0,1]$. Suppose that for each $x \in [0,1]$ the series $\sum_{k=1}^{\infty} f_k(x)$ converges to a real number, so that we can define the function $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Prove that for almost all $x \in [0,1]$ we have

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x).$$

(3) Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$, $f$, $g$ be measurable functions on a measure space $(X, \mathscr{M}, \mu)$, each of which is finite $\mu$-almost everywhere. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges to $f$ in measure, and $\{g_n\}_{n=1}^{\infty}$ converges to $g$ in measure.

(a) Show that $\{f_n + g_n\}_{n=1}^{\infty}$ converges to $f+g$ in measure.

(b) Show that $\{f_n g_n\}_{n=1}^{\infty}$ converges to $fg$ in measure if $\mu(X) < \infty$.

(4) Find all $f \in L^2([1,3])$ such that for every $n = 0,1,2,3,\dots$ we have

$$\int_1^3 x^{2n} f(x)dx = 0.$$

(5) Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on $[0,1]$ such that:

(a) There exists a constant $C > 0$ such that for each $n \in \mathbb{N}$, $|g_n(x)| \le C$ for almost all $x \in [0,1]$.

(b) For every $a \in [0,1]$, $\lim_{n\to\infty} \int_0^a g_n(x)dx = 0$.

Prove that for each $f \in L^1([0,1])$ we have

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)dx = 0.$$

(6) For each $k \in \mathbb{N}$, let $f_k : (-1,1) \to \mathbb{R}$ be a continuous function. Suppose that there exists a constant $C > 0$ such that for every $x \in (-1,1)$ and every $k \in \mathbb{N}$ we have $|f_k(x)| \le C$. For each $k \in \mathbb{N}$ define $g_k : (-1,1) \to \mathbb{R}$ by

$$g_k(x) = \int_0^x f_k(y)dy.$$

Show that the sequence $\{g_k\}_{k=1}^\infty$ has a uniformly convergent subsequence.

(7) Let $U_1, U_2, U_3, \ldots$ be open subsets of $[0,1]$, and suppose that $m\left(\bigcap_{n=1}^\infty U_n\right) = 0$. Prove or find a counterexample: there must exist $n \in \mathbb{N}$ such that $m\left(\overline{U_n}\right) < 1$, where $\overline{U_n}$ denotes the closure of $U_n$ in the usual topology on $[0,1]$.

(8) Let $(X, \mathscr{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\mathscr{N} \subseteq \mathscr{M}$ be a $\sigma$-algebra of subsets of $X$, and let $\nu$ be the restriction of $\mu$ to $\mathscr{N}$. Suppose that $f : X \to \mathbb{R}$ is nonnegative, $\mathscr{M}$-measurable and $\mu$-integrable. Prove that there exists a nonnegative function $g : X \to \mathbb{R}$ which is $\mathscr{N}$-measurable, $\nu$-integrable and such that

$$\int_E g d\nu = \int_E f d\mu \qquad \text{for all } E \in \mathscr{N}.$$

(9) Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a Hilbert space $H$. Let $\{u_k\}_{k=1}^\infty$ be a sequence in $H$ satisfying for each $n \in \mathbb{N}$

$$\lim_{k\to\infty} \langle u_k, e_n \rangle = 0.$$

Suppose that there exists $C > 0$ such that $\|u_k\| \le C$ for all $k \in \mathbb{N}$. Prove that for each $w \in H$ we have

$$\lim_{k\to\infty} \langle u_k, w \rangle = 0.$$

(10) Let $K : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and let $f, g \in L^1(\mathbb{R})$. Let

$$F(x) = \int_{\mathbb{R}} K(xy) f(y) dy, \quad G(x) = \int_{\mathbb{R}} K(xy) g(y) dy.$$

Show that $F$ and $G$ are bounded continuous functions which satisfy

$$\int_{\mathbb{R}} f(x) G(x) dx = \int_{\mathbb{R}} F(x) g(x) dx.$$