Real Analysis Qualifying Exam – January 2019

Name: ______________________________ ID number: ______________

Instructions:

- **USE A SEPARATE SHEET OF PAPER FOR EACH PROBLEM**. Make sure to **write your ID number** on the top right corner of each sheet of paper, but **do not write your name**. Also, **clearly identify the problem** being solved on each sheet.
- **USE YOUR TIME WISELY.** You will have 3 hours for the exam. If you get stuck on a problem, move on and come back to it later.
- **DO NOT ERASE.** Just cross out the stuff you no longer want. You may later realize that what you had was actually useful!
- **JUSTIFY YOUR ANSWERS THOROUGHLY.** For any theorem that you wish to cite, you should either give its name or a statement of the theorem.
- **YOU ONLY NEED TO SOLVE 5 OF THE PROBLEMS. If you attempt more than 5 problems, clearly indicate which problems you are choosing to be graded.** Each problem is worth 20 points.
- In order to receive full credit, you must show appropriate, legible work.
- **NOTATION**: $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Unless otherwise specified, the standard metrics/topologies/measures are always assumed for all spaces involved. m denotes Lebesgue measure on the real line. As customary, the integral of f with respect to Lebesgue measure may be written as $\int f(x)dx$ instead of $\int f dm$. $L^p(X,\mathscr{M},\mu)$ denotes the space of $\mathscr{M}$-measurable functions f such that $|f|^p$ is μ-integrable (with the necessary modification in the case $p=\infty$), and oftentimes we write just $L^p(X)$ when the measure and the σ-algebra are clear from context; the norm of f in $L^p(X,\mathscr{M},\mu)$ is denoted $\|f\|_p$.

Problem	Points	Problem	Points
1		6	
2		7	
3		8	
4		9	
5		10	

(1) Let f be a nonnegative, measurable function on the real line, such that the function $g(x) = \sum_{n=1}^{\infty} f(x+n)$ is integrable on the real line. Show that $f = 0$ almost everywhere.

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(2) Suppose that $f \in L^1[0,1]$. For $x \in [0,1]$, let

$$F(x) = \int_0^x f(t)dt.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\varphi(0) = 0$. Show that there exists $g \in L^1[0,1]$ such that for every $x \in [0,1]$ we have

$$\varphi\big(F(x)\big) = \int_0^x g(t)dt.$$

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(3) Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions on a finite measure space $(X, \mathscr{M}, \mu)$. Recall that $(f_n)_{n=1}^{\infty}$ is said to be *uniformly integrable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f_n| < \varepsilon$ for all measurable sets $E \subseteq X$ satisfying $\mu(E) < \delta$ and all n. Prove that if $(f_n)_{n=1}^{\infty}$ is uniformly integrable, $\sup_n \|f_n\|_1 < \infty$, and $(f_n)_{n=1}^{\infty}$ converges in measure to 0, then $\lim_{n\to\infty} \|f_n\|_1 = 0$.

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(4) Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function such that $\int_0^1 f(t)e^{nt}dt = 0$ for every $n = 0, 1, 2, \ldots$ Prove that $f(t) = 0$ for almost every $t \in [0,1]$.

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(5) Prove that the following limit exists and compute its value:

$$\lim_{n\to\infty} \int_0^n \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x} dx.$$

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(6) Let $f : [0,1] \to \mathbb{R}$ be a measurable function satisfying $0 < f(x) < \infty$ for each $x \in [0,1]$. Show that

$$\left[\int_0^1 f(x)dx \right] \cdot \left[\int_0^1 \frac{1}{f(x)} dx \right] \geq 1.$$

(7) Let H be a Hilbert space, and let $(v_n)_{n=1}^{\infty}$ be an orthonormal sequence in H. Show that if $\varphi : H \to \mathbb{R}$ is a bounded linear functional, then

$$\lim_{n\to\infty} \varphi(v_n) = 0.$$

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(8) A real-valued sequence (x_n) is called *monotone* if it is either increasing (i.e. $x_{n+1} \geq x_n$ for all n) or decreasing (i.e. $x_{n+1} \leq x_n$ for all n). Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on $[0,1]$. Suppose that for each $x \in [0,1]$, the sequence $\big(f_n(x)\big)_{n=1}^{\infty}$ is eventually monotone (that is, there exists $N_x \in \mathbb{N}$ such that the sequence $\big(f_n(x)\big)_{n=N_x}^{\infty}$ is monotone). Show that there exists an open interval $I \subseteq [0,1]$ and $N \in \mathbb{N}$ such that the sequences

$$\big(f_n(x)\big)_{n=N}^{\infty}, \qquad x \in I$$

are either all increasing or all decreasing.

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(9) Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, let $\mathscr{N}$ be a sub-σ-algebra of $\mathscr{M}$, and let ν be the restriction of μ to $\mathscr{N}$. Given a μ-integrable function f, show that there exists a $\mathscr{N}$-measurable function f_0 satisfying

$$\int_X fg\,d\mu = \int_X f_0 g\,d\nu$$

for every $\mathscr{N}$-measurable function g such that fg is integrable.

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(10) Let $f \in L^1(\mathbb{R})$. With $h > 0$ fixed, define a function φ_h on $\mathbb{R}$ by setting

$$\varphi_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt.$$

(a) Show that φ_h is continuous.

(b) Show that $\varphi_h \in L^1(\mathbb{R})$ and $\|\varphi_h\|_1 \leq \|f\|_1$.